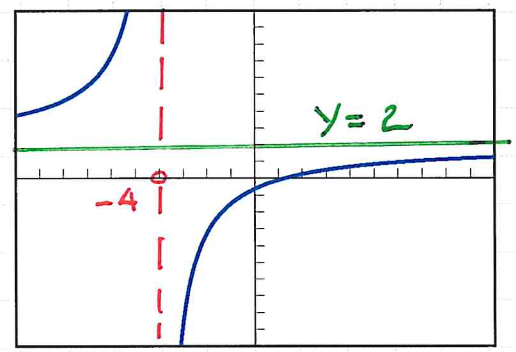
2.6 Rational Functions (Part 1)

Definition: A rational function is the quotient of two polynomials.

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}$$

Ex. ① Consider the graph of $g(x) = \dfrac{4x-5}{2x+8}$

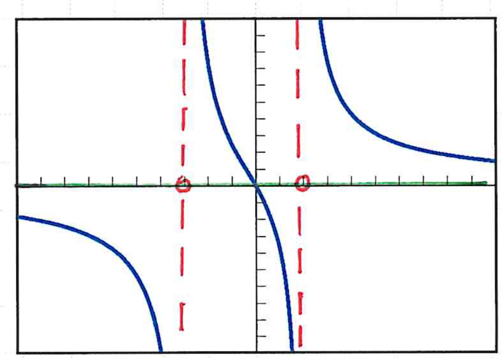


Note that the function is undefined when $x = -4$. Why? Division by 0.

This results in a Vertical Asymptote

VA: $x = -4$ HA: $y = 2$

Ex. ② Consider the graph of $h(x) = \dfrac{15x}{x^2 + x - 6}$



$h(x)$ is undefined when $D(x) = 0$.

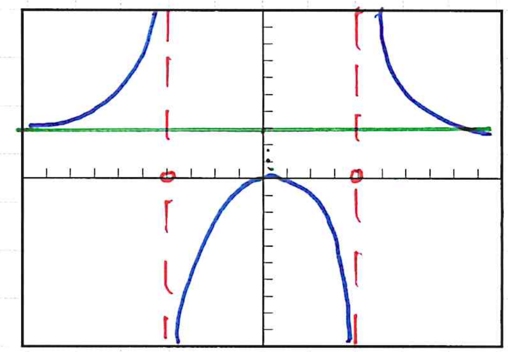
So, solve: $x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$x = -3$ or $x = 2$

VA: $x = -3$, $x = 2$ HA: $y = 0$

Ex. ③ Use a calculator to sketch the graph of $m(x) = \frac{3x^2}{x^2-16}$



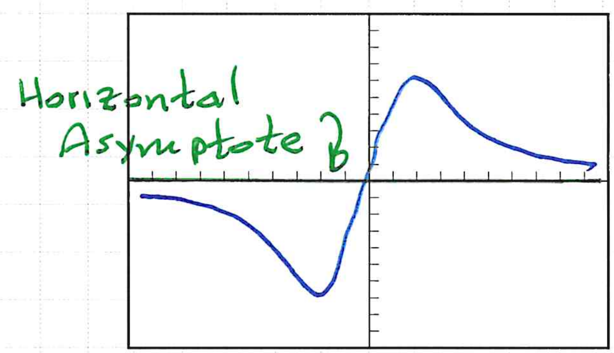
$m(x)$ is undefined when

$x^2 - 16 = 0 \Rightarrow (x+4)(x-4) = 0$

$x = -4, \; x = 4$

HA: $y = 3$ VA: $x = -4$, $x = 4$

Ex. ④ Use a calculator to sketch the graph of $n(x) = \frac{15x}{x^2+1}$



$n(x)$ is undefined when

$x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm\sqrt{-1}$

No solutions. No VA!

HA: $y = 0$ VA: None

Conclusion: When simplified completely, the rational function $f(x) = \frac{N(x)}{D(x)}$ has a VA at the ZEROS of $D(x)$.

Definition: The line $y = b$ is a horizontal asymptote (HA) of f

if $f(x) \to b$ as $x \to \infty$ or $x \to -\infty$.

Conclusion: The graph of $f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0}$

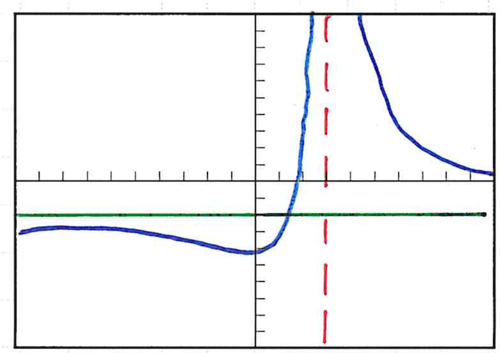
has at most one horizontal asymptote determined by comparing the degrees of $N(x)$ and $D(x)$.

a) If $n < m$ the graph of f has the line $y = 0$ as its HA.

b) If $n = m$ the graph of f has the line $y = a_n / b_m$ as its HA.

c) If $n > m$ the graph of f does not have a HA.

Ex. 5 Sketch: $r(x) = \frac{28x - 44 - 2x^2}{x^2 - 6x + 9}$



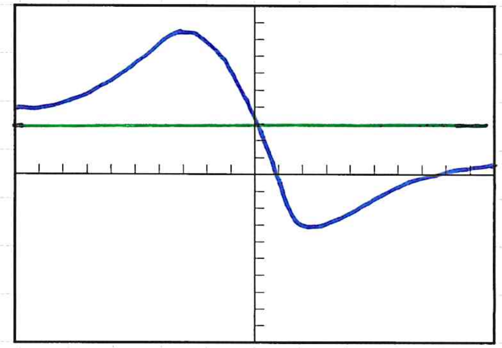
VA: $x^2 - 6x + 9 = 0 \Rightarrow (x-3)(x-3) = 0$

HA: $n = 2$, $m = 2$

$y = \frac{-2}{1} = -2$

VA: $x = 3$ HA: $y = -2$

Ex. 6 Sketch: $g(x) = \dfrac{3x^2 - 24x + 12}{x^2 + 4}$



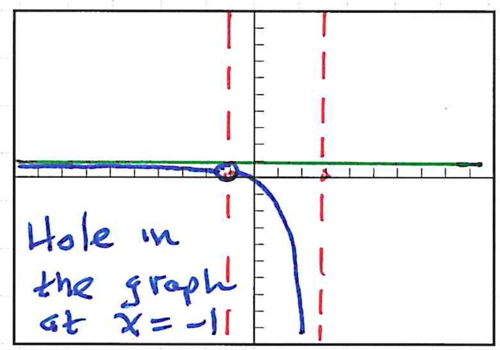
VA: $x^2 + 4 = 0 \Rightarrow x = \pm\sqrt{-4}$ No real solution

HA: $n = 2$, $m = 2$

$y = \dfrac{3}{1} = 3$

VA: None HA: $y = 3$

Ex. 7 Sketch: $r(x) = \dfrac{x^2 - 1}{x^2 - 2x - 3} = \dfrac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}(x-3)}$



VA: $x^2 - 2x - 3 = 0 \Rightarrow (x+1)(x-3) = 0$

$x = -1$ $x = 3$

HA: $n = 2$, $m = 2$

$y = \dfrac{1}{1} = 1$

VA: $x = -1$
$x = 3$ HA: $y = 1$