B.3   Solving Equations Algebraically and Graphically

Equations and Solutions of Equations

An **equation** in x is a statement that two algebraic expressions are equal. For example,

$$3x - 5 = 7, \quad x^2 - x - 6 = 0, \quad \text{and} \quad \sqrt{2x} = 4$$

are equations. To **solve** an equation in x means to find all values of x for which the equation is true. Such values are **solutions.** For instance, $x = 4$ is a solution of the equation $3x - 5 = 7$, because $3(4) - 5 = 7$ is a true statement.

The solutions of an equation depend on the kinds of numbers being considered. For instance, in the set of rational numbers, $x^2 = 10$ has no solution because there is no rational number whose square is 10. In the set of real numbers, however, the equation has two solutions: $x = \sqrt{10}$ and $x = -\sqrt{10}$.

An equation can be classified as an *identity*, a *conditional*, or a *contradiction*, as shown in the following table.

Equation	Definition	Example
Identity	An equation that is true for *every* real number in the domain of the variable	$\dfrac{x}{3x^2} = \dfrac{1}{3x}$ is a true statement for any nonzero real value of x.
Conditional	An equation that is true for just *some* (but not all) of the real numbers in the domain of the variable	$x^2 - 9 = 0$ is a true statement for $x = 3$ and $x = -3$, but not for any other real values.
Contradiction	An equation that is *false* for every real number in the domain of the variable	$2x + 1 = 2x - 3$ is a false statement for any real value of x.

A **linear equation in one variable** x is an equation that can be written in the standard form $ax + b = 0$, where a and b are real numbers, with $a \neq 0$.

To solve an equation involving fractional expressions, find the least common denominator (LCD) of all terms in the equation and multiply every term by this LCD. This procedure clears the equation of fractions.

Example 1   Solving an Equation Involving Fractions

Solve $\dfrac{x}{3} + \dfrac{3x}{4} = 2$.

Solution

$$\dfrac{x}{3} + \dfrac{3x}{4} = 2 \qquad \text{Write original equation.}$$

$$(12)\dfrac{x}{3} + (12)\dfrac{3x}{4} = (12)2 \qquad \text{Multiply each term by the LCD of 12.}$$

$$4x + 9x = 24 \qquad \text{Divide out and multiply.}$$

$$13x = 24 \qquad \text{Combine like terms.}$$

$$x = \dfrac{24}{13} \qquad \text{Divide each side by 13.}$$

✓CHECKPOINT   Now try Exercise 23.

Study Tip

For a review of solving one- and two-step linear equations, see Appendix E at this textbook's *Companion Website*.

After solving an equation, you should check each solution in the original equation. For instance, you can check the solution to Example 1 as follows.

$$\frac{x}{3} + \frac{3x}{4} = 2 \qquad \text{Write original equation.}$$

$$\frac{\frac{24}{13}}{3} + \frac{3\left(\frac{24}{13}\right)}{4} \stackrel{?}{=} 2 \qquad \text{Substitute } \tfrac{24}{13} \text{ for } x.$$

$$\frac{8}{13} + \frac{18}{13} \stackrel{?}{=} 2 \qquad \text{Simplify.}$$

$$2 = 2 \qquad \text{Solution checks. } \checkmark$$

When multiplying or dividing an equation by a *variable* expression, it is possible to introduce an **extraneous solution**—one that does not satisfy the original equation. The next example demonstrates the importance of checking your solution when you have multiplied or divided by a variable expression.

Example 2   An Equation with an Extraneous Solution

Solve $\dfrac{1}{x - 2} = \dfrac{3}{x + 2} - \dfrac{6x}{x^2 - 4}$.

Solution

The LCD is $x^2 - 4 = (x + 2)(x - 2)$. Multiplying each term by the LCD and simplifying produces the following.

$$\frac{1}{x - 2}(x + 2)(x - 2) = \frac{3}{x + 2}(x + 2)(x - 2) - \frac{6x}{x^2 - 4}(x + 2)(x - 2)$$

$$x + 2 = 3(x - 2) - 6x, \quad x \neq \pm 2$$

$$x + 2 = 3x - 6 - 6x$$

$$4x = -8$$

$$x = -2 \qquad \text{Extraneous solution}$$

In the original equation, $x = -2$ yields a denominator of zero. So, $x = -2$ is an extraneous solution, and the original equation has *no solution*.

✓CHECKPOINT   Now try Exercise 39.

Study Tip

Recall that the least common denominator (LCD) of several rational expressions consists of the product of all prime factors in the denominators, with each factor given the highest power of its occurrence in any denominator.

Intercepts and Solutions

In Appendix B.2, you learned that the intercepts of a graph are the points at which the graph intersects the x- or y-axis.

Definitions of Intercepts

1. The point $(a, 0)$ is called an **x-intercept** of the graph of an equation when it is a solution point of the equation. To find the x-intercept(s), set y equal to 0 and solve the equation for x.
2. The point $(0, b)$ is called a **y-intercept** of the graph of an equation when it is a solution point of the equation. To find the y-intercept(s), set x equal to 0 and solve the equation for y.

Sometimes it is convenient to denote the x-intercept as simply the x-coordinate of the point $(a, 0)$ rather than the point itself. Unless it is necessary to make a distinction, "intercept" will be used to mean either the point or the coordinate.

It is possible for a graph to have no intercepts, one intercept, or several intercepts. For instance, consider the four graphs shown in Figure B.27.
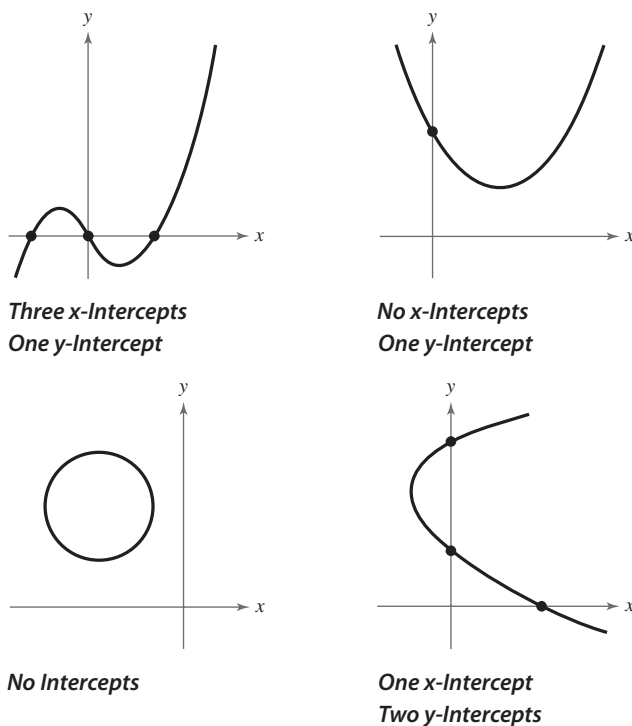

Three x-Intercepts
One y-Intercept

No x-Intercepts
One y-Intercept

No Intercepts

One x-Intercept
Two y-Intercepts

Figure B.27

Example 3   Finding x- and y-Intercepts

Find the x- and y-intercepts of the graph of $2x + 3y = 5$.

Solution

To find the x-intercept, let $y = 0$ and solve for x. This produces

$$2x = 5 \quad \Longrightarrow \quad x = \tfrac{5}{2}$$

which implies that the graph has one x-intercept at $\left(\tfrac{5}{2}, 0\right)$. To find the y-intercept, let $x = 0$ and solve for y. This produces

$$3y = 5 \quad \Longrightarrow \quad y = \tfrac{5}{3}$$

which implies that the graph has one y-intercept at $\left(0, \tfrac{5}{3}\right)$. See Figure B.28.
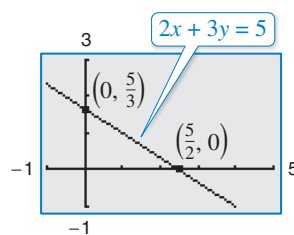

Figure B.28

✓*CHECKPOINT*   Now try Exercise 41.

The concepts of x-intercepts and solutions of equations are closely related. In fact, the following statements are equivalent.

1. The point $(a, 0)$ is an *x-intercept* of the graph of $y = f(x)$.

2. The number a is a *solution* of the equation $f(x) = 0$.

The close connection among x-intercepts and solutions is crucial to your study of algebra. You can take advantage of this connection in two ways. Use your algebraic "equation-solving skills" to find the x-intercepts of a graph and your "graphing skills" to approximate the solutions of an equation.

Finding Solutions Graphically

Polynomial equations of degree 1 or 2 can be solved in relatively straightforward ways. Solving polynomial equations of higher degrees can, however, be quite difficult, especially when you rely only on algebraic techniques. For such equations, a graphing utility can be very helpful.

Graphical Approximations of Solutions of an Equation

1. Write the equation in *general form*, $f(x) = 0$, with the nonzero terms on one side of the equation and zero on the other side.

2. Use a graphing utility to graph the function $y = f(x)$. Be sure the viewing window shows all the relevant features of the graph.

3. Use the *zero* or *root* feature or the *zoom* and *trace* features of the graphing utility to approximate the x-intercepts of the graph of f.

Chapter 2 shows techniques for determining the number of solutions of a polynomial equation. For now, you should know that a polynomial equation of degree n cannot have more than n different solutions.

Example 4 Finding Solutions of an Equation Graphically

Use a graphing utility to approximate the solutions of

$$2x^3 - 3x + 2 = 0.$$

Solution

Graph the function $y = 2x^3 - 3x + 2$. You can see from the graph that there is one x-intercept. It lies between -2 and -1 and is approximately -1.5. By using the *zero* or *root* feature of a graphing utility, you can improve the approximation. Choose a left bound of $x = -2$ (see Figure B.29) and a right bound of $x = -1$ (see Figure B.30). To two-decimal-place accuracy, the solution is $x \approx -1.48$, as shown in Figure B.31. Check this approximation on your calculator. You will find that the value of y is

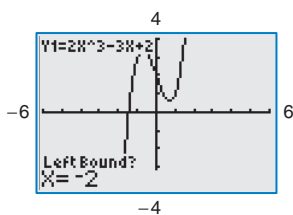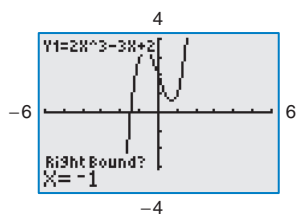$$y = 2(-1.48)^3 - 3(-1.48) + 2 \approx -0.04.$$

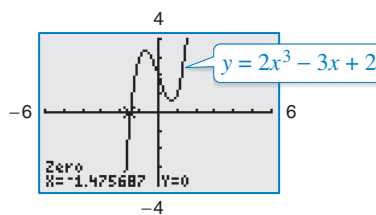
Figure B.29   **Figure B.30**   **Figure B.31**

✔CHECKPOINT   Now try Exercise 69.

You can also use a graphing utility's *zoom* and *trace* features to approximate the solution of an equation. Here are some suggestions for using the *zoom-in* feature of a graphing utility.

1. With each successive zoom-in, adjust the *x*-scale (if necessary) so that the resulting viewing window shows at least one scale mark on each side of the solution.

2. The accuracy of the approximation will always be such that the error is less than the distance between two scale marks. For instance, to approximate the zero to the nearest hundredth, set the *x*-scale to 0.01. To approximate the zero to the nearest thousandth, set the *x*-scale to 0.001.

3. The graphing utility's *trace* feature can sometimes be used to add one more decimal place of accuracy without changing the viewing window.

Unless stated otherwise, all real solutions in this text will be approximated with an error of *at most* 0.01.

Example 5 Approximating Solutions of an Equation Graphically

Use a graphing utility to approximate the solutions of

$$x^2 + 3 = 5x.$$

Solution

In general form, this equation is

$$x^2 - 5x + 3 = 0. \qquad \text{\textcolor{magenta}{Equation in general form}}$$

So, you can begin by graphing

$$y = x^2 - 5x + 3 \qquad \text{\textcolor{magenta}{Function to be graphed}}$$

as shown in Figure B.32. This graph has two *x*-intercepts, and by using the *zoom* and *trace* features you can approximate the corresponding solutions to be

$$x \approx 0.70 \quad \text{and} \quad x \approx 4.30$$

as shown in Figures B.33 and B.34.
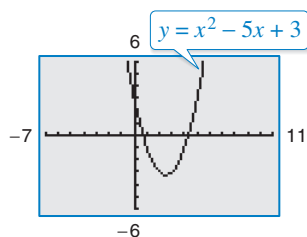

Figure B.32

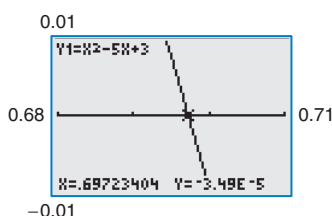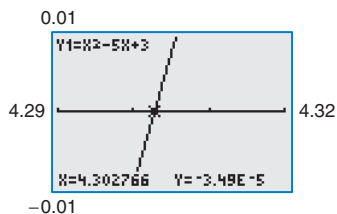**Figure B.33**

Figure B.34

✔*CHECKPOINT*   Now try Exercise 79.

Technology Tip

Use the *zero* or *root* feature of a graphing utility to approximate the solutions of the equation in Example 5 to see that it yields a similar result.

Points of Intersection of Two Graphs

An ordered pair that is a solution of two different equations is called a **point of intersection** of the graphs of the two equations. For instance, in Figure B.35 you can see that the graphs of the following equations have two points of intersection.

$y = x + 2$ Equation 1

$y = x^2 - 2x - 2$ Equation 2

The point $(-1, 1)$ is a solution of both equations, and the point $(4, 6)$ is a solution of both equations. To check this algebraically, substitute

$$x = -1 \quad \text{and} \quad x = 4$$

into each equation.

Figure B.35

Check that $(-1, 1)$ *is a solution.*

Equation 1: $y = -1 + 2 = 1$ Solution checks. ✓

Equation 2: $y = (-1)^2 - 2(-1) - 2 = 1$ Solution checks. ✓

Check that $(4, 6)$ *is a solution.*

Equation 1: $y = 4 + 2 = 6$ Solution checks. ✓

Equation 2: $y = (4)^2 - 2(4) - 2 = 6$ Solution checks. ✓

To find the points of intersection of the graphs of two equations, solve each equation for y (or x) and set the two results equal to each other. The resulting equation will be an equation in one variable, which can be solved using standard procedures, as shown in Example 6.

Example 6   Finding Points of Intersection

Find any points of intersection of the graphs of

$$2x - 3y = -2 \quad \text{and} \quad 4x - y = 6.$$

Algebraic Solution

To begin, solve each equation for y to obtain

$$y = \frac{2}{3}x + \frac{2}{3} \quad \text{and} \quad y = 4x - 6.$$

Next, set the two expressions for y equal to each other and solve the resulting equation for x, as follows.

$\frac{2}{3}x + \frac{2}{3} = 4x - 6$ Equate expressions for y.

$2x + 2 = 12x - 18$ Multiply each side by 3.

$-10x = -20$ Subtract $12x$ and 2 from each side.

$x = 2$ Divide each side by -10.

When $x = 2$, the y-value of each of the original equations is 2. So, the point of intersection is $(2, 2)$.

✓CHECKPOINT   Now try Exercise 87.
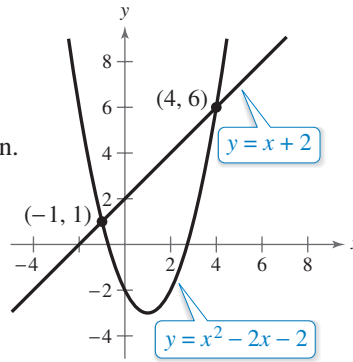
Graphical Solution

To begin, solve each equation for y to obtain

$$y_1 = \frac{2}{3}x + \frac{2}{3} \quad \text{and} \quad y_2 = 4x - 6.$$

Then use a graphing utility to graph both equations in the same viewing window, as shown in Figure B.36. Use the *intersect* feature to approximate the point of intersection.
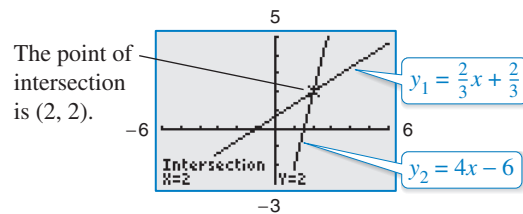
The point of intersection is $(2, 2)$.

$y_1 = \frac{2}{3}x + \frac{2}{3}$

$y_2 = 4x - 6$

Intersection X=2 Y=2

Figure B.36

Another way to approximate the points of intersection of two graphs is to graph both equations with a graphing utility and use the *zoom* and *trace* features to find the point or points at which the two graphs intersect.

Example 7   Approximating Points of Intersection Graphically

Use a graphing utility to approximate any points of intersection of the graphs of the following equations.

$$y = x^2 - 3x - 4 \qquad \qquad \text{Equation 1 (quadratic function)}$$

$$y = x^3 + 3x^2 - 2x - 1 \qquad \text{Equation 2 (cubic function)}$$

Solution

Begin by using the graphing utility to graph both functions, as shown in Figure B.37. From this display, you can see that the two graphs have only one point of intersection. Then, using the *zoom* and *trace* features, approximate the point of intersection to be

$$(-2.17, 7.25) \qquad \qquad \text{Point of intersection}$$

as shown in Figure B.38.
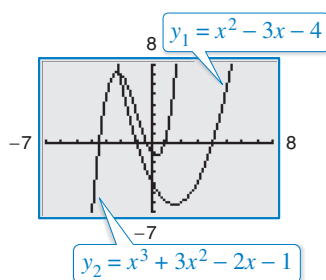
Figure B.37
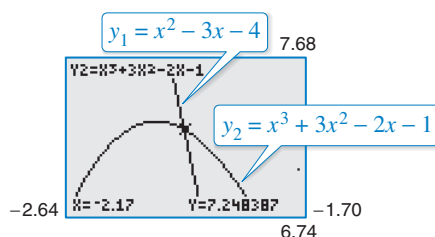
Figure B.38

To test the reasonableness of this approximation, you can evaluate both functions at $x = -2.17$.

Quadratic Function

$$y = (-2.17)^2 - 3(-2.17) - 4$$

$$\approx 7.22$$

Cubic Function

$$y = (-2.17)^3 + 3(-2.17)^2 - 2(-2.17) - 1$$

$$\approx 7.25$$

Because both functions yield approximately the same y-value, you can conclude that the approximate coordinates of the point of intersection are

$$x \approx -2.17 \quad \text{and} \quad y \approx 7.25.$$

✓CHECKPOINT   Now try Exercise 97.

Technology Tip

Use the *intersect* feature of a graphing utility to approximate the point of intersection in Example 7 to see that it yields a similar result.

Solving Polynomial Equations Algebraically

Polynomial equations can be classified by their *degree*. The degree of a polynomial equation is the highest degree of its terms.

Degree	*Name*	*Example*
First	Linear equation	$6x + 2 = 4$
Second	Quadratic equation	$2x^2 - 5x + 3 = 0$
Third	Cubic equation	$x^3 - x = 0$
Fourth	Quartic equation	$x^4 - 3x^2 + 2 = 0$
Fifth	Quintic equation	$x^5 - 12x^2 + 7x + 4 = 0$

You should be familiar with the following four methods for solving quadratic equations *algebraically.*

Solving a Quadratic Equation

Factoring: If $ab = 0$, then $a = 0$ or $b = 0$.   Zero-Factor Property

Example: $x^2 - x - 6 = 0$

$(x - 3)(x + 2) = 0$

$x - 3 = 0$ $\Longrightarrow$ $x = 3$

$x + 2 = 0$ $\Longrightarrow$ $x = -2$

Extracting Square Roots: If $u^2 = c$, then $u = \pm\sqrt{c}$.

Example: $(x + 3)^2 = 16$

$x + 3 = \pm 4$

$x = -3 \pm 4$

$x = 1$ or $x = -7$

Completing the Square: If $x^2 + bx = c$, then

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

$$\left(x + \frac{b}{2}\right)^2 = c + \frac{b^2}{4}.$$

Example: $x^2 + 6x = 5$

$x^2 + 6x + 3^2 = 5 + 3^2$

$(x + 3)^2 = 14$

$x + 3 = \pm\sqrt{14}$

$x = -3 \pm \sqrt{14}$

Quadratic Formula: If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example: $2x^2 + 3x - 1 = 0$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-1)}}{2(2)}$$

$$x = \frac{-3 \pm \sqrt{17}}{4}$$

The methods used to solve quadratic equations can sometimes be extended to polynomial equations of higher degree, as shown in the next two examples.

Example 8   Solving a Polynomial Equation by Factoring

Solve $2x^3 - 6x^2 - 6x + 18 = 0$.

Solution

This equation has a common factor of 2. You can simplify the equation by first dividing each side of the equation by 2.

$$2x^3 - 6x^2 - 6x + 18 = 0 \qquad \text{Write original equation.}$$

$$x^3 - 3x^2 - 3x + 9 = 0 \qquad \text{Divide each side by 2.}$$

$$x^2(x - 3) - 3(x - 3) = 0 \qquad \text{Group terms.}$$

$$(x - 3)(x^2 - 3) = 0 \qquad \text{Factor by grouping.}$$

$$x - 3 = 0 \quad \Longrightarrow \quad x = 3 \qquad \text{Set 1st factor equal to 0.}$$

$$x^2 - 3 = 0 \quad \Longrightarrow \quad x = \pm\sqrt{3} \qquad \text{Set 2nd factor equal to 0.}$$

The equation has three solutions: $x = 3$, $x = \sqrt{3}$, and $x = -\sqrt{3}$. Check these solutions in the original equation. Figure B.39 verifies the solutions graphically.

✓ **CHECKPOINT**   Now try Exercise 155.

> **Study Tip**
>
> Many cubic polynomial equations can be solved using factoring by grouping, as illustrated in Example 8.

Figure B.39

Occasionally, mathematical models involve equations that are of **quadratic type.** In general, an equation is of quadratic type when it can be written in the form

$$au^2 + bu + c = 0$$

where $a \neq 0$ and u is an algebraic expression.

Example 9   Solving an Equation of Quadratic Type

Solve $x^4 - 3x^2 + 2 = 0$.

Solution

This equation is of quadratic type with $u = x^2$.

$$(x^2)^2 - 3(x^2) + 2 = 0$$

To solve this equation, you can use the Quadratic Formula.

$$x^4 - 3x^2 + 2 = 0 \qquad \text{Write original equation.}$$

$$\overset{u^2}{\overbrace{(x^2)^2}} - \overset{3u}{\overbrace{3(x^2)}} + 2 = 0 \qquad \text{Write in quadratic form.}$$

$$x^2 = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(2)}}{2(1)} \qquad \text{Quadratic Formula}$$

$$x^2 = \frac{3 \pm 1}{2} \qquad \text{Simplify.}$$

$$x^2 = 2 \quad \Longrightarrow \quad x = \pm\sqrt{2} \qquad \text{Solutions}$$

$$x^2 = 1 \quad \Longrightarrow \quad x = \pm 1 \qquad \text{Solutions}$$

The equation has four solutions: $x = -1$, $x = 1$, $x = \sqrt{2}$, and $x = -\sqrt{2}$. Check these solutions in the original equation. Figure B.40 verifies the solutions graphically.
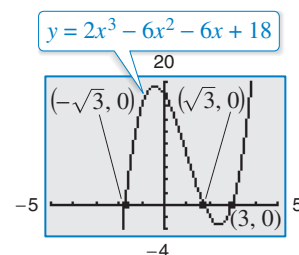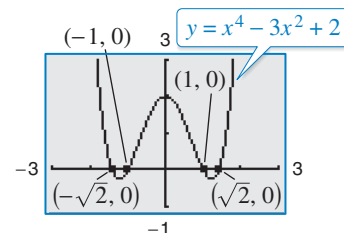
✓ **CHECKPOINT**   Now try Exercise 157.

Figure B.40

Equations Involving Radicals

An equation involving a radical expression can often be cleared of radicals by raising each side of the equation to an appropriate power. When using this procedure, remember to check for extraneous solutions.

Example 10   Solving Equations Involving Radicals

a. $\sqrt{2x + 7} - x = 2$ Original equation

   $\sqrt{2x + 7} = x + 2$ Isolate radical.

   $2x + 7 = x^2 + 4x + 4$ Square each side.

   $x^2 + 2x - 3 = 0$ Write in general form.

   $(x + 3)(x - 1) = 0$ Factor.

   $x + 3 = 0 \implies x = -3$ Set 1st factor equal to 0.

   $x - 1 = 0 \implies x = 1$ Set 2nd factor equal to 0.

By checking these values, you can determine that the only solution is $x = 1$. Figure B.41 verifies the solution graphically.

$y = \sqrt{2x + 7} - x - 2$

(1, 0)

Figure B.41

b. $\sqrt{2x + 6} - \sqrt{x + 4} = 1$ Original equation

   $\sqrt{2x + 6} = 1 + \sqrt{x + 4}$ Isolate $\sqrt{2x + 6}$.

   $2x + 6 = 1 + 2\sqrt{x + 4} + (x + 4)$ Square each side.

   $x + 1 = 2\sqrt{x + 4}$ Isolate $2\sqrt{x + 4}$.

   $x^2 + 2x + 1 = 4(x + 4)$ Square each side.

   $x^2 - 2x - 15 = 0$ Write in general form.

   $(x - 5)(x + 3) = 0$ Factor.

   $x - 5 = 0 \implies x = 5$ Set 1st factor equal to 0.

   $x + 3 = 0 \implies x = -3$ Set 2nd factor equal to 0.

By checking these values, you can determine that the only solution is $x = 5$. Figure B.42 verifies the solution graphically.
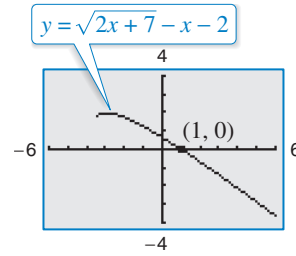
$y = \sqrt{2x + 6} - \sqrt{x + 4} - 1$

(5, 0)
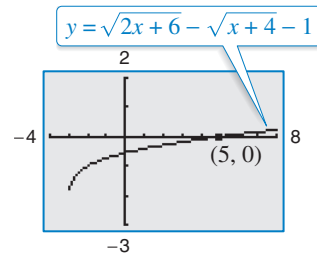
Figure B.42

✓**CHECKPOINT**   Now try Exercise 169.

Example 11   Solving an Equation Involving a Rational Exponent

Solve $(x + 1)^{2/3} = 4$.

Solution

$(x + 1)^{2/3} = 4$ Write original equation.

$\sqrt[3]{(x + 1)^2} = 4$ Rewrite in radical form.

$(x + 1)^2 = 64$ Cube each side.

$x + 1 = \pm 8$ Take square root of each side.

$x = 7, x = -9$ Subtract 1 from each side.

The solutions are $x = 7$ and $x = -9$. Check these in the original equation. Figure B.43 verifies the solutions graphically.

$y = \sqrt[3]{(x + 1)^2} - 4$

(−9, 0)
(7, 0)
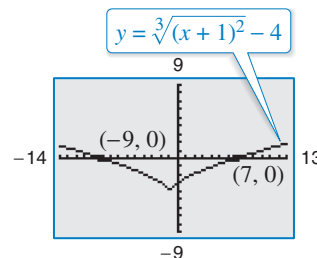
Figure B.43

✓**CHECKPOINT**   Now try Exercise 175.

Equations Involving Fractions or Absolute Values

As demonstrated in Example 1, you can solve an equation involving fractions algebraically by multiplying each side of the equation by the least common denominator of all terms in the equation to clear the equation of fractions.

Example 12   Solving an Equation Involving Fractions

Solve $\dfrac{2}{x} = \dfrac{3}{x-2} - 1$.

Solution

For this equation, the least common denominator of the three terms is

$$x(x-2)$$

so you can begin by multiplying each term of the equation by this expression.

$$\dfrac{2}{x} = \dfrac{3}{x-2} - 1 \qquad\qquad\qquad \text{Write original equation.}$$

$$x(x-2)\dfrac{2}{x} = x(x-2)\dfrac{3}{x-2} - x(x-2)(1) \qquad \text{Multiply each term by the LCD.}$$

$$2(x-2) = 3x - x(x-2), \quad x \neq 0, 2 \qquad \text{Simplify.}$$

$$2x - 4 = -x^2 + 5x \qquad\qquad\qquad \text{Simplify.}$$

$$x^2 - 3x - 4 = 0 \qquad\qquad\qquad \text{Write in general form.}$$

$$(x-4)(x+1) = 0 \qquad\qquad\qquad \text{Factor.}$$

$$x - 4 = 0 \implies x = 4 \qquad\qquad \text{Set 1st factor equal to 0.}$$

$$x + 1 = 0 \implies x = -1 \qquad\qquad \text{Set 2nd factor equal to 0.}$$

Check these solutions in the original equation as follows.

Check $x = 4$

$$\dfrac{2}{x} = \dfrac{3}{x-2} - 1$$

$$\dfrac{2}{4} \overset{?}{=} \dfrac{3}{4-2} - 1$$

$$\dfrac{1}{2} \overset{?}{=} \dfrac{3}{2} - 1$$

$$\dfrac{1}{2} = \dfrac{1}{2} \quad \checkmark$$

Check $x = -1$

$$\dfrac{2}{x} = \dfrac{3}{x-2} - 1$$

$$\dfrac{2}{-1} \overset{?}{=} \dfrac{3}{-1-2} - 1$$

$$-2 \overset{?}{=} -1 - 1$$

$$-2 = -2 \quad \checkmark$$

So, the solutions are $x = 4$ and $x = -1$.

✓CHECKPOINT   Now try Exercise 179.

Example 13   Solving an Equation Involving an Absolute Value

Solve $|x^2 - 3x| = -4x + 6$.

Solution

To solve an equation involving an absolute value algebraically, you must consider the fact that the expression inside the absolute value symbols can be positive or negative. This results in *two* separate equations, each of which must be solved.

First Equation

$$x^2 - 3x = -4x + 6 \qquad\qquad \text{Use positive expression.}$$

$$x^2 + x - 6 = 0 \qquad\qquad \text{Write in general form.}$$

$$(x + 3)(x - 2) = 0 \qquad\qquad \text{Factor.}$$

$$x + 3 = 0 \implies x = -3 \qquad \text{Set 1st factor equal to 0.}$$

$$x - 2 = 0 \implies x = 2 \qquad \text{Set 2nd factor equal to 0.}$$

Second Equation

$$-(x^2 - 3x) = -4x + 6 \qquad\qquad \text{Use negative expression.}$$

$$x^2 - 7x + 6 = 0 \qquad\qquad \text{Write in general form.}$$

$$(x - 1)(x - 6) = 0 \qquad\qquad \text{Factor.}$$

$$x - 1 = 0 \implies x = 1 \qquad \text{Set 1st factor equal to 0.}$$

$$x - 6 = 0 \implies x = 6 \qquad \text{Set 2nd factor equal to 0.}$$

Check

$$|(-3)^2 - 3(-3)| \overset{?}{=} -4(-3) + 6 \qquad \text{Substitute } -3 \text{ for } x.$$

$$18 = 18 \qquad\qquad -3 \text{ checks. } \checkmark$$

$$|2^2 - 3(2)| \overset{?}{=} -4(2) + 6 \qquad \text{Substitute 2 for } x.$$

$$2 \neq -2 \qquad\qquad 2 \text{ does not check.}$$

$$|1^2 - 3(1)| \overset{?}{=} -4(1) + 6 \qquad \text{Substitute 1 for } x.$$

$$2 = 2 \qquad\qquad 1 \text{ checks. } \checkmark$$

$$|6^2 - 3(6)| \overset{?}{=} -4(6) + 6 \qquad \text{Substitute 6 for } x.$$

$$18 \neq -18 \qquad\qquad 6 \text{ does not check.}$$

The solutions are

$$x = -3 \quad \text{and} \quad x = 1.$$

Figure B.44 shows the solutions graphically.

$$y = |x^2 - 3x| + 4x - 6$$
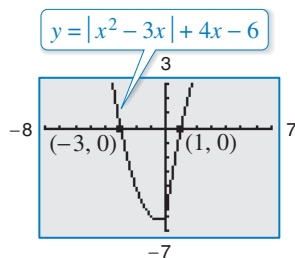
Figure B.44

B.3 Exercises

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank.

1. An _____ is a statement that two algebraic expressions are equal.

2. To find all values for which an equation is true is to _____ the equation.

3. When solving an equation, it is possible to introduce an _____ solution, which is a value that does not satisfy the original equation.

4. An ordered pair that is a solution of two different equations represents a _____ of the graphs of the two equations.

Procedures and Problem Solving

Checking Solutions In Exercises 5–10, determine whether each value of x is a solution of the equation.

Equation	Values
5. $\dfrac{5}{2x} - \dfrac{4}{x} = 3$	(a) $x = -\frac{1}{2}$ (b) $x = 4$
	(c) $x = 0$ (d) $x = \frac{1}{4}$
6. $\dfrac{x}{2} + \dfrac{6x}{7} = \dfrac{19}{14}$	(a) $x = -2$ (b) $x = 1$
	(c) $x = \frac{1}{2}$ (d) $x = 7$
7. $3 + \dfrac{1}{x + 2} = 4$	(a) $x = -1$ (b) $x = -2$
	(c) $x = 0$ (d) $x = 5$
8. $\dfrac{(x + 5)(x - 3)}{2} = 24$	(a) $x = -3$ (b) $x = -2$
	(c) $x = 7$ (d) $x = 9$
9. $\dfrac{\sqrt{x + 4}}{6} + 3 = 4$	(a) $x = -3$ (b) $x = 0$
	(c) $x = 21$ (d) $x = 32$
10. $\dfrac{\sqrt[3]{x - 8}}{3} = -\dfrac{2}{3}$	(a) $x = -16$ (b) $x = 0$
	(c) $x = 9$ (d) $x = 16$

Classifying an Equation In Exercises 11–16, determine whether the equation is an identity or a conditional equation.

11. $2(x - 1) = 2x - 2$

12. $-7(x - 3) + 4x = 3(7 - x)$

13. $x^2 - 8x + 5 = (x - 4)^2 - 11$

14. $x^2 + 2(3x - 2) = x^2 + 6x - 4$

15. $3 + \dfrac{1}{x + 1} = \dfrac{4x}{x + 1}$

16. $\dfrac{5}{x} + \dfrac{3}{x} = 24$

Solving an Equation In Exercises 17–40, solve the equation (if possible). Use a graphing utility to verify your solution.

17. $3x - 5 = 2x + 7$

18. $5x + 3 = 6 - 2x$

19. $4y + 2 - 5y = 7 - 6y$

20. $5y + 1 = 8y - 5 + 6y$

21. $3(y - 5) = 3 + 5y$

22. $5(z - 4) + 4z = 5 - 6z$

✓ 23. $\dfrac{x}{5} - \dfrac{x}{2} = 3$

24. $\dfrac{5x}{4} + \dfrac{1}{2} = x - \dfrac{1}{2}$

25. $\dfrac{3}{2}(z + 5) - \dfrac{1}{4}(z + 24) = 0$

26. $\dfrac{3x}{2} + \dfrac{1}{4}(x - 2) = 10$

27. $\dfrac{2(z - 4)}{5} + 5 = 10z$

28. $\dfrac{5}{3} + 2(y + 1) = \dfrac{10}{3}$

29. $\dfrac{100 - 4u}{3} = \dfrac{5u + 6}{4} + 6$

30. $\dfrac{17 + y}{y} + \dfrac{32 + y}{y} = 100$

31. $\dfrac{5x - 4}{5x + 4} = \dfrac{2}{3}$

32. $\dfrac{10x + 3}{5x + 6} = \dfrac{1}{2}$

33. $\dfrac{1}{x - 3} + \dfrac{1}{x + 3} = \dfrac{10}{x^2 - 9}$

34. $\dfrac{1}{x - 2} + \dfrac{3}{x + 3} = \dfrac{4}{x^2 + x - 6}$

35. $\dfrac{7}{2x + 1} - \dfrac{8x}{2x - 1} = -4$

36. $\dfrac{x}{x + 4} + \dfrac{4}{x + 4} + 2 = 0$

37. $\dfrac{1}{x} + \dfrac{2}{x - 5} = 0$ **38.** $3 = 2 + \dfrac{2}{z + 2}$

✓ **39.** $\dfrac{3}{x^2 - 3x} + \dfrac{4}{x} = \dfrac{1}{x - 3}$ **40.** $\dfrac{6}{x} - \dfrac{2}{x + 3} = \dfrac{3(x + 5)}{x(x + 3)}$

Finding x- and y-Intercepts In Exercises 41–52, find the x- and y-intercepts of the graph of the equation algebraically.

✓ **41.** $y = x - 5$ **42.** $y = -\tfrac{3}{4}x - 3$

43. $y = x^2 + x - 2$ **44.** $y = 4 - x^2$

45. $y = x\sqrt{x + 2}$ **46.** $y = -\tfrac{1}{2}x\sqrt{x + 3}$

47. $y = \dfrac{4}{x}$ **48.** $y = \dfrac{3x - 1}{4x}$

49. $y = |x - 2| - 4$ **50.** $y = 3 - \tfrac{1}{2}|x + 1|$

51. $xy - 2y - x + 1 = 0$ **52.** $xy - x + 4y = 0$

Comparing x-Intercepts and Solutions of an Equation In Exercises 53–56, use a graphing utility to graph the equation and approximate any x-intercepts. Set $y = 0$ and solve the resulting equation. Compare the results with the x-intercepts of the graph.

53. $y = 2(x - 1) - 4$ **54.** $y = 4(x + 3) - 2$

55. $y = 20 - (3x - 10)$ **56.** $y = 10 + 2(x - 2)$

Checking Solutions In Exercises 57–62, the solution(s) of the equation are given. Verify the solution(s) both algebraically and graphically.

Equation	Solution(s)
57. $y = 5(4 - x)$	$x = 4$
58. $y = 3(x - 5) + 9$	$x = 2$
59. $y = x^3 - 6x^2 + 5x$	$x = 0, 5, 1$
60. $y = x^3 - 9x^2 + 18x$	$x = 0, 3, 6$
61. $y = \dfrac{x + 2}{3} - \dfrac{x - 1}{5} - 1$	$x = 1$
62. $y = x - 3 - \dfrac{10}{x}$	$x = -2, 5$

Finding Solutions of an Equation Graphically In Exercises 63–86, write the equation in the form $y = 0$ (if necessary) and use a graphing utility to approximate the solutions of the equation. Use a calculator to verify your answer.

63. $2.7x - 0.4x = 1.2$ **64.** $3.5x - 8 = 0.5x$

65. $25(x - 3) = 12(x + 2) - 10$

66. $1200 = 300 + 2(x - 500)$

67. $\dfrac{3x}{2} + \dfrac{1}{4}(x - 2) = 10$ **68.** $\dfrac{2x}{3} + \dfrac{1}{2}(x - 5) = 6$

✓ **69.** $x^3 - x + 4 = 0$

70. $x^3 - 9x^2 + 26x - 27 = 0$

71. $\dfrac{2x}{3} = 10 - \dfrac{24}{x}$ **72.** $\dfrac{x - 3}{25} = \dfrac{x - 5}{12}$

73. $\dfrac{3}{x + 2} - \dfrac{4}{x - 2} = 5$ **74.** $\dfrac{6}{x} + \dfrac{8}{x + 5} = 3$

75. $(x + 2)^2 = x^2 - 6x + 1$

76. $(x + 1)^2 + 2(x - 2) = (x + 1)(x - 2)$

77. $2x^3 - x^2 - 18x + 9 = 0$

78. $4x^3 + 12x^2 - 8x - 24 = 0$

✓ **79.** $x^4 = 2x^2 - 1$ **80.** $5 = 3x^{1/3} + 2x^{2/3}$

81. $\dfrac{2}{x + 2} = 3$ **82.** $\dfrac{5}{x} = 1 + \dfrac{3}{x + 2}$
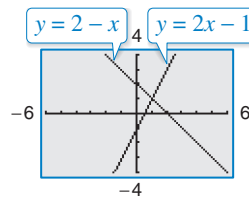
83. $|x - 3| = 4$ **84.** $|x + 1| = 6$

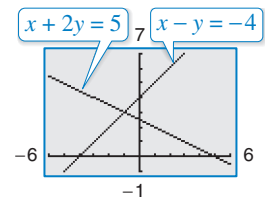**85.** $\sqrt{x - 2} = 3$ **86.** $\sqrt{x - 4} = 8$

Finding Points of Intersection In Exercises 87–92, determine any point(s) of intersection of the equations algebraically. Then use a graphing utility to verify your results.
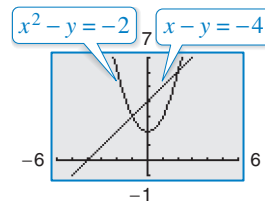
✓ **87.** $y = 2 - x$ **88.** $x - y = -4$
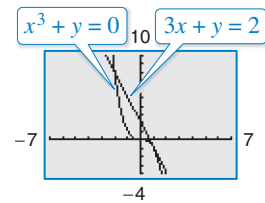 $y = 2x - 1$ $x + 2y = 5$

$y = 2 - x$ $y = 2x - 1$

$x + 2y = 5$ $x - y = -4$

89. $x - y = -4$ **90.** $3x + y = 2$
      $x^2 - y = -2$ $x^3 + y = 0$

$x^2 - y = -2$ $x - y = -4$

$x^3 + y = 0$ $3x + y = 2$

91. $y = x^2 - x + 1$ **92.** $y = -x^2 + 3x + 1$
      $y = x^2 + 2x + 4$ $y = -x^2 - 2x - 4$

$y = x^2 + 2x + 4$ $y = x^2 - x + 1$
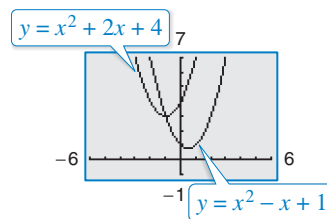
$y = -x^2 + 3x + 1$ $y = -x^2 - 2x - 4$
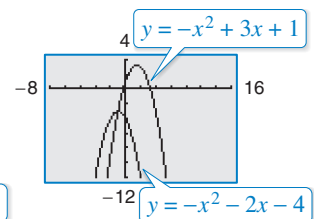
Approximating Points of Intersection Graphically In Exercises 93–98, use a graphing utility to approximate any points of intersection (accurate to three decimal places) of the graphs of the equations. Verify your results algebraically. Test your approximations for reasonableness.

93. $y = 9 - 2x$
$y = x - 3$

94. $y = \frac{1}{3}x + 2$
$y = \frac{5}{2}x - 11$

95. $y = 4 - x^2$
$y = 2x - 1$

96. $y = x^3 - 3$
$y = 5 - 2x$

✓ **97.** $y = 2x^2$
$y = x^4 - 2x^2$

98. $y = -x$
$y = 2x - x^2$

Solving a Quadratic Equation by Factoring In Exercises 99–108, solve the quadratic equation by factoring. Check your solutions in the original equation.

99. $6x^2 + 3x = 0$

100. $9x^2 - 1 = 0$

101. $x^2 - 2x - 8 = 0$

102. $x^2 - 10x + 9 = 0$

103. $3 + 5x - 2x^2 = 0$

104. $2x^2 = 19x + 33$

105. $x^2 + 4x = 12$

106. $-x^2 + 8x = 12$

107. $(x + a)^2 - b^2 = 0$

108. $x^2 + 2ax + a^2 = 0$

Solving an Equation by Extracting Square Roots In Exercises 109–118, solve the equation by extracting square roots. List both the exact solutions and the decimal solutions rounded to two decimal places.

109. $x^2 = 49$

110. $x^2 = 144$

111. $(x - 12)^2 = 16$

112. $(x - 5)^2 = 25$

113. $(3x - 1)^2 + 6 = 0$

114. $(2x + 3)^2 + 25 = 0$

115. $(2x - 1)^2 = 12$

116. $(4x + 7)^2 = 44$

117. $(x - 7)^2 = (x + 3)^2$

118. $(x + 5)^2 = (x + 4)^2$

Completing the Square In Exercises 119–128, solve the quadratic equation by completing the square. Verify your answer graphically.

119. $x^2 + 4x - 32 = 0$

120. $x^2 - 2x - 3 = 0$

121. $x^2 + 6x + 2 = 0$

122. $x^2 + 8x + 14 = 0$

123. $9x^2 - 18x + 3 = 0$

124. $4x^2 - 4x - 99 = 0$

125. $-6 + 2x - x^2 = 0$

126. $-x^2 + x - 1 = 0$

127. $2x^2 + 5x - 8 = 0$

128. $9x^2 - 12x - 14 = 0$

Using the Quadratic Formula In Exercises 129–138, use the Quadratic Formula to solve the equation. Use a graphing utility to verify your solutions graphically.

129. $2 + 2x - x^2 = 0$

130. $x^2 - 10x + 22 = 0$

131. $x^2 + 8x - 4 = 0$

132. $4x^2 - 4x - 4 = 0$

133. $x^2 + 3x = -8$

134. $x^2 + 16 = -5x$

135. $28x - 49x^2 = 4$

136. $9x^2 + 24x + 16 = 0$

137. $3x^2 + 16x + 17 = 0$ **138.** $9x^2 - 6x - 37 = 0$

Choose the Best Method In Exercises 139–148, solve the equation using any convenient method. Use a graphing utility to verify your solutions graphically.

139. $x^2 - 2x - 1 = 0$

140. $11x^2 + 33x = 0$

141. $(x + 3)^2 = 81$

142. $(x - 1)^2 = -1$

143. $x^2 - 14x + 49 = 0$

144. $x^2 - 2x + \frac{13}{4} = 0$

145. $x^2 - x - \frac{11}{4} = 0$

146. $x^2 + 3x - \frac{3}{4} = 0$

147. $(x + 1)^2 = x^2$

148. $a^2x^2 - b^2 = 0, \ a \neq 0$

Solving Equations Algebraically In Exercises 149–166, find all solutions of the equation algebraically. Use a graphing utility to verify the solutions graphically.

149. $4x^4 - 16x^2 = 0$

150. $8x^4 - 18x^2 = 0$

151. $5x^3 + 30x^2 + 45x = 0$

152. $9x^4 - 24x^3 + 16x^2 = 0$

153. $4x^4 - 18x^2 = 0$

154. $20x^3 - 125x = 0$

✓ **155.** $x^3 - 3x^2 - x + 3 = 0$

156. $x^4 + 2x^3 - 8x - 16 = 0$

✓ **157.** $x^4 - 4x^2 + 3 = 0$

158. $x^4 + 5x^2 - 36 = 0$

159. $4x^4 - 65x^2 + 16 = 0$ **160.** $36t^4 + 29t^2 - 7 = 0$

161. $\dfrac{1}{t^2} + \dfrac{8}{t} + 15 = 0$

162. $6 - \dfrac{1}{x} - \dfrac{1}{x^2} = 0$

163. $6\left(\dfrac{s}{s + 1}\right)^2 + 5\left(\dfrac{s}{s + 1}\right) - 6 = 0$

164. $8\left(\dfrac{t}{t - 1}\right)^2 - 2\left(\dfrac{t}{t - 1}\right) - 3 = 0$

165. $2x + 9\sqrt{x} - 5 = 0$ **166.** $6x - 7\sqrt{x} - 3 = 0$

Solving Equations Algebraically In Exercises 167–186, find all solutions of the equation algebraically. Check your solutions both algebraically and graphically.

167. $\sqrt{x - 10} - 4 = 0$

168. $\sqrt{2x + 5} + 3 = 0$

✓ **169.** $\sqrt{x + 1} - 3x = 1$

170. $\sqrt{x + 5} - 2x = 3$

171. $\sqrt[3]{2x + 1} + 8 = 0$

172. $\sqrt[3]{4x - 3} + 2 = 0$

173. $\sqrt{x} - \sqrt{x - 5} = 1$

174. $\sqrt{x} + \sqrt{x - 20} = 10$

✓ **175.** $(x - 5)^{2/3} = 16$

176. $(x^2 - x - 22)^{4/3} = 16$

177. $3x(x - 1)^{1/2} + 2(x - 1)^{3/2} = 0$

178. $4x^2(x - 1)^{1/3} + 6x(x - 1)^{4/3} = 0$

✓ **179.** $\dfrac{1}{x} - \dfrac{1}{x + 1} = 3$

180. $\dfrac{x}{x^2 - 4} + \dfrac{1}{x + 2} = 3$

181. $x = \dfrac{3}{x} + \dfrac{1}{2}$

182. $4x + 1 = \dfrac{3}{x}$

183. $|2x - 1| = 5$

184. $|3x + 2| = 7$

✓ **185.** $|x| = x^2 + x - 3$

186. $|x - 10| = x^2 - 10x$

Comparing *x*-Intercepts and Solutions of an Equation
In Exercises 187–194, (a) use a graphing utility to graph the equation, (b) use the graph to approximate any *x*-intercepts of the graph, (c) set $y = 0$ and solve the resulting equation, and (d) compare the result of part (c) with the *x*-intercepts of the graph.

187. $y = x^3 - 2x^2 - 3x$ **188.** $y = x^4 - 10x^2 + 9$

189. $y = \sqrt{11x - 30} - x$ **190.** $y = 2x - \sqrt{15 - 4x}$

191. $y = \dfrac{1}{x} - \dfrac{4}{x - 1} - 1$ **192.** $y = x + \dfrac{9}{x + 1} - 5$

193. $y = |x + 1| - 2$ **194.** $y = |x - 2| - 3$

195. *Why you should learn it* (p. B21)   The populations (in thousands) of Maryland M and Arizona A from 1990 through 2008 can be modeled by

$$M = 51.1t + 4785, \quad 0 \le t \le 18$$

$$A = 162.0t + 3522, \quad 0 \le t \le 18$$

where t represents the year, with $t = 0$ corresponding to 1990.   (Source: U.S. Census Bureau)

(a) Use a graphing utility to graph each model in the same viewing window over the appropriate domain. Approximate the point of intersection. Round your result to one decimal place. Explain the meaning of the coordinates of the point.

(b) Find the point of intersection algebraically. Round your result to one decimal place. What does the point of intersection represent?

(c) Explain the meaning of the slopes of both models and what they tell you about the population growth rates.

(d) Use the models to estimate the population of each state in 2014. Do the values seem reasonable? Explain.

196. Education   The average salaries S (in thousands of dollars) of secondary classroom teachers in the United States from 2000 through 2007 can be approximated by the model

$$S = 0.010t^2 + 1.01t + 42.9, \ 0 \le t \le 7$$

where t represents the year, with $t = 0$ corresponding to 2000.   (Source: National Education Association)

(a) Determine algebraically when the average salary of a secondary classroom teacher was $47,000.

(b) Verify your answer to part (a) by creating a table of values for the model.

(c) Use a graphing utility to graph the model.

(d) Use the model to predict when the average salary will reach $58,000.

(e) Do you believe the model could be used to predict the average salaries for years beyond 2007? Explain your reasoning.

197. Biology   The metabolic rate of an ectothermic organism increases with increasing temperature within a certain range. Experimental data for oxygen consumption C (in microliters per gram per hour) of a beetle at certain temperatures yielded the model

$$C = 0.45x^2 - 1.65x + 50.75, \quad 10 \le x \le 25$$

where x is the air temperature in degrees Celsius.

(a) Use a graphing utility to graph the consumption model over the specified domain.

(b) Use the graph to approximate the air temperature resulting in oxygen consumption of 150 microliters per gram per hour.

(c) The temperature is increased from 10°C to 20°C. The oxygen consumption is increased by approximately what factor?

198. Thermodynamics   The temperature T (in degrees Fahrenheit) of saturated steam increases as pressure increases. This relationship is approximated by

$$T = 75.82 - 2.11x + 43.51\sqrt{x}, \quad 5 \le x \le 40$$

where x is the absolute pressure in pounds per square inch.

(a) Use a graphing utility to graph the model over the specified domain.

(b) The temperature of steam at sea level $(x = 14.696)$ is 212°F. Evaluate the model at this pressure and verify the result graphically.

(c) Use the model to approximate the pressure for a steam temperature of 240°F.

Conclusions

True or False?   In Exercises 199 and 200, determine whether the statement is true or false. Justify your answer.

199. Two linear equations can have either one point of intersection or no points of intersection.

200. An equation can never have more than one extraneous solution.

201. Think About It   Find c such that $x = 3$ is a solution to the linear equation $2x - 5c = 10 + 3c - 3x$.

202. Think About It   Find c such that $x = 2$ is a solution to the linear equation $5x + 2c = 12 + 4x - 2c$.

203. Exploration   Given that a and b are nonzero real numbers, determine the solutions of the equations.

(a) $ax^2 + bx = 0$

(b) $ax^2 - ax = 0$