B.2 Graphs of Equations

The Graph of an Equation

News magazines often show graphs comparing the rate of inflation, the federal deficit, or the unemployment rate to the time of year. Businesses use graphs to report monthly sales statistics. Such graphs provide geometric pictures of the way one quantity changes with respect to another. Frequently, the relationship between two quantities is expressed as an **equation.** This appendix introduces the basic procedure for determining the geometric picture associated with an equation.

For an equation in the variables x and y, a point (a, b) is a **solution point** when substitution of a for x and b for y satisfies the equation. Most equations have *infinitely many* solution points. For example, the equation

$$3x + y = 5$$

has solution points

$$(0, 5), \quad (1, 2), \quad (2, -1), \quad (3, -4)$$

and so on. The set of all solution points of an equation is the **graph of the equation.**

What you should learn
- Sketch graphs of equations by point plotting.
- Graph equations using a graphing utility.
- Use graphs of equations to solve real-life problems.

Why you should learn it
The graph of an equation can help you see relationships between real-life quantities. For example, in Exercise 66 on page B20, a graph can be used to estimate the life expectancies of children born in the years 1948 and 2020.

Example 1 Determining Solution Points

Determine whether (a) $(2, 13)$ and (b) $(-1, -3)$ lie on the graph of

$$y = 10x - 7.$$

Solution

a. $y = 10x - 7$ Write original equation.

$13 \overset{?}{=} 10(2) - 7$ Substitute 2 for x and 13 for y.

$13 = 13$ (2, 13) is a solution. ✓

The point $(2, 13)$ *does* lie on the graph of $y = 10x - 7$ because it is a solution point of the equation.

b. $y = 10x - 7$ Write original equation.

$-3 \overset{?}{=} 10(-1) - 7$ Substitute -1 for x and -3 for y.

$-3 \neq -17$ $(-1, -3)$ is not a solution.

The point $(-1, -3)$ *does not* lie on the graph of $y = 10x - 7$ because it is not a solution point of the equation.

✓CHECKPOINT Now try Exercise 7.

The basic technique used for sketching the graph of an equation is the point-plotting method.

Sketching the Graph of an Equation by Point Plotting

1. If possible, rewrite the equation so that one of the variables is isolated on one side of the equation.

2. Make a table of values showing several solution points.

3. Plot these points on a rectangular coordinate system.

4. Connect the points with a smooth curve or line.

Example 2   Sketching a Graph by Point Plotting

Use point plotting and graph paper to sketch the graph of $3x + y = 6$.

Solution

In this case you can isolate the variable y.

$y = 6 - 3x$ Solve equation for y.

Using negative and positive values of x, and $x = 0$, you can obtain the following table of values (solution points).

x	-1	0	1	2	3
$y = 6 - 3x$	9	6	3	0	-3
Solution point	$(-1, 9)$	$(0, 6)$	$(1, 3)$	$(2, 0)$	$(3, -3)$

Figure B.15

Next, plot the solution points and connect them, as shown in Figure B.15. It appears that the graph is a straight line. You will study lines extensively in Section 1.1.

✓CHECKPOINT   Now try Exercise 11.

The points at which a graph touches or crosses an axis are called the **intercepts** of the graph. For instance, in Example 2 the point

$(0, 6)$ y-intercept

is the y-intercept of the graph because the graph crosses the y-axis at that point. The point

$(2, 0)$ x-intercept

is the x-intercept of the graph because the graph crosses the x-axis at that point.

Example 3   Sketching a Graph by Point Plotting

Use point plotting and graph paper to sketch the graph of

$y = x^2 - 2.$

Solution

Because the equation is already solved for y, make a table of values by choosing several convenient values of x and calculating the corresponding values of y.

x	-2	-1	0	1	2	3
$y = x^2 - 2$	2	-1	-2	-1	2	7
Solution point	$(-2, 2)$	$(-1, -1)$	$(0, -2)$	$(1, -1)$	$(2, 2)$	$(3, 7)$

(a)

(b)

Figure B.16

Next, plot the solution points, as shown in Figure B.16(a). Finally, connect the points with a smooth curve, as shown in Figure B.16(b). This graph is called a *parabola*. You will study parabolas in Section 2.1.

✓CHECKPOINT   Now try Exercise 13.

In this text, you will study two basic ways to create graphs: *by hand* and *using a graphing utility*. For instance, the graphs in Figures B.15 and B.16 were sketched by hand, and the graph in Figure B.19 (on the next page) was created using a graphing utility.
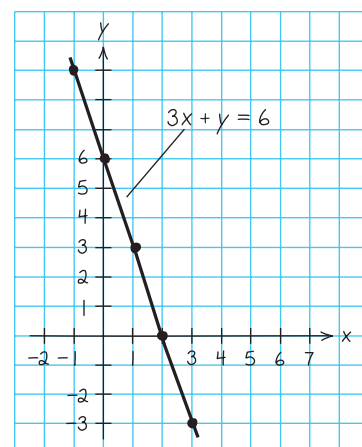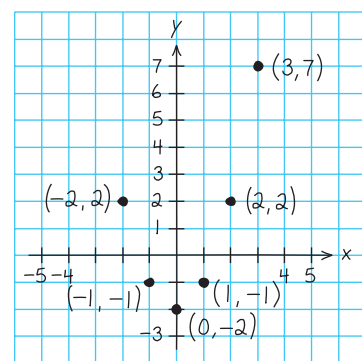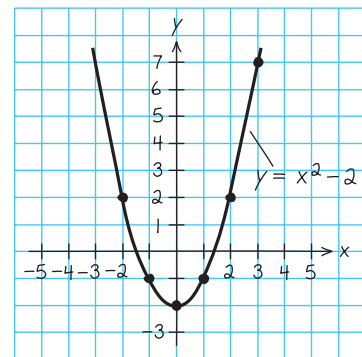
Using a Graphing Utility

One of the disadvantages of the point-plotting method is that to get a good idea about the shape of a graph, you need to plot *many* points. With only a few points, you could misrepresent the graph of an equation. For instance, consider the equation

$$y = \frac{1}{30}x(x^4 - 10x^2 + 39).$$

When you plot the points $(-3, -3)$, $(-1, -1)$, $(0, 0)$, $(1, 1)$, and $(3, 3)$, as shown in Figure B.17(a), you might think that the graph of the equation is a line. This is not correct. By plotting several more points and connecting the points with a smooth curve, you can see that the actual graph is not a line, as shown in Figure B.17(b).

(a)

(b)
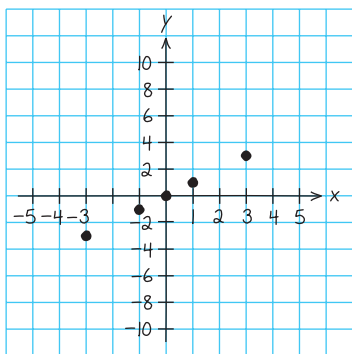
Figure B.17
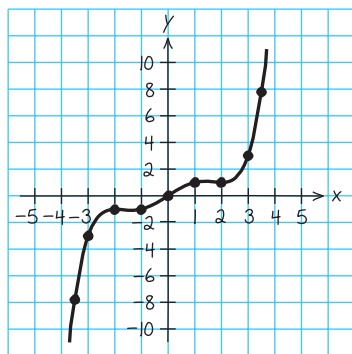
From this, you can see that the point-plotting method leaves you with a dilemma. This method can be very inaccurate when only a few points are plotted, and it is very time-consuming to plot a dozen (or more) points. Technology can help solve this dilemma. Plotting several (even several hundred) points on a rectangular coordinate system is something that a computer or calculator can do easily. For instance, you can enter the equation $y = \frac{1}{30}x(x^4 - 10x^2 + 39)$ in a graphing utility (see Figure B.18) to obtain the graph shown in Figure B.19.
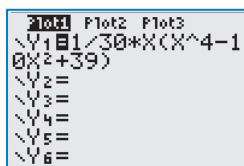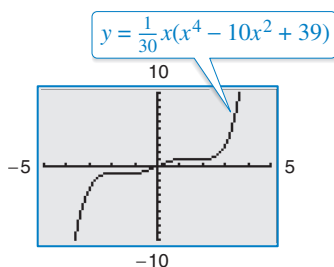

Figure B.18

$$y = \frac{1}{30}x(x^4 - 10x^2 + 39)$$

Figure B.19

Using a Graphing Utility to Graph an Equation

To graph an equation involving x and y on a graphing utility, do the following.

1. Rewrite the equation so that y is isolated on the left side.

2. Enter the equation in the graphing utility.

3. Determine a *viewing window* that shows all important features of the graph.

4. Graph the equation.

Technology Tip

Many graphing utilities are capable of creating a table of values such as the following, which shows some points of the graph in Figure B.17(b). For instructions on how to use the *table* feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website*.

Technology Tip

By choosing different viewing windows for a graph, it is possible to obtain very different impressions of the graph's shape. For instance, Figure B.20 shows a different viewing window for the graph of the equation in Figure B.19. Note how Figure B.20 does not show *all* of the important features of the graph as does Figure B.19. For instructions on how to set up a viewing window, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website*.
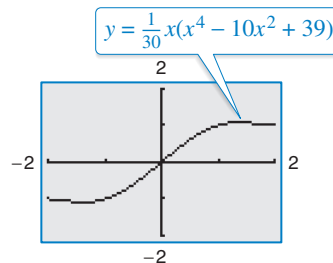
$$y = \frac{1}{30}x(x^4 - 10x^2 + 39)$$

Figure B.20

Example 4   Using a Graphing Utility to Graph an Equation

To graph

$$y = -\frac{x^3}{2} + 2x$$

enter the equation in a graphing utility. Then use a standard viewing window (see Figure B.21) to obtain the graph shown in Figure B.22.
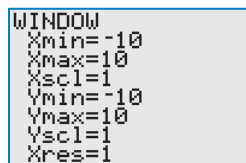
```
WINDOW
  Xmin=-10
  Xmax=10
  Xscl=1
  Ymin=-10
  Ymax=10
  Yscl=1
  Xres=1
```
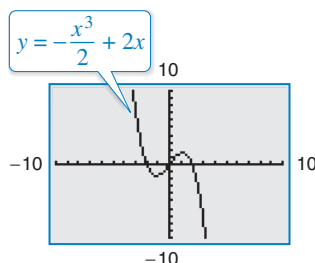
Figure B.21

$y = -\frac{x^3}{2} + 2x$

Figure B.22

✓CHECKPOINT   Now try Exercise 43.

Example 5   Using a Graphing Utility to Graph a Circle

Use a graphing utility to graph $x^2 + y^2 = 9$.

Solution

The graph of $x^2 + y^2 = 9$ is a circle whose center is the origin and whose radius is 3. To graph the equation, begin by solving the equation for y.

$x^2 + y^2 = 9$	Write original equation.
$y^2 = 9 - x^2$	Subtract x^2 from each side.
$y = \pm\sqrt{9 - x^2}$	Take the square root of each side.

Remember that when you take the square root of a variable expression, you must account for both the positive and negative solutions. The graph of $y = \sqrt{9 - x^2}$ is the upper semicircle. The graph of $y = -\sqrt{9 - x^2}$ is the lower semicircle. Enter *both* equations in your graphing utility and generate the resulting graphs. In Figure B.23, note that for a standard viewing window, the two graphs do not appear to form a circle. You can overcome this problem by using a *square setting*, in which the horizontal and vertical tick marks have equal spacing, as shown in Figure B.24. On many graphing utilities, a square setting can be obtained by using a y to x ratio of 2 to 3. For instance, in Figure B.24, the y to x ratio is

$$\frac{Y_{max} - Y_{min}}{X_{max} - X_{min}} = \frac{4 - (-4)}{6 - (-6)} = \frac{8}{12} = \frac{2}{3}.$$
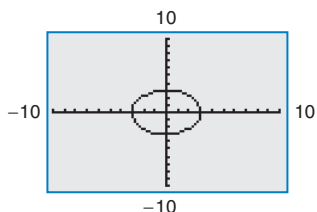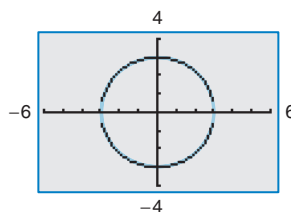
Figure B.23

Figure B.24

✓CHECKPOINT   Now try Exercise 57.

Technology Tip

The standard viewing window on many graphing utilities does not give a true geometric perspective because the screen is rectangular, which distorts the image. That is, perpendicular lines will not appear to be perpendicular and circles will not appear to be circular. To overcome this, you can use a *square setting*, as demonstrated in Example 5.

Technology Tip

Notice that when you graph a circle by graphing two separate equations for y, your graphing utility may not connect the two semicircles. This is because some graphing utilities are limited in their resolution. So, in this text, a blue curve is placed behind the graphing utility's display to indicate where the graph should appear.

Applications

Throughout this course, you will learn that there are many ways to approach a problem. Two of the three common approaches are illustrated in Example 6.

An Algebraic Approach: Use the rules of algebra.

A Graphical Approach: Draw and use a graph.

A Numerical Approach: Construct and use a table.

You should develop the habit of using at least two approaches to solve every problem in order to build your intuition and to check that your answer is reasonable.

The following two applications show how to develop mathematical models to represent real-world situations. You will see that both a graphing utility and algebra can be used to understand and solve the problems posed.

Example 6   Running a Marathon

A runner runs at a constant rate of 4.9 miles per hour. The verbal model and algebraic equation relating distance run and elapsed time are as follows.

Verbal Model: $\boxed{\text{Distance}} = \boxed{\text{Rate}} \cdot \boxed{\text{Time}}$

Equation: $d = 4.9t$

a. Determine how far the runner can run in 3.1 hours.

b. Determine how long it will take the runner to run a 26.2-mile marathon.

Algebraic Solution

a. To begin, find how far the runner can run in 3.1 hours by substituting 3.1 for t in the equation.

$d = 4.9t$ Write original equation.

$ = 4.9(3.1)$ Substitute 3.1 for t.

$ \approx 15.2$ Use a calculator.

So, the runner can run about 15.2 miles in 3.1 hours. Use estimation to check your answer. Because 4.9 is about 5 and 3.1 is about 3, the distance is about

$5(3) = 15.$

So, 15.2 is reasonable.

b. You can find how long it will take to run a 26.2-mile marathon as follows. (For help with solving linear equations, see Appendix E at this textbook's *Companion Website.*)

$d = 4.9t$ Write original equation.

$26.2 = 4.9t$ Substitute 26.2 for d.

$\dfrac{26.2}{4.9} = t$ Divide each side by 4.9.

$5.3 \approx t$ Use a calculator.

So, it will take the runner about 5.3 hours to run 26.2 miles.

✓CHECKPOINT   Now try Exercise 63.

Graphical Solution

a. Use a graphing utility to graph $d = 4.9t$. (Represent d by y and t by x.) Choose a viewing window that shows the graph at $x = 3.1$.

Use the *value* feature to estimate that when $x = 3.1$, the distance is $y \approx 15.2$ miles.

Figure B.25

b. Adjust the viewing window so that it shows the graph at $y = 26.2$.

Use the *zoom* and *trace* features to estimate that when $y \approx 26.2$, the time is $x \approx 5.3$ hours.
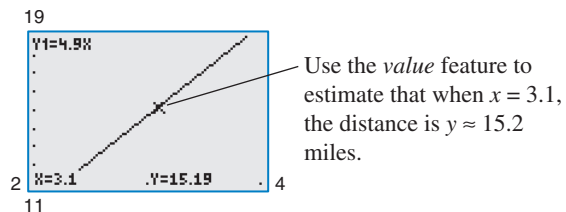
Figure B.26

Note that the viewing window on your graphing utility may differ slightly from those shown in Figures B.25 and B.26.

Example 7   **Monthly Wage**

You receive a monthly salary of $2000 plus a commission of 10% of sales. The verbal model and algebraic equation relating the wages, the salary, and the commission are as follows.

Verbal Model: Wages = Salary + Commission on sales

Equation: $y = 2000 + 0.1x$

a. Sales are $1480 in August. What are your wages for that month?

b. You receive $2225 for September. What are your sales for that month?

Numerical Solution	Graphical Solution
a. Enter $y = 2000 + 0.1x$ in a graphing utility. Then use the *table* feature of the graphing utility to create a table. Start the table at $x = 1400$ with a table step of 10.	**a.** Use a graphing utility to graph $y = 2000 + 0.1x$. Choose a viewing window that shows the graph at $x = 1480$.

When $x = 1480$, the wages are $y = \$2148$.
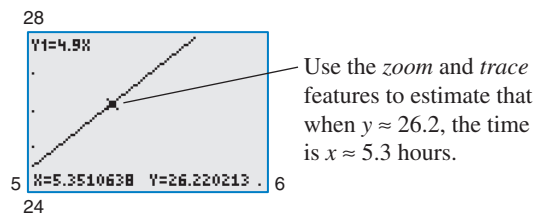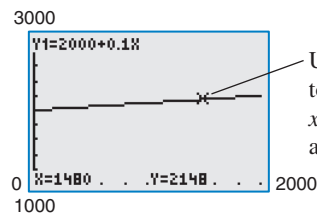

Use the *value* feature to estimate that when $x = 1480$, the wages are $y = \$2148$.

b. Adjust the table to start at $x = 2000$ with a table step of 100.

b. Use the graphing utility to find the value along the x-axis (sales) that corresponds to a y-value of 2225 (wages). Adjust the viewing window so that it shows the graph at $y = 2225$.

From the table, you can see that wages of $2225 result from sales between $2200 and $2300.

Use the *zoom* and *trace* features to estimate that when $y = 2225$, the sales are $x = \$2250$.

You can improve the estimate by starting the table at $x = 2200$ with a table step of 10.

From the table, you can see that wages of $2225 result from sales of $2250.

✓CHECKPOINT   Now try Exercise 65.

Remember to use a different approach to check that your answer is reasonable. For instance, to check the numerical solution to Example 7, use a graphical approach as shown above or use an algebraic approach as follows.

**a.** Substitute 1480 for x in the original equation and solve for y.

$$y = 2000 + 0.1(1480) = \$2148$$

b. Substitute 2225 for y in the original equation and solve for x.

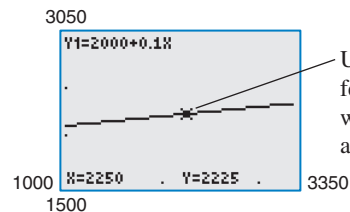$$2225 = 2000 + 0.1x \implies x = \$2250$$

B.2   Exercises

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. For an equation in x and y, if substitution of a for x and b for y satisfies the equation, then the point (a, b) is a _____ .

2. The set of all solution points of an equation is the _____ of the equation.

3. Name three common approaches you can use to solve problems mathematically.

4. List the steps for sketching the graph of an equation by point plotting.

Procedures and Problem Solving

Determining Solution Points   In Exercises 5–10, determine whether each point lies on the graph of the equation.

Equation	Points
5. $y = \sqrt{x + 4}$	(a) $(0, 2)$ (b) $(12, 4)$
6. $y = x^2 - 3x + 2$	(a) $(2, 0)$ (b) $(-2, 8)$
✓ 7. $y = 4 - |x - 2|$	(a) $(1, 5)$ (b) $(1.2, 3.2)$
8. $2x - y - 3 = 0$	(a) $(1, 2)$ (b) $(1, -1)$
9. $x^2 + y^2 = 20$	(a) $(3, -2)$ (b) $(-4, 2)$
10. $y = \frac{1}{3}x^3 - 2x^2$	(a) $\left(2, -\frac{16}{3}\right)$ (b) $(-3, 9)$

Sketching a Graph by Point Plotting   In Exercises 11–14, complete the table. Use the resulting solution points to sketch the graph of the equation. Use a graphing utility to verify the graph.

✓ **11.** $3x - 2y = 2$

x	-2	0	$\frac{2}{3}$	1	2
y					
Solution point					

12. $8x + 4y = 24$

x	-3	-1	0	2	3
y					
Solution point					

✓ **13.** $2x + y = x^2$

x	-1	0	1	2	3
y					
Solution point					

14. $6x - 2y = -2x^2$

x	-4	-3	-2	0	1
y					
Solution point					

Matching an Equation with Its Graph   In Exercises 15–18, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]

(a)

(b)

(c)

(d)
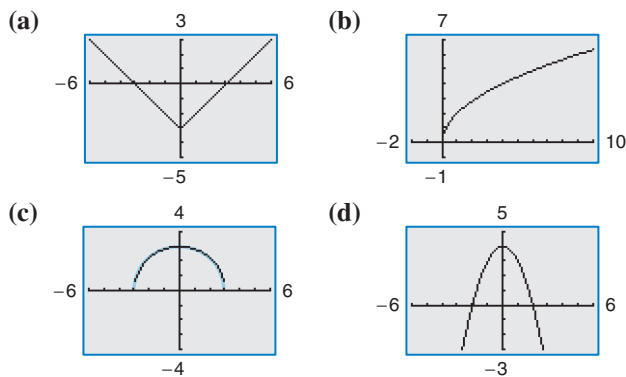
15. $y = 2\sqrt{x}$ **16.** $y = 4 - x^2$

17. $y = \sqrt{9 - x^2}$ **18.** $y = |x| - 3$

Sketching the Graph of an Equation   In Exercises 19–32, sketch the graph of the equation.

19. $y = -4x + 1$ **20.** $y = 2x - 3$

21. $y = 2 - x^2$ **22.** $y = x^2 - 1$

23. $y = x^2 - 3x$ **24.** $y = -x^2 - 4x$

25. $y = x^3 + 2$ **26.** $y = x^3 - 3$

27. $y = \sqrt{x - 3}$ **28.** $y = \sqrt{1 - x}$

29. $y = |x - 2|$ **30.** $y = 4 - |x|$

31. $x = y^2 - 1$ **32.** $x = y^2 + 4$

Using a Graphing Utility to Graph an Equation In Exercises 33–46, use a graphing utility to graph the equation. Use a standard viewing window. Approximate any *x*- or *y*-intercepts of the graph.

33. $y = x - 7$

34. $y = x + 1$

35. $y = 3 - \frac{1}{2}x$

36. $y = \frac{2}{3}x - 1$

37. $y = \dfrac{2x}{x - 1}$

38. $y = \dfrac{6}{x}$

39. $y = x\sqrt{x + 3}$

40. $y = (6 - x)\sqrt{x}$

41. $y = \sqrt[3]{x - 8}$

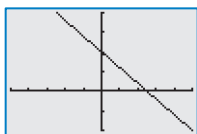**42.** $y = \sqrt[3]{x + 1}$

✓ **43.** $x^2 - y = 4x - 3$

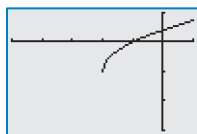**44.** $2y - x^2 + 8 = 2x$

45. $y - 4x = x^2(x - 4)$

46. $x^3 + y = 1$

Describing the Viewing Window of a Graphing Utility In Exercises 47 and 48, describe the viewing window of the graph shown.

47. $y = -10x + 50$

48. $y = \sqrt{x + 2} - 1$

Verifying a Rule of Algebra In Exercises 49–52, explain how to use a graphing utility to verify that $y_1 = y_2$. Identify the rule of algebra that is illustrated.

49. $y_1 = \frac{1}{4}(x^2 - 8)$

$y_2 = \frac{1}{4}x^2 - 2$

50. $y_1 = \frac{1}{2}x + (x + 1)$

$y_2 = \frac{3}{2}x + 1$

51. $y_1 = \frac{1}{5}[10(x^2 - 1)]$

$y_2 = 2(x^2 - 1)$

52. $y_1 = (x - 3) \cdot \dfrac{1}{x - 3}$

$y_2 = 1$

Using a Graphing Utility to Graph an Equation In Exercises 53–56, use a graphing utility to graph the equation. Use the *trace* feature of the graphing utility to approximate the unknown coordinate of each solution point accurate to two decimal places. (*Hint:* You may need to use the *zoom* feature of the graphing utility to obtain the required accuracy.)

53. $y = \sqrt{5 - x}$

(a) $(3, y)$

(b) $(x, 3)$

54. $y = x^3(x - 3)$

(a) $(2.25, y)$

(b) $(x, 20)$

55. $y = x^5 - 5x$

(a) $(-0.5, y)$

(b) $(x, -4)$

56. $y = |x^2 - 6x + 5|$

(a) $(2, y)$

(b) $(x, 1.5)$

Using a Graphing Utility to Graph a Circle In Exercises 57–60, solve for *y* and use a graphing utility to graph each of the resulting equations in the same viewing window. (Adjust the viewing window so that the circle appears circular.)

✓ **57.** $x^2 + y^2 = 16$

58. $x^2 + y^2 = 36$

59. $(x - 1)^2 + (y - 2)^2 = 9$

60. $(x - 3)^2 + (y - 1)^2 = 25$

Determining Solution Points In Exercises 61 and 62, determine which point lies on the graph of the circle. (There may be more than one correct answer.)

61. $(x - 1)^2 + (y - 2)^2 = 25$

(a) $(1, 3)$

(b) $(-2, 6)$

(c) $(5, -1)$

(d) $\left(0, 2 + 2\sqrt{6}\right)$

62. $(x + 2)^2 + (y - 3)^2 = 25$

(a) $(-2, 3)$

(b) $(0, 0)$

(c) $(1, -1)$

(d) $\left(-1, 3 - 2\sqrt{6}\right)$

✓ **63. MODELING DATA**

A manufacturing plant purchases a new molding machine for $225,000. The depreciated value (decreased value) *y* after *t* years is $y = 225{,}000 - 20{,}000t$, for $0 \le t \le 8$.

(a) Use the constraints of the model and a graphing utility to graph the equation using an appropriate viewing window.

(b) Use the *value* feature or the *zoom* and *trace* features of the graphing utility to determine the value of *y* when $t = 5.8$. Verify your answer algebraically.

(c) Use the *value* feature or the *zoom* and *trace* features of the graphing utility to determine the value of *y* when $t = 2.35$. Verify your answer algebraically.

64. MODELING DATA

You buy a personal watercraft for $8100. The depreciated value *y* after *t* years is $y = 8100 - 929t$, for $0 \le t \le 6$.

(a) Use the constraints of the model and a graphing utility to graph the equation using an appropriate viewing window.

(b) Use the *zoom* and *trace* features of the graphing utility to determine the value of *t* when $y = 5545.25$. Verify your answer algebraically.

(c) Use the *value* feature or the *zoom* and *trace* features of the graphing utility to determine the value of *y* when $t = 5.5$. Verify your answer algebraically.

✓ **65. MODELING DATA**

The table shows the median (middle) sales prices y (in thousands of dollars) of new one-family homes in the southern United States from 2000 through 2008. (Sources: U.S. Census Bureau; U.S. Department of Housing and Urban Development)

Year	Median sales price, y
2000	148.0
2001	155.4
2002	163.4
2003	168.1
2004	181.1
2005	197.3
2006	208.2
2007	217.7
2008	203.7

A model that represents the data is given by $y = -0.4221t^3 + 4.690t^2 - 3.47t + 150.9$, $0 \le t \le 8$, where t represents the year, with $t = 0$ corresponding to 2000.

(a) Use the model and the *table* feature of a graphing utility to find the median sales prices from 2000 through 2008. How well does the model fit the data? Explain.

(b) Use the graphing utility to graph the data from the table and the model in the same viewing window. How well does the model fit the data? Explain.

(c) Use the model to estimate the median sales prices in 2012 and 2014. Do the values seem reasonable? Explain.

(d) Use the *zoom* and *trace* features of the graphing utility to determine during which year(s) the median sales price was approximately $150,000.

66. *Why you should learn it* (p. B12)   The table shows the life expectancy y of a child (at birth) in the United States for each of the selected years from 1930 through 2000. (Source: U.S. National Center for Health Statistics)

Year	Life expectancy, y
1930	59.7
1940	62.9
1950	68.2
1960	69.7
1970	70.8
1980	73.7
1990	75.4
2000	77.0

A model that represents the data is given by

$$y = \frac{59.617 + 1.18t}{1 + 0.012t}, \quad 0 \le t \le 70$$

where t is the time in years, with $t = 0$ corresponding to 1930.

(a) Use a graphing utility to graph the data from the table above and the model in the same viewing window. How well does the model fit the data? Explain.

(b) Find the y-intercept of the graph of the model. What does it represent in the context of the problem?

(c) Use the *zoom* and *trace* features of the graphing utility to determine the year when the life expectancy was 73.2. Verify your answer algebraically.

(d) Determine the life expectancy in 1948 both graphically and algebraically.

(e) Use the model to estimate the life expectancy of a child born in 2020.

Conclusions

True or False?   In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.

67. A parabola can have only one x-intercept.

68. The graph of a linear equation can have either no x-intercepts or only one x-intercept.

69. Writing   Your employer offers you a choice of wage scales: a monthly salary of $3000 plus commission of 7% of sales or a salary of $3400 plus a 5% commission. Write a short paragraph discussing how you would choose your option. At what sales level would the options yield the same salary?

70. Finance   You open a savings account and deposit $200. Every week you withdraw $50. The account balance y after t weeks is $y = -50t + 200$, for $0 \le t \le 4$.

(a) Use point plotting and graph paper to sketch the graph of $y = -50t + 200$.

(b) Use a graphing utility to graph $y = -50t + 200$.

(c) Explain how to find an appropriate viewing window for the graph of the equation.

(d) Find the y-intercept of the graph of the model. What does it represent in the context of the problem?