11.7 Ex

$$\sum_{n=1}^{\infty} e^{-n} \cdot n!$$

Solution Let try the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{e^{-(n+1)} \cdot (n+1)!}{e^{-n} \cdot n!}$$

$$= \lim_{n\to\infty} \frac{e^{-n} \cdot e^{-1} (n+1) \cdot n!}{e^{-n} \cdot n!}$$

$$= \lim_{n\to\infty} e^{-1} \cdot (n+1)$$

$$= \infty$$

Therefore by the Ratio Test the series diverges.

11.7) Ex)

$$\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$$

Solution Take the series $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges. Using the Limit Comparison Test we get

$$\lim_{n\to\infty} \frac{\frac{n-1}{n^2+n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2-n}{n^2+n}$$

$$\stackrel{\text{OTE}}{=} \lim_{n\to\infty} \frac{n^2}{n^2}$$

$$= 1$$

Thus the series diverges by the L. C. T.

11.7) Ex)

$$\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$$

Solution Lets Examine

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{(3)^{n+2}}{2^{3n+3}}}{\frac{3^{n+1}}{2^{3n}}}$$

$$= \lim_{n\to\infty} \frac{3^{n+2}}{2^{3n+3}} \cdot \frac{2^{3n}}{3^{n+1}}$$

$$= \lim_{n\to\infty} \frac{3}{2^3}$$

$$= \frac{3}{8} < 1$$

So by the Ratio Test the series coverges absolutely, hence the series converges.

11.7] Ex]

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$$

Solution: ① Let $b_n = \frac{1}{\sqrt{n}-1}$ and $b_n = \frac{1}{\sqrt{n+1}-1}$

clearly $b_{n+1} \leq b_n$. So $\{b_n\}$ is decreasing.

$\lim_{n\to\infty} \frac{1}{\sqrt{n}-1} = 0$. So by the Alternating Series Test the series converges.

② $\sum_{n=2}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}-1} \right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$. Take $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$

which diverges by the p-series test.

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}-1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}-1}$$

$$\overset{DTE}{=} \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}}$$

$$= 1$$

Therefore by the Limit Comparison Test $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges. So the original series $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$ converges conditionally.

11.7 Ex1

$$\sum_{n=1}^{\infty} \frac{\tan\left(\frac{1}{n}\right)}{n}$$

Solution: Lets compare it to the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges by the p-series test.

$$\lim_{n\to\infty} \frac{\frac{\tan\left(\frac{1}{n}\right)}{n}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} n \cdot \tan\left(\frac{1}{n}\right) \quad (\infty \cdot 0)$$

$$= \lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}} \quad \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{\sec^2\left(\frac{1}{n}\right)\cdot\left(-\frac{1}{n^2}\right)}{\left(-\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \sec^2\left(\frac{1}{n}\right)$$

$$= 1$$

Therefore by the Limit Comparison Test the series converges.

11.7) Ex)

$$\sum_{n=1}^{\infty} (-1)^n \cdot 2^{\frac{1}{n}}$$

Solution:

Take $\lim_{n\to\infty} 2^{\frac{1}{n}} = 2^0 = 1$, so $\lim_{n\to\infty} (-1)^n \cdot 2^{\frac{1}{n}}$ does not exist. Therefore by the Test for Divergence the series diverges.

11.7) Ex)

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$$

Solution: Lets use the Root Test.

Take $\lim_{n\to\infty} \left| \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} \right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{\frac{n^2}{n}}$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-n}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$= \frac{1}{e} < 1$$

Thus by the Root Test the series converges absolutely, and hence it converges.

11.7 Ex

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \cdot \ln n}$$ Alternating Series

Solution: Let $b_n = \frac{1}{n \cdot \ln n}$

① $\lim_{n \to \infty} b_n = 0$

Clearly, ② $b_{n+1} \leq b_n$

So the series converges by the Alternating Series Test.

How about absolutely? Lets try the Ratio Test

$$\lim_{n \to \infty} \frac{\frac{1}{(n+1) \cdot \ln(n+1)}}{\frac{1}{n \cdot \ln n}} = \lim_{n \to \infty} \frac{n \cdot \ln n}{n+1 \cdot \ln(n+1)}$$

$$= \lim_{n \to \infty} \frac{n}{n+1} \cdot \lim_{n \to \infty} \frac{\ln n}{\ln(n+1)} \quad \frac{\infty}{\infty}$$

$$\overset{L'H}{=} 1 \cdot \lim_{n \to \infty} \frac{\frac{1}{n}}{\frac{1}{n+1}}$$

$$= 1 \cdot \lim_{n \to \infty} \frac{n+1}{n}$$

$$= 1$$

11.7 Ex Continued

No Conclusion! What's next?
How about Integral Test? Since $\frac{1}{n \cdot \ln n}$ is positive, continuous, and decreasing we let $f(x) = \frac{1}{x \cdot \ln x}$

$$\int_2^\infty \frac{1}{x \cdot \ln x}\,dx = \lim_{t \to \infty} \int_2^t \frac{1}{x \cdot \ln x}\,dx$$

let $u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \lim_{t \to \infty} \int_{\ln 2}^{\ln t} \frac{1}{u}\,du$$

New Limits

$u(2) = \ln 2$

$u(t) = \ln t$

$$= \lim_{t \to \infty} \Big[\ln u\Big]_{\ln 2}^{\ln t}$$

$$= \lim_{t \to \infty} \Big[\underbrace{\ln(\ln t)}^{\infty} - \ln(\ln 2)\Big]$$

$$= \infty$$

Hence the series $\sum_{n=2}^{\infty} \frac{1}{n \cdot \ln n}$ diverges.

Thus the original series converges conditionally

11.7) Ex)

$$\sum_{n=1}^{\infty} \frac{1}{2n^2+3n+1}$$

Solution: $\frac{1}{2n^2+3n+1}$ is positive, continuous and decreasing. Integral Test!

$$\int_1^\infty \frac{1}{2x^2+3x+1}\,dx = \lim_{t\to\infty} \int_1^t \left(\frac{2}{2x+1} - \frac{1}{x+1}\right) dx$$

$$= \lim_{t\to\infty} \Big[\ln(2x+1) - \ln(x+1)\Big]_1^t$$

$$= \lim_{t\to\infty} \left[\ln\left(\frac{2x+1}{x+1}\right)\right]_1^t$$

$$= \lim_{t\to\infty} \left[\ln\left(\frac{2t+1}{t+1}\right) - \ln\left(\frac{3}{2}\right)\right]$$

$$\overset{DTE}{=} \ln 2 - \ln \frac{3}{2}$$

$$= \ln \frac{4}{3}$$

Hence by the Integral Test the Series converges.