16. Plot the wave function \( \psi_{1s}(r) \) (see Eq. 42.22) and the radial probability density function \( P_{1s}(r) \) (see Eq. 42.25) for hydrogen. Let \( r \) range from 0 to \( 1.5a_0 \), where \( a_0 \) is the Bohr radius.

\[
\psi_{1s}(r) = \frac{1}{\sqrt{\pi a_0}} e^{-r/a_0}
\]

is the ground state hydrogen wave function.

\[
P_{1s}(r) = \frac{4r^2}{a_0^3} e^{-2r/a_0}
\]

is the ground state radial probability distribution function.

18. The wave function for an electron in the \( 2p \) state of hydrogen is

\[
\psi_{2p} = \frac{1}{\sqrt{3} (2a_0)^{3/2} a_0} r e^{-r/2a_0}
\]

What is the most likely distance from the nucleus to find an electron in the \( 2p \) state?

\[
\psi = \frac{1}{\sqrt{3} (2a_0)^{3/2} a_0} r e^{-r/2a_0}
\]

so

\[
P_r = 4\pi r^2 |\psi|^2 = 4\pi r^2 \frac{r^2}{24a_0^2} e^{-r/2a_0}
\]

Set

\[
\frac{dP_r}{dr} = 4\pi \left[ 4r^2 e^{-r/2a_0} + r^4 \left( \frac{1}{a_0} \right) e^{-r/2a_0} \right] = 0
\]

Solving for \( r \), this is a maximum at \( r = 4a_0 \).
19. For a spherically symmetric state of a hydrogen atom, the Schrödinger equation in spherical coordinates is

\[- \frac{\hbar^2}{2m} \left( \frac{d^2 \psi}{dr^2} + \frac{2}{r} \frac{d \psi}{dr} \right) - \frac{k_e^2}{r} \psi = E \psi\]

Show that the 1s wave function for an electron in hydrogen, \( \psi(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} \), satisfies the Schrödinger equation.

P42.19

\[\psi = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} \quad \frac{2}{r} \frac{d \psi}{dr} = -\frac{2}{r} \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} = -\frac{2}{r a_0} \psi\]

\[\frac{d^2 \psi}{dr^2} = -\frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} = -\frac{1}{a_0^2} \psi\]

Substitution into the Schrödinger equation to test the validity of the solution yields

\[- \frac{\hbar^2}{2m} \left( \frac{1}{a_0^2} - \frac{2}{ra_0} \right) \psi - \frac{e^2}{4\pi a_0} \psi = E \psi\]

But \( a_0 = \frac{\hbar^2 (4\pi \in\xi_0)}{m_e^2} \)

so \( -\frac{e^2}{8\pi a_0} = E \)

or \( E = -\frac{k_e^2}{2a_0^2} \)

This is true, so the Schrödinger equation is satisfied.

20. In an experiment, electrons are fired at a sample of neutral hydrogen atoms and observations are made of how the incident particles scatter. A large set of trials can be thought of as containing 1000 observations of the electron in the ground state of a hydrogen atom being momentarily at a distance \( a_0/2 \) from the nucleus. How many times is the atomic electron observed at a distance \( 2a_0 \) from the nucleus in this set of trials?

P42.20 The hydrogen ground-state radial probability density is

\[P(r) = 4\pi r^2 |\psi_{1s}|^2 = \frac{4r^2}{a_0^2} \exp\left(-\frac{2r}{a_0}\right)\]

The number of observations at \( 2a_0 \) is, by proportion

\[N = \frac{1000 \cdot P(2a_0)}{P(a_0/2)} = \frac{1000 \cdot (2a_0)^2}{(a_0/2)^2} \cdot \frac{e^{-4a_0/a_0}}{e^{-a_0/a_0}} = 1000 \cdot 16 \cdot e^{-3} = 797 \text{ times}\]
50. As the Earth moves around the Sun, its orbits are quantized. (a) Follow the steps of Bohr’s analysis of the hydrogen atom to show that the allowed radii of the Earth’s orbit are given by

\[ r = \frac{n^2 \hbar^2}{GM_S M_E^2} \]

where \( M_S \) is the mass of the Sun, \( M_E \) is the mass of the Earth, and \( n \) is an integer quantum number. (b) Calculate the numerical value of \( n \). (c) Find the distance between the orbit for quantum number \( n \) and the next orbit out from the Sun corresponding to the quantum number \( n + 1 \). Discuss the significance of your results.

P42.50

(a) Using the same procedure that was used in the Bohr model of the hydrogen atom, we apply Newton’s second law to the Earth. We simply replace the Coulomb force by the gravitational force exerted by the Sun on the Earth and find

\[ G \frac{M_S M_E}{r^2} = M_E \frac{v^2}{r} \]  

(1)

where \( v \) is the orbital speed of the Earth. Next, we apply the postulate that angular momentum of the Earth is quantized in multiples of \( \hbar \):

\[ M_E vr = nh \quad (n = 1, 2, 3, \ldots) \]

Solving for \( v \) gives

\[ v = \frac{nh}{M_E r} \]  

(2)

Substituting (2) into (1), we find

\[ r = \frac{n^2 \hbar^2}{GM_S M_E^2} \]  

(3)

(b) Solving (3) for \( n \) gives

\[ n = \sqrt{GM_S r} \frac{M_E}{\hbar} \]

(4)

Taking \( M_S = 1.99 \times 10^{30} \text{ kg} \), and \( M_E = 5.98 \times 10^{24} \text{ kg} \), \( r = 1.496 \times 10^{11} \text{ m} \), \( G = 6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2 \), and \( \hbar = 1.055 \times 10^{-34} \text{ Js} \), we find

\[ n = \frac{2.53 \times 10^{24}}{GM_S r} \frac{M_E}{\hbar} \]

(c) We can use (3) to determine the radii for the orbits corresponding to the quantum numbers \( n \) and \( n + 1 \):

\[ r_n = \frac{n^2 \hbar^2}{GM_S M_E^2} \quad \text{and} \quad r_{n+1} = \frac{(n+1)^2 \hbar^2}{GM_S M_E^2} \]

Hence, the separation between these two orbits is

\[ \Delta r = \frac{\hbar^2}{GM_S M_E^2} \left[ (n+1)^2 - n^2 \right] = \frac{\hbar^2}{GM_S M_E^2} (2n+1) \]

Since \( n \) is very large, we can neglect the number 1 in the parentheses and express the separation as

\[ \Delta r \approx \frac{\hbar^2}{GM_S M_E^2} (2n) = \frac{1.18 \times 10^{-30}}{\text{m}} \]

This number is much smaller than the radius of an atomic nucleus \(( \sim 10^{-15} \text{ m})\), so the distance between quantized orbits of the Earth is too small to observe.
54. **Review problem.** (a) How much energy is required to cause an electron in hydrogen to move from the $n = 1$ state to the $n = 2$ state? (b) Suppose the electron gains this energy through collisions among hydrogen atoms at a high temperature. At what temperature would the average atomic kinetic energy $3k_B T/2$, where $k_B$ is the Boltzmann constant, be great enough to excite the electron?

\[ E = E_2 - E_1 = \frac{(-13.6 \text{ eV})}{4} - \frac{(-13.6 \text{ eV})}{1} = 10.2 \text{ eV} = 1.63 \times 10^{-18} \text{ J} \]

\[ E = \frac{3}{2} k_B T \text{ or } T = \frac{2E}{3k_B} = \frac{2(1.63 \times 10^{-18} \text{ J})}{3(1.38 \times 10^{-23} \text{ J/K})} = 7.88 \times 10^4 \text{ K} \]

62. The force on a magnetic moment $\mu$ in a nonuniform magnetic field $B_z$ is given by $F_z = \mu (dB_z/dz)$. If a beam of silver atoms travels a horizontal distance of 1.00 m through such a field and each atom has a speed of 100 m/s, how strong must be the field gradient $dB_z/dz$ in order to deflect the beam 1.00 mm?

\[ \Delta z = \frac{at^2}{2} = \frac{1}{2} \left( \frac{F_z}{m_0} \right) t^2 = \frac{\mu z (dB_z/dz)}{2m_0} \left( \frac{\Delta x}{v} \right)^2 \]

\[ \Delta x^2 = \frac{e\hbar}{2m_e} \]

\[ \frac{dB_z}{dz} = \frac{2m_0 (\Delta z) v^2 (2m_e)}{\Delta x^2 e\hbar} = \frac{2(108)(1.66 \times 10^{-27} \text{ kg})(10^{-3} \text{ m})(10^4 \text{ m}^2/s^2)(2 \times 9.11 \times 10^{-31} \text{ kg})}{(1.00 \text{ m}^2)(1.60 \times 10^{-19} \text{ C})(1.05 \times 10^{-34} \text{ J/s})} \]

\[ \frac{dB_z}{dz} = 0.389 \text{ T/m} \]