Mathematical Modeling for the Community College Curriculum: Examples from the COVID-19 Pandemic

YING LIN
SANTA ROSA JUNIOR COLLEGE
Challenges of Online Teaching

• Social presence
• Engagement with the content
• Assessment of learning
## Mathematics Related to COVID-19

<table>
<thead>
<tr>
<th>Logarithms and exponential functions</th>
<th>Type I and Type II Errors</th>
</tr>
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<td>Conditional Probability and Bayes Formula</td>
<td>Rate of Change and Limit</td>
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<tr>
<td>Expected Value</td>
<td>System of Differential Equations and Stability Analysis</td>
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Examples: Exponential Decay

- Based on the data reported in (van Doremalen et al., 2020), find a linear model that predicts $\log(\text{viral load})$ from time. Then convert this model to an exponential model. What does your model predict about the half life of CoV-19 on each surface?
Examples: Logarithmic scale

Source: Ueki et al. (2020)
COVID Testing and Probability

Elon Musk
@elonmusk

Something extremely bogus is going on. Was tested for covid four times today. Two tests came back negative, two came back positive. Same machine, same test, same nurse. Rapid antigen test from BD.

9:47 PM · Nov 12, 2020 · Twitter for iPhone

126.5K Retweets  40.8K Quote Tweets  481K Likes
Conditional Probability: Specificity and Sensitivity

- Specificity: $P(\text{Test} = - \mid \text{Infected} = \text{no})$
  - Low specificity: high false positives
- Sensitivity: $P(\text{Test} = + \mid \text{Infected} = \text{yes})$
  - Low sensitivity: high false negatives

<table>
<thead>
<tr>
<th></th>
<th>Nasal swab + PCR</th>
<th>Chest CT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specificity</td>
<td>90%</td>
<td>25%</td>
</tr>
<tr>
<td>Sensitivity</td>
<td>70%</td>
<td>97%</td>
</tr>
</tbody>
</table>

Source: Ai et al (2020)
Example: Bayes Formula

1. Suppose you receive a positive test result after taking Nasal Swab/PCR test for COVID-19 (Specificity: 90%, Sensitivity: 70%). Assume 2% of the people in your area are infected. What is the probability that you are infected given your test result?

2. Suppose you went to a party before taking this test, and 40% of the people at the party were infected. What is your chance of being infected after testing positive?
How likely was Elon Musk infected? Given his mixed test results: (-, +, -, +)

Assume the prior probability $P(\text{Infected} = \text{yes}) = 0.4$

Specificity and Sensitivity estimate from the previous table

$$
P(I = \text{yes} | \{T_i = t_i\}) = \frac{P(I = \text{yes}, \{T_i = t_i\})}{P(\{T_i = t_i\})}$$

$$
= \frac{P(\{T_i = t_i\} | I = \text{yes}) \cdot P(I = \text{yes})}{P(\{T_i = t_i\} | I = \text{yes}) \cdot P(I = \text{yes}) + P(\{T_i = t_i\} | I = \text{no}) \cdot P(I = \text{no})}$$

$$
= \frac{\prod_{i=1}^{4} P(T_i = t_i | I = \text{yes}) \cdot P(I = \text{yes})}{\prod_{i=1}^{4} P(T_i = t_i | I = \text{yes}) \cdot P(I = \text{yes}) + \prod_{i=1}^{4} P(T_i = t_i | I = \text{no}) \cdot P(I = \text{no})}$$

$$
= \frac{0.3^2 \cdot 0.7^2 \cdot 0.4}{0.3^2 \cdot 0.7^2 \cdot 0.4 + 0.9^2 \cdot 0.1^2 \cdot 0.6}

= 0.784$$
Colleges Turn To Wastewater Testing In An Effort To Flush Out The Coronavirus

October 26, 2020 · 5:01 AM ET
Heard on Morning Edition
Pooled Testing

• Does it reduce the average number of tests?
• How are false positive and false negative rates impacted?

How pooled testing works

1. People are broken up into groups and a group is tested together.
2. A combined sample from the group either tests negative or positive.
3. If positive, people are tested individually to find the positive cases.

SOURCE USA TODAY research
Karl Gelles/USA TODAY
Expected Number of Tests in a Pooled Testing Regime

k: size of the pool, p: incidence of infection

<table>
<thead>
<tr>
<th>Number of Tests (X)</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(1 - p)^k$</td>
</tr>
<tr>
<td>$k + 1$</td>
<td>$1 - (1 - p)^k$</td>
</tr>
</tbody>
</table>

Expected Number of Tests:

$$E(X) = f(p) = (1 - p)^k + (k + 1)[1 - (1 - p)^k]$$

Graph showing expected number of tests for different values of $k$. The graph indicates that as $k$ increases, the expected number of tests decreases.
## Type I, Type II Errors and Vaccine Trials

<table>
<thead>
<tr>
<th></th>
<th>Reject $H_0$</th>
<th>Do not reject $H_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_0$ true</td>
<td><strong>TYPE I ERROR</strong> $\alpha$</td>
<td>☺ $1 - \alpha$</td>
</tr>
<tr>
<td>Vaccinated group</td>
<td>Vaccinated group has the same infection rate as the placebo group</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>☺ $1 - \beta$</td>
</tr>
<tr>
<td>$H_0$ false</td>
<td>☺ Power of test = $1 - \beta$</td>
<td><strong>TYPE II ERROR</strong> $\beta$</td>
</tr>
<tr>
<td>Vaccinated group</td>
<td>Vaccinated group has a lower infection rate</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>☺ $1 - \alpha$</td>
</tr>
</tbody>
</table>
Recent news headlines about vaccines...

**PFIZER AND BIONTECH CONCLUDE PHASE 3 STUDY OF COVID-19 VACCINE CANDIDATE, MEETING ALL PRIMARY EFFICACY ENDPOINTS**

Wednesday, November 18, 2020 - 06:59am

**Moderna Announces Primary Efficacy Analysis in Phase 3 COVE Study for Its COVID-19 Vaccine Candidate and Filing Today with U.S. FDA for Emergency Use Authorization**

November 30, 2020 at 6:59 AM EST

Source: Pfizer and Moderna
Vaccine Trial Design:
**Determine Sample Size**

- \( \alpha = 0.03 \)
- \( \text{Power} = 1 - \beta \)
  - \( = 0.9039 \)
- \( p_0 = 0.02 \)
- \( p_1 = 0.008 \)
- \( n = 1000 \)

Infection rate after vaccination: \( p_1 \)

\( H_0: p = p_0 \) (infection rate in general population)
The Susceptible-Infected-Resistant Model in Epidemiology

- Also known as “compartmental models”
- First proposed by Kermack and McKendrick (1927)
- Systems of non-linear, 1st order ordinary differential equations with time as independent variable
- Many variations (SIS, SIRS, SEIR, etc.)
The Susceptible-Infected-Resistant Model in Epidemiology

Rate of moving from S to I
\[ = \beta \cdot S \cdot I \]

Rate of moving from I to R
\[ = \gamma \cdot I \]  \(\gamma: \text{recovery rate}\)

Additional constraint: \(S + I + R = 1\)
(No birth or death in the population.)

Initial condition: \((S(0), I(0), R(0))\)

Can be solved using numerical ODE solver

\[
\begin{align*}
\frac{dS}{dt} &= -\beta SI \\
\frac{dI}{dt} &= \beta SI - \gamma I \\
\frac{dR}{dt} &= \gamma I
\end{align*}
\]
Two Ways to Visualize Solutions

AS ORBITS IN PHASE SPACE

AS FUNCTIONS OF TIME
Stages of an Epidemic: Analyze Rates of Change

\[ \frac{dI}{dt} = \beta SI - \gamma I \]

- \( \beta \): contact rate
- \( \gamma \): recovery rate

Susceptible is gradually depleted until \( \beta S - \gamma = 0 \). Infected reaches maximum.

If \( S \to 0 \), \( \frac{dI}{dt} \approx -\gamma I \)

Resembles exponential decay

When Infected is small compared to Susceptible,

\[ \frac{dI}{dt} \approx (\beta S(0) - \gamma)I \]

growth is approximately exponential
Flatten the Curve!

Flattening the Curve by lowering the contact rate parameter $\beta$
The Basic Reproduction Number

\[ R_0 = \frac{\beta \cdot S(0)}{\gamma} \]

- Interpretation: the number of secondary infections caused by an infected individual
- \( R_0 > 1 \) implies \( \frac{dI}{dt} \big|_{t=0} > 0 \): There is epidemic
- \( R_0 < 1 \) implies \( \frac{dI}{dt} \big|_{t=0} < 0 \): No epidemic

\[
\begin{align*}
\frac{dS}{dt} &= -\beta SI \\
\frac{dI}{dt} &= \beta SI - \gamma I \\
\frac{dR}{dt} &= \gamma I
\end{align*}
\]

\( \beta \): contact rate
\( \gamma \): recovery rate
Effect of Mass Vaccination

Initial Values

\[(S(0), I(0), R(0)) = (0.99, 0.01, 0)\]

\[(S(0), I(0), R(0)) = (0.29, 0.01, 0.70)\]

\[\beta = 0.20, \gamma = 0.05, R_0 = 3.96\]

\[\beta = 0.20, \gamma = 0.05, R_0 = 1.16\]
Further Analysis: Critical Points and Stability

• Does the solution depend on initial values?

• Since $S + I + R = 1$, we can drop R and analyze the solution on the S-I plane.
Orbits of Solutions in the S-I Plane

- Equilibria all occur at \((S_\infty, 0)\), when the Infected disappears.
- Linearizing the system near \((S_\infty, 0)\) shows it’s stable.
- Interpretation as “herd immunity”?

\[
\begin{align*}
\frac{dS}{dt} &= -\beta SI \\
\frac{dI}{dt} &= \beta SI - \gamma I \\
\frac{dR}{dt} &= \gamma I
\end{align*}
\]
Does “herd immunity” just happen by itself without intervention?

No herd immunity

Herd immunity achieved

Susceptible  Infected  Immune  Disease transmission

Source: GAO adaptation of NIH graphic.  |  GAO-20-646SP
Loss of Resistance: Possibility of Reinfection or Mutation?

- S-I-R model with feedback loop

\[
\frac{dS}{dt} = -\beta SI + \alpha R \\
\frac{dI}{dt} = \beta SI - \gamma I \\
\frac{dR}{dt} = \gamma I - \alpha R \\
S + I + R = 1
\]
S-I-R with Loss of Resistance

Herd Immunity

(a) \( \alpha = 0.05, \beta = 0.02, \gamma = 0.05 \)

No Herd Immunity

(b) \( \alpha = 0.02, \beta = 0.20, \gamma = 0.05 \)
Herd Immunity is Stable only if $\beta < \gamma$
(Critical Point #1)
Herd Immunity Not Achievable: Critical Point #2 is stable under certain conditions
Resources

• Slides, references, the complete report and source code are available on my webpage: 
  http://srjcstaff.santarosa.edu/~ylin/

• Open source software:
  • https://www.sagemath.org/
  • https://www.geogebra.org/

Coronavirus-like surface in spherical coordinates: 
\[ \rho = r_0 + \sin(k \cdot (\phi + \theta)) \]
Plotted with SageMath