Cramer’s Rule

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1 Introduction

In solving for unknown values in electrical circuits, you will occasionally have two or more linear equations which are coupled. Coupling implies that there is more than one variable in at least one equation. For example:

\[ \begin{align*}
2v_1 + v_2 &= 0 \\
-v_1 + 3v_2 &= 5
\end{align*} \]

are coupled because both \( v_1 \) and \( v_2 \) show up in at least one equation. In high school, often the only way to solve this was to solve for one of the variables in one equation, plug it into the other equation, and then back substitute. If you had three equations with three unknowns, that process could get fairly complicated and the possibility for error increased greatly. Four equations, and the problem seemed hopeless.

Some of you have been introduced to the idea of writing systems of equations in matrix form. The equation above, for example, would look like:

\[
\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}
\]

If you remember how to do matrix multiplication, you will see that this does in fact represent the equations given above. At this point, you were told to premultiply both sides of the equation by the inverse of the matrix. This leaves the vector on the left - known as the solution vector, all by itself. That is:

\[
\text{inv} \left( \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \right)^{-1} = \frac{1}{7} \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}
\]

\[
\begin{align*}
\frac{1}{7} \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 5 \end{bmatrix} \\
\frac{1}{7} \begin{bmatrix} -5 \\ 10 \end{bmatrix} &= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}
\end{align*}
\]

so \( v_1 = -\frac{5}{7} \) and \( v_2 = \frac{10}{7} \). The advantage of this method is that it allows you to solve for all the variables at once. Taking the inverse of a matrix larger than 2x2, however, can be a daunting task. Fortunately, there is a simpler way to solve linear systems of equations.
2 Cramer’s Rule

Cramer’s rule uses determinants instead of the inverse to solve linear systems of equations. Its major disadvantage is that you can only solve for one variable at a time - this is why most computer programs do not use this rule to solve systems of equations. For people, however, it is generally the easiest way to solve systems of equations.

To understand how the rule works, you need to understand what the columns in the left-hand matrix are doing. Each column multiplies by the same variable in the left-hand vector. In the example above, both the 2 and the -1 multiply the \( v_1 \), though in different equations. Similarly, the 1 and the 3 multiply \( v_2 \). To use Cramer’s rule, you must replace the column of the variable for which you are solving by the left-hand vector (hereafter called the constant vector). The final step of the rule is to divide the determinant of your new matrix by the determinant of the original left-hand matrix. For example, to solve for \( v_1 \) above you would:

\[
\Delta = \det \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = (2)(3) - (1)(1) = 7
\]

Replace the first column with solution vector

\[
\begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}
\]

Take determinant of this new matrix

\[
\Delta_1 = \det \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix} = (0)(3) - (5)(1) = -5
\]

Divide the new determinant over the original

\[
v_1 = \frac{\Delta_1}{\Delta} = \frac{-5}{7}
\]

which is the same answer as above. The \( \Delta \) stands for the determinant of the original matrix, while \( \Delta_i \) is the determinant of the matrix with the \( i \)th column replaced by the constant vector. Solving for \( v_2 \) gives:

\[
v_2 = \frac{\Delta_2}{\Delta} = \frac{\det \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix}}{7} = \frac{(2)(5) - (-1)(0)}{7} = \frac{10}{7}
\]

3 Circuit Example

In the circuit below, assume that you are given the values of all the resistors and of the voltage source \( V_s \). Use Cramer’s Rule to find - symbolically - the voltages across the resistors.

Figure 1: Figure for example problem.

There are four unknowns which means you need four equations. For this example, I am going to use two KCL and two KVL equations. The top line gives what I come up with applying either \( i_{in} = i_{out} \) (for KCL) or
going counter-clockwise around the loop (for KVL). The middle lines involve putting the unknowns on the left side, the known values on the right, and then replacing currents with their Ohm’s law equivalent. The bottom line gives the equation with the variables in order:

\[
\begin{align*}
\text{KCL @ 2} & \quad \begin{align*}
i_a &= i_b + i_c \\
i_a - i_b - i_c &= 0 \\
\frac{v}{R_a} - \frac{v}{R_c} - \frac{v}{R_b} &= 0 \\
R_b R_c V_a - R_a R_c V_b - R_a R_b V_c &= 0
\end{align*} \\
\text{KCL @ 3} & \quad \\
\begin{align*}
i_c &= i_d \\
i_c - i_d &= 0 \\
\frac{v}{R_c} - \frac{v}{R_d} &= 0 \\
R_d V_c - R_c V_d &= 0
\end{align*}
\end{align*}
\]

\[
\begin{align*}
\text{KVL around I} & \quad V_a - V_b - V_c = 0 \\
V_a + V_b &= V_c
\end{align*}
\]

\[
\begin{align*}
\text{KVL around II} & \quad V_b - V_d - V_c = 0 \\
V_b - V_c - V_d &= 0
\end{align*}
\]

In matrix form, this gives:

\[
\begin{bmatrix}
R_b R_c & -R_a R_c & -R_a R_b & 0 \\
0 & 0 & R_d & -R_c \\
1 & 1 & 0 & 0 \\
0 & 1 & -1 & -1
\end{bmatrix}
\begin{bmatrix}
V_a \\
V_b \\
V_c \\
V_d
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

To apply Cramer’s rule, first find the determinant of the original matrix:

\[
\Delta = \det
\begin{bmatrix}
R_b R_c & -R_a R_c & -R_a R_b & 0 \\
0 & 0 & R_d & -R_c \\
1 & 1 & 0 & 0 \\
0 & 1 & -1 & -1
\end{bmatrix}
\]

Use the first column to do the expansion:

\[
\Delta = (+1)(R_b R_c) \det
\begin{bmatrix}
0 & -R_a R_c & -R_a R_b & 0 \\
1 & 0 & 0 & R_d \\
0 & 1 & -1 & -1
\end{bmatrix}
\]

\[
\Delta = (+1)(1) \det
\begin{bmatrix}
-R_a R_c & -R_a R_b & 0 \\
0 & R_d & -R_c \\
1 & -1 & -1
\end{bmatrix}
\]

\[
\Delta = R_b R_c \left(0 + 0 + R_c - 0 - (0) - (-R_d)\right) + \left(0 \left(R_a R_c R_d + R_a R_b R_c + 0 - (0) - (-R_a R_c R_c) - (0)\right)\right)
\]

\[
\Delta = R_c \left(R_a R_b + R_a R_c + R_a R_d + R_b R_c + R_b R_d\right)
\]

To solve for \(V_a\), find the determinant of the matrix with the first column replaced by the constant vector:

\[
\Delta_1 = \det
\begin{bmatrix}
0 & -R_a R_c & -R_a R_b & 0 \\
0 & 0 & R_d & -R_c \\
V_a & 1 & 0 & 0 \\
0 & 1 & -1 & -1
\end{bmatrix}
\]

Use the first column to do the expansion:

\[
\Delta_1 = (+1)(V_a) \det
\begin{bmatrix}
-R_a R_c & -R_a R_b & 0 \\
0 & R_d & -R_c \\
1 & -1 & -1
\end{bmatrix}
\]

\[
\Delta_1 = V_a \left(R_a R_c R_d + R_a R_b R_c + R_a R_c R_c\right)
\]

\[
\Delta_1 = V_a R_c \left(R_a R_d + R_a R_b + R_a R_c\right)
\]

3
Finally, find $V_a$ by dividing $\Delta_1$ by $\Delta$:

$$V_a = \frac{\Delta_1}{\Delta}$$

$$V_a = \frac{V_s R_c (R_a R_d + R_b R_b + R_c)}{R_c (R_a R_b + R_a R_c + R_a R_d + R_b R_c)}$$

$$V_a = V_s R_a R_b + R_a R_c + R_a R_d + R_b R_c + R_b R_d$$

To find $V_b$, the process is much the same. Since $\Delta$ is known, you only have to find $\Delta_2$:

$$\Delta_2 = \text{det} \left( \begin{bmatrix} R_b R_c & 0 & -R_a R_b & 0 \\ 0 & 0 & R_d & -R_c \\ V_s & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix} \right)$$

$$\Delta_2 = (-1)(V_s) \text{det} \left( \begin{bmatrix} R_b R_c & -R_a R_b & 0 \\ 0 & R_d & -R_c \\ 0 & -1 & -1 \end{bmatrix} \right)$$

$$\Delta_2 = (-1)(V_s) (-R_b R_c R_d - R_b R_c R_e)$$

$$\Delta_2 = V_s R_e (R_b R_d + R_b R_c)$$

$$V_b = \frac{\Delta_2}{\Delta} = \frac{V_s R_c (R_b R_d + R_b R_c)}{R_c (R_a R_b + R_a R_c + R_a R_d + R_b R_c + R_b R_d + R_c R_d)}$$

$$V_b = V_s \frac{R_b R_c}{R_a R_b + R_a R_c + R_a R_d + R_b R_c + R_b R_d}$$

$V_c$ and $V_d$ can be found in the same way. They are:

$$V_c = V_s \frac{R_b R_c}{R_a R_b + R_a R_c + R_a R_d + R_b R_c + R_b R_d}$$

$$V_d = V_s \frac{R_b R_d}{R_a R_b + R_a R_c + R_a R_d + R_b R_c + R_b R_d}$$