Physics 41 HW Set 1 Chapter 15

Serway 8\textsuperscript{th} OC: 2, 14, 17  CQ: 4, 8  P: 4, 5, 8, 18, 20, 29, 31, 34, 39, 43, 51, 52

Discussion Problems: 32, 57, 59, 67, 74

OC

\*\textbf{OQ15.2} Answer (d) is the only false statement. At the equilibrium position, \( x = 0 \), the elastic potential energy \( (PE_s = kx^2/2) \) is a minimum and the kinetic energy is a maximum.

\textbf{OQ15.14} Answer (c). The equilibrium position is 15 cm below the starting point. The motion is symmetric about the equilibrium position, so the two turning points are 30 cm apart.

\*\textbf{OQ15.17} Answer (a). Higher frequency. When it supports your weight, the center of the diving board flexes down less than the end does when it supports your weight—this is similar to a spring that stretches a smaller distance for the same force: its spring constant is greater because the displacement is smaller. Therefore, the stiffness constant describing the center of the board is greater than the stiffness constant describing the end. And then \( f = \left(\frac{1}{2\pi}\right)\sqrt{\frac{k}{m}} \) is greater for you bouncing on the center of the board.

CQ

\textbf{CQ15.4} The motion will be periodic—that is, it will repeat, though it is not harmonic at large angles. The period is nearly constant as the angular amplitude increases through small values; then the period becomes noticeably larger as \( \theta \) increases farther.

P: 4, 5, 8, 18, 20, 29, 31, 34, 39, 43, 51, 52

Discussion Problems: 32, 57, 59, 67, 74

\textbf{P4} In an engine, a piston oscillates with simple harmonic motion so that its position varies according to the expression \( x = (5.00 \text{ cm}) \cos(2t + \frac{\pi}{6}) \)

where \( x \) is in centimeters and \( t \) is in seconds. At \( t = 0 \), find (a) the position of the piston, (b) its velocity, and (c) its acceleration. (d) Find the period and amplitude of the motion.

\begin{align*}
(a) \quad & x = (5.00 \text{ cm}) \cos\left(2t + \frac{\pi}{6}\right) \quad \text{At } t = 0, \quad x = (5.00 \text{ cm}) \cos\left(\frac{\pi}{6}\right) = 4.33 \text{ cm} \\
(b) \quad & v = \frac{dx}{dt} = -(10.0 \text{ cm/s}) \sin\left(2t + \frac{\pi}{6}\right) \quad \text{At } t = 0, \quad v = -5.00 \text{ cm/s} \\
(c) \quad & a = \frac{dv}{dt} = -(20.0 \text{ cm/s}^2) \cos\left(2t + \frac{\pi}{6}\right) \quad \text{At } t = 0, \quad a = -17.3 \text{ cm/s}^2 \\
(d) \quad & A = 5.00 \text{ cm} \quad \text{and} \quad T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = 3.14 \text{ s}
\end{align*}
The position of a particle is given by the expression $x = (4.00 \text{ m}) \cos(3.00 \pi t + \pi)$, where $x$ is in meters and $t$ is in seconds. Determine (a) the frequency and period of the motion, (b) the amplitude of the motion, (c) the phase constant, and (d) the position of the particle at $t = 0.250 \text{ s}$.

$x = (4.00 \text{ m}) \cos(3.00 \pi t + \pi)$ Compare this with $x = A \cos(\omega t + \phi)$ to find

(a) $\omega = 2\pi f = 3.00\pi$
   or $f = 1.50 \text{ Hz}$, $T = \frac{1}{f} = 0.667 \text{ s}$

(b) $A = 4.00 \text{ m}$

(c) $\phi = \pi \text{ rad}$

(d) $x(t = 0.250 \text{ s}) = (4.00 \text{ m}) \cos(1.75\pi) = 2.83 \text{ m}$

P8. A simple harmonic oscillator takes 12.0 s to undergo five complete vibrations. Find (a) the period of its motion, (b) the frequency in hertz, and (c) the angular frequency in radians per second.

(a) $T = \frac{12.0 \text{ s}}{5} = 2.40 \text{ s}$

(b) $f = \frac{1}{T} = \frac{1}{2.40} = 0.417 \text{ Hz}$

(c) $\omega = 2\pi f = 2\pi(0.417) = 2.62 \text{ rad/s}$

P18. A 200-g block is attached to a horizontal spring and executes simple harmonic motion with a period of 0.250 s. If the total energy of the system is 2.00 J, find (a) the force constant of the spring and (b) the amplitude of the motion.

$m = 200 \text{ g}, T = 0.250 \text{ s}, E = 2.00 \text{ J}$; $\omega = \frac{2\pi}{T} = \frac{2\pi}{0.250} = 25.1 \text{ rad/s}$

(a) $k = m\omega^2 = 0.200 \text{ kg}(25.1 \text{ rad/s})^2 = 126 \text{ N/m}$

(b) $E = \frac{1}{2}kA^2 \Rightarrow A = \frac{\sqrt{2E}}{\sqrt{k}} = \sqrt{\frac{2(2.00)}{126}} = 0.178 \text{ m}$
P20. A 2.00-kg object is attached to a spring and placed on a horizontal, smooth surface. A horizontal force of 20.0 N is required to hold the object at rest when it is pulled 0.200 m from its equilibrium position (the origin of the x axis). The object is now released from rest with an initial position of \( x_i = 0.200 \) m, and it subsequently undergoes simple harmonic oscillations. Find (a) the force constant of the spring, (b) the frequency of the oscillations, and (c) the maximum speed of the object. Where does this maximum speed occur? (d) Find the maximum acceleration of the object. Where does it occur? (e) Find the total energy of the oscillating system. Find (f) the speed and (g) the acceleration of the object when its position is equal to one third of the maximum value.

(a) \( k = \frac{|F|}{x} = \frac{20.0 \text{ N}}{0.200 \text{ m}} = 100 \text{ N/m} \)

(b) \( \omega = \sqrt{\frac{k}{m}} = \sqrt{50} \text{ rad/s} \) so \( f = \frac{\omega}{2\pi} = \frac{1.13 \text{ Hz}}{2\pi} \)

(c) \( v_{\max} = \omega A = \sqrt{50}(0.200) = 1.41 \text{ m/s} \) at \( x = 0 \)

(d) \( a_{\max} = \omega^2 A = 50(0.200) = 10 \text{ m/s}^2 \) at \( x = \pm A \)

(e) \( E = \frac{1}{2} kA^2 = \frac{1}{2}(100)(0.200)^2 = 2.00 \text{ J} \)

(f) \( v = \omega \sqrt{A^2 - x^2} = \sqrt{50} \sqrt{\frac{0.200}{3}} = 1.33 \text{ m/s} \)

(g) \( a = \omega^2 x = 50 \left( \frac{0.200}{3} \right) = 3.33 \text{ m/s}^2 \)

P29. A physical pendulum in the form of a planar body moves in simple harmonic motion with a frequency of 0.450 Hz. If the pendulum has a mass of 2.20 kg and the pivot is located 0.350 m from the center of mass, determine the moment of inertia of the pendulum about the pivot point.

\[ f = 0.450 \text{ Hz}, \quad d = 0.350 \text{ m}, \quad \text{and} \quad m = 2.20 \text{ kg} \]

\[ T = \frac{1}{f} \]

\[ T = 2\pi \sqrt{\frac{I}{mgd}} \]

\[ T^2 = \frac{4\pi^2 I}{mgd} \]

\[ I = \frac{m gd}{4\pi^2} \left( \frac{1}{f} \right)^2 = \frac{2.20(9.80)(0.350)}{4\pi^2 (0.450 \text{ s}^{-1})^2} = 0.944 \text{ kg\cdot m}^2 \]
A simple pendulum has a mass of 0.250 kg and a length of 1.00 m. It is displaced through an angle of 15.0° and then released. What are (a) the maximum speed, (b) the maximum angular acceleration, and (c) the maximum restoring force?  

What If? Solve this problem by using the simple harmonic motion model for the motion of the pendulum, and then solve the problem more precisely by using more general principles. Using the simple harmonic motion model:

\[ A = x_0 = 1 \text{ m} \times 15^\circ = 0.262 \text{ m} \]

\[ \omega = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.8 \text{ m/s}^2}{1 \text{ m}}} = 3.13 \text{ rad/s} \]

(a) \[ v_{\text{max}} = A\omega = 0.262 \text{ m} \times 3.13 \text{ rad/s} = 0.820 \text{ m/s} \]

(b) \[ a_{\text{max}} = A\omega^2 = 0.262 \text{ m} \times (3.13 \text{ rad/s})^2 = 2.57 \text{ m/s}^2 \]

\[ a_{\text{tan}} = \omega^2 \]

\[ \alpha = \frac{a_{\text{tan}}}{r} = \frac{2.57 \text{ m/s}^2}{1 \text{ m}} = 2.57 \text{ rad/s}^2 \]

(c) \[ F = m a = 0.25 \text{ kg} \times 2.57 \text{ m/s}^2 = 0.641 \text{ N} \]

More precisely,

(a) \[ mgh = \frac{1}{2}mv^2 \quad \text{and} \quad h = L(1 - \cos \theta) \]

\[ \therefore v_{\text{max}} = \sqrt{2gL(1 - \cos \theta)} = 0.817 \text{ m/s} \]

(b) \[ \alpha = m g L \sin \theta \]

\[ \alpha_{\text{max}} = \frac{m g L \sin \theta}{m L^2} = \frac{g}{L} \sin \theta = \frac{2.54 \text{ rad/s}^2}{1 \text{ m}} \]

(c) \[ \ell_{\text{max}} = m g \sin \theta = 0.250(9.80)(\sin 15^\circ) = 0.634 \text{ N} \]
34. A very light rigid rod with a length of 0.500 m extends straight out from one end of a meter stick. The stick is suspended from a pivot at the far end of the rod and is set into oscillation. (a) Determine the period of oscillation. *Suggestion:* Use the parallel-axis theorem from Section 10.5. (b) By what percentage does the period differ from the period of a simple pendulum 1.00 m long?

\[ P15.34 \]

(a) The parallel-axis theorem:

\[
I = I_{cm} + Md^2 = \frac{1}{12} ML^2 + Md^2 = \frac{1}{12} M (1.00 \text{ m})^2 + M (1.00 \text{ m})^2
\]

\[
= M \left( \frac{13}{12} \text{ m}^2 \right)
\]

\[
T = 2\pi \sqrt{\frac{I}{Mgd}} = 2\pi \sqrt{\frac{M (13 \text{ m}^2)}{12Mg (1.00 \text{ m})}} = 2\pi \sqrt{\frac{13 \text{ m}}{12 (9.80 \text{ m/s}^2)}} = 2.09 \text{ s}
\]

(b) For the simple pendulum

\[
T = 2\pi \sqrt{\frac{1.00 \text{ m}}{9.80 \text{ m/s}^2}} = 2.01 \text{ s}
\]

\[
\text{difference} = \frac{2.09 \text{ s} - 2.01 \text{ s}}{2.01 \text{ s}} = 4.08\%
\]

43. A 2.00-kg object attached to a spring moves without friction and is driven by an external force \( F = (3.00 \text{ N}) \sin(2\pi t) \). If the force constant of the spring is 20.0 N/m, determine (a) the period and (b) the amplitude of the motion.

\[ P15.43 \]

(a) \( \omega_0 = \sqrt{\frac{m}{k}} = \frac{2.00}{20.0} = 0.316 \text{ s}^{-1} \)

(b) From \( F = 3.00 \sin(2\pi t) \), the angular frequency of the force is \( \omega = 2\pi = 6.28 \text{ s}^{-1} \)

(c) From equation 15.36, the amplitude \( A \) of a driven oscillator, with \( b = 0 \), gives

\[
A = \frac{F_0/m}{\omega^2 - \omega_0^2} = \frac{3.00/2.00}{(6.28)^2 - (3.16)^2} = 0.0509 \text{ m} = 5.09 \text{ cm}
\]
39. An 10.6-kg object oscillates at the end of a vertical spring which has a spring constant of $2.05 \times 10^4$ N/m. The effect of air resistance is represented by the damping coefficient $b = 3.00$ N·s/m. (a) Calculate the frequency of the damped oscillation. (b) By what percentage does the amplitude of the oscillation decrease in each cycle? (c) Find the time interval that elapses while the energy of the system drops to 5.00% of its initial value.

P15.39

The frequency if undamped would be $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2.05 \times 10^4 \text{ N/m}}{10.6 \text{ kg}}} = 44.0/\text{s}$.

(a) With damping

$$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} = \sqrt{\left(\frac{44}{s}\right)^2 - \left(\frac{3 \text{ kg}}{2 \times 10.6 \text{ kg}}\right)^2} = \sqrt{1933.96 - 0.02} = 44.0 \frac{1}{s}$$

$$f = \frac{\omega}{2\pi} = \frac{44.0}{2\pi} = \boxed{7.00 \text{ Hz}}$$

(b) In $x = A_0 e^{-b t / 2m} \cos(\omega t + \phi)$ over one cycle, a time $T = \frac{2\pi}{\omega}$, the amplitude changes from $A_0$ to $A_0 e^{-b\pi / 2m}$ for a fractional decrease of

$$\frac{A_0 - A_0 e^{-b\pi / 2m}}{A_0} = 1 - e^{-3\pi / 10.6} = 1 - e^{-0.0202} = 1 - 0.97998 = 0.0200 = \boxed{2.00\%}$$

(c) The energy is proportional to the square of the amplitude, so its fractional rate of decrease is twice as fast:

$$E = \frac{1}{2} kA^2 = \frac{1}{2} kA_0^2 e^{-2b t / 2m} = E_0 e^{-b t / m}$$

We specify

$$0.05 E_0 = E_0 e^{-3t / 10.6}$$

$$0.05 = e^{-3t / 10.6}$$

$$e^{3t / 10.6} = 20$$

$$\frac{3t}{10.6} = \ln 20 = 3.00$$

$$t = \frac{3 \times 10.6}{3} = 10.6 \text{ s}$$
A small ball of mass $M$ is attached to the end of a uniform rod of equal mass $M$ and length $L$ that is pivoted at the top (Fig. P15.51). (a) Determine the tensions in the rod at the pivot and at the point $P$ when the system is stationary. (b) Calculate the period of oscillation for small displacements from equilibrium, and determine this period for $L = 2.00\, \text{m}$. (*Suggestions:* Model the object at the end of the rod as a particle and use Eq. 15.28.)

Let $F$ represent the tension in the rod.

(a) At the pivot, $F = M\, g + M\, g = 2M\, g$

A fraction of the rod’s weight $M\, g\left(\frac{y}{L}\right)$ as well as the weight of the ball pulls down on point $P$. Thus, the tension in the rod at point $P$ is

$$F = M\, g\left(\frac{y}{L}\right) + M\, g = M\, g\left(1 + \frac{y}{L}\right).$$

(b) Relative to the pivot, $I = I_{\text{rod}} + I_{\text{ball}} = \frac{1}{3} M L^2 + M L^2 = \frac{4}{3} M L^2$

For the physical pendulum, $T = 2\pi \sqrt{\frac{I}{m\, gd}}$ where $m = 2M$ and $d$ is the distance from the pivot to the center of mass of the rod and ball combination. Therefore,

$$d = \frac{M\left(\frac{L}{2}\right) + M\, L}{M + M} = \frac{3L}{4} \quad \text{and} \quad T = 2\pi \sqrt{\frac{\frac{4}{3} M L^2}{(2M)\, g\left(\frac{3L}{4}\right)}} = \frac{4\pi}{3} \sqrt{\frac{2L}{g}}.$$

For $L = 2.00\, \text{m}$, $T = \frac{4\pi}{3} \sqrt{\frac{2(2.00\, \text{m})}{9.80\, \text{m/s}^2}} = 2.68\, \text{s}$.
(a) The amplitude is the magnitude of the maximum displacement from equilibrium (at \( x = 0 \)). Thus, \([A = 2.00 \, \text{cm}]\)

(b) The period is the time for one full cycle of the motion. Therefore, \([T = 4.00 \, \text{s}]\)

(c) The angular frequency is
\[
\omega = \frac{2\pi}{T} = \frac{2\pi}{4.00 \, \text{s}} = \frac{\pi}{2} \, \text{rad/s}.
\]

(d) The maximum speed is
\[
v_{\text{max}} = \omega A = \left(\frac{\pi}{2} \, \text{rad/s}\right)(2.00 \, \text{cm}) = \pi \, \text{cm/s}
\]

(e) The maximum acceleration is
\[
a_{\text{max}} = \omega^2 A = \left(\frac{\pi}{2} \, \text{rad/s}\right)^2(2.00 \, \text{cm}) = 4.93 \, \text{cm/s}^2
\]

(f) The general equation for position as a function of time for an object undergoing simple harmonic motion with \( x = 0 \) when \( t = 0 \) and \( x \) increasing positively is \( x = A \sin \omega t \). For this oscillator, this becomes
\[
x = 2.00 \sin \left(\frac{\pi}{2} t\right), \text{ where } x \text{ is in centimeters and } t \text{ in seconds.}
\]
Discussion Problems: 32, 57, 59, 67, 74

32. Consider the physical pendulum of Figure 15.18. (a) If its moment of inertia about an axis passing through its center of mass and parallel to the axis passing through its pivot point is $I_{CM}$, show that its period is 

$$ T = 2\pi \sqrt{\frac{I_{CM} + md^2}{mgd}} $$

where $d$ is the distance between the pivot point and center of mass. (b) Show that the period has a minimum value when $d$ satisfies $md^2 = I_{CM}$.

P15.32

(a) The parallel axis theorem says directly $I = I_{CM} + md^2$

so 

$$ T = 2\pi \sqrt{\frac{I}{mgd}} = 2\pi \sqrt{\frac{I_{CM} + md^2}{mgd}} $$

(b) When $d$ is very large $T \to 2\pi \sqrt{\frac{d}{g}}$ gets large.

When $d$ is very small $T \to 2\pi \sqrt{\frac{I_{CM}}{mgd}}$ gets large.

So there must be a minimum, found by

$$ \frac{dT}{dd} = 0 = \frac{d}{dd} 2\pi \left( I_{CM} + md^2 \right)^{1/2} \left( mgd \right)^{-1/2} $$

$$ = 2\pi \left( I_{CM} + md^2 \right)^{1/2} \left( -\frac{1}{2} \right) \left( mgd \right)^{-3/2} mg + 2\pi \left( mgd \right)^{-1/2} \left( \frac{1}{2} \right) \left( I_{CM} + md^2 \right)^{-1/2} 2md $$

$$ = -\pi \left( I_{CM} + md^2 \right) mg \left( I_{CM} + md^2 \right)^{1/2} (mgd)^{3/2} + \frac{2\pi md mgd}{(I_{CM} + md^2)^{1/2} (mgd)^{3/2}} = 0 $$

This requires

$$ -I_{CM} - md^2 + 2md^2 = 0 $$

or $I_{CM} = md^2$. 

A horizontal plank of mass \( m \) and length \( L \) is pivoted at one end. The plank’s other end is supported by a spring of force constant \( k \). The plank is displaced by a small angle \( \theta \) from its horizontal equilibrium position and released. (a) Show that it moves with simple harmonic motion with an angular frequency \( \omega = \sqrt{3k/m} \). (b) Evaluate the frequency if the mass is 5.00 kg and the spring has a force constant of 100 N/m.

P15.57 (a) The problem tells us that the plank and spring are at equilibrium when the plank is horizontal. Then:

\[
\sum \tau = 0 - mg \left( \frac{L}{2} \right) + kx_0L \rightarrow x_0 = \frac{mg}{2k}
\]

where \( x_0 \) is the equilibrium compression distance from the unstretched spring.

The plank is then displaced upwards by a small angle, a distance: \( L \sin \theta \) above equilibrium. The system will oscillate about the equilibrium position with an amplitude: \( x = L \sin \theta - x_0 \). Both the plank and the spring produce clockwise and negative torques:

\[
\sum \tau = -mg \left( \frac{L}{2} \right) \cos \theta - kxL \cos \theta = -mg \frac{L}{2} \cos \theta - k \left( L \sin \theta - x_0 \right) L \cos \theta
\]

\[
= -mg \frac{L}{2} \cos \theta - k \left( L \sin \theta - \frac{mg}{2k} \right) L \cos \theta
\]

\[
= -mg \frac{L}{2} \cos \theta + mg \frac{L}{2} \cos \theta - kL^2 \sin \theta \cos \theta
\]

\[
= -k \theta L^2
\]

where we substituted \( x_0 = \frac{mg}{2k} \) and used the small angle approximations \( \cos \theta = 1 \), \( \sin \theta = \theta \).

Using the parallel axis thm for the plank rotated about the pivot we get: \( I = \frac{1}{3} mL^2 \).

Combining we derive:

\[
\sum \tau = I \alpha = \frac{1}{3} mL^2 \frac{d^2 \theta}{dt^2} = -k \theta L^2.
\]

Rearranging:

\[
\frac{d^2 \theta}{dt^2} = -\frac{3k}{m} \theta.
\]

The angular acceleration is opposite in direction and proportional to the displacement, so we have simple harmonic motion with \( \omega = \sqrt{\frac{3k}{m}} \). Notice how the torque due to the plank canceled out!! Why does this make sense?

(b) \( \omega = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{3k}{m}} = \frac{1}{2\pi} \sqrt{\frac{3(100 \text{ N/m})}{5.00 \text{ kg}}} = 1.23 \text{ Hz} \)
59. **Review problem.** A particle of mass 4.00 kg is attached to a spring with a force constant of 100 N/m. It is oscillating on a horizontal frictionless surface with an amplitude of 2.00 m. A 6.00-kg object is dropped vertically on top of the 4.00-kg object as it passes through its equilibrium point. The two objects stick together. (a) By how much does the amplitude of the vibrating system change as a result of the collision? (b) By how much does the period change? (c) By how much does the energy change? (d) Account for the change in energy.

As it passes through equilibrium, the 4-kg object has speed

\[ v_{\text{max}} = \omega A = \sqrt{\frac{k}{m}} A = \sqrt{\frac{100 \text{ N/m}}{4 \text{ kg}}} \times 2 \text{ m} = 10 \text{ m/s}. \]

In the completely inelastic collision momentum of the two-object system is conserved. So the new 10-kg object starts its oscillation with speed given by

\[ v_{\text{max}} = \frac{m_1 v_{\text{max}, 1} + m_2 v_{\text{max}, 2}}{m_1 + m_2} = \frac{4 \text{ kg}(10 \text{ m/s}) + 6 \text{ kg}(0 \text{ m/s})}{10 \text{ kg}} = 4.0 \text{ m/s}. \]

(a) The new amplitude is given by

\[ \frac{1}{2} m v_{\text{max}}^2 = \frac{1}{2} k A^2 \]

\[ 10 \text{ kg}(4 \text{ m/s})^2 = (100 \text{ N/m}) A^2 \]

\[ A = 1.26 \text{ m} \]

Thus the amplitude has **decreased by** \( 2.00 \text{ m} - 1.26 \text{ m} = 0.735 \text{ m} \)

(b) The old period was

\[ T = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{4 \text{ kg}}{100 \text{ N/m}}} = 1.26 \text{ s} \]

The new period is

\[ T = 2\pi \sqrt{\frac{10 \text{ kg}}{100 \text{ N/m}}} = 1.99 \text{ s} \]

The period has **increased by** \( 1.99 \text{ m} - 1.26 \text{ m} = 0.730 \text{ s} \)

(c) The old energy was

\[ \frac{1}{2} m v_{\text{max}}^2 = \frac{1}{2}(4 \text{ kg})(10 \text{ m/s})^2 = 200 \text{ J} \]

The new mechanical energy is

\[ \frac{1}{2}(10 \text{ kg})(4 \text{ m/s})^2 = 80 \text{ J} \]

The energy has **decreased by** 120 J.

(d) The missing mechanical energy has turned into internal energy in the completely inelastic collision.
A block of mass $m$ is connected to two springs of force constants $k_1$ and $k_2$ as shown in each case, the block moves on a frictionless table after it is displaced from equilibrium and released. Show that in the two cases the block exhibits simple harmonic motion with periods

(a) $T = 2\pi \sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$

(b) $T = 2\pi \sqrt{\frac{m}{k_1 + k_2}}$

When the mass is displaced a distance $x$ from equilibrium, spring 1 is stretched a distance $x_1$ and spring 2 is stretched a distance $x_2$. By Newton’s third law, we expect $k_1 x_1 = k_2 x_2$. When this is combined with the requirement that $x = x_1 + x_2$, we find

$$x_1 = \left[ \frac{k_2}{k_1 + k_2} \right] x$$

The force on either spring is given by

$$F = \left[ k_1 \frac{k_2}{k_1 + k_2} \right] x = ma$$

where $a$ is the acceleration of the mass $m$.

This is in the form

$$F = k_{\text{eff}} x = ma$$

and

$$T = 2\pi \sqrt{\frac{m}{k_{\text{eff}}}} = 2\pi \sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$$

(b) In this case each spring is distorted by the distance $x$ which the mass is displaced. Therefore, the restoring force is

$$F = -(k_1 + k_2) x \quad \text{and} \quad k_{\text{eff}} = k_1 + k_2$$

so that

$$T = \frac{2\pi m}{\sqrt{(k_1 + k_2)}}$$
**74. Review problem.** Imagine that a hole is drilled through the center of the Earth to the other side. An object of mass \( m \) at a distance \( r \) from the center of the Earth is pulled toward the center of the Earth only by the mass within the sphere of radius \( r \). (a) Write Newton’s law of gravitation for an object at the distance \( r \) from the center of the Earth, and show that the force on it is of Hooke’s law form, \( F = -kr \), where the effective force constant is \( k = \frac{4}{3} \pi \rho Gm \). Here \( \rho \) is the density of the Earth, assumed uniform, and \( G \) is the gravitational constant. (b) Show that a sack of mail dropped into the hole will execute simple harmonic motion if it moves without friction. When will it arrive at the other side of the Earth?

(a) Newton’s law of universal gravitation is

\[
F = -\frac{GMm}{r^2} = -\frac{Gm}{r^2} \left( \frac{4}{3} \pi r^3 \right) \rho
\]

Thus,

\[
F = -\left( \frac{4}{3} \pi G \rho m \right) r
\]

Which is of Hooke’s law form with

\[
k = \frac{4}{3} \pi G \rho m
\]

(b) The sack of mail moves without friction according to

\[
a = -\left( \frac{4}{3} \pi G \rho m \right) r = -\omega^2 r
\]

Since acceleration is a negative constant times excursion from equilibrium, it executes SHM with

\[
\omega = \sqrt{\frac{4\pi G \rho}{3}}
\]

and period

\[
T = \frac{2\pi}{\omega} = \sqrt{\frac{3\pi}{4\rho G}}
\]

The time for a one-way trip through the Earth is

\[
T = 2\sqrt{\frac{3\pi}{4\rho G}}
\]

We have also

\[
g = \frac{GM}{R_e^2} = \frac{G4\pi R_e^3 \rho}{3R_e^2} = \frac{4}{3} \pi G \rho R_e
\]

so

\[
\frac{4\rho G}{3} = \frac{g}{(\pi R_e)} \quad \text{and} \quad \frac{T}{2} = \pi \sqrt{\frac{R_e}{g}} = \pi \sqrt{\frac{6.37 \times 10^8 \text{ m}}{9.8 \text{ m/s}^2}} = 2.53 \times 10^3 \text{ s} = \frac{42.2 \text{ m in}}{}
\]

Recall from physics 40 Chapter 13 that the period of an object in orbit around the Earth at the surface is twice this 84 minutes and the orbital velocity is 7.9 km/s.