Math 1B

Test 3 Review

1. Determine whether $\sum_{n=0}^{\infty} \left(\frac{3}{\pi}\right)^n$ converges or diverges. If it converges, find the sum.

Solution:

$$\sum_{n=0}^{\infty} \left(\frac{3}{\pi}\right)^n = 1 + \frac{3}{\pi} + \frac{3^2}{\pi^2} + \frac{3^3}{\pi^3} + \cdots$$

the series is geometric with a common ratio $r = \frac{3}{\pi} < 1$. So

$$S = \frac{a}{1-r}$$

$$S = \frac{1}{1-\frac{3}{\pi}}$$

$$S = \frac{1}{\frac{\pi-3}{\pi}}$$

$$S = \frac{\pi}{\pi-3}$$ convergent!

2. Consider the telescoping series $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^2+2n}$.

a) Find a formula a_i for the decomposition of the general term of the given infinite series.

b) Express the series as a partial sum s_n in sigma notation, $s_n = \displaystyle\sum_{i=1}^{n} a_i$.

c) Expand the partial sum s_n and simplify.

d) Find the sum of the infinite series, if it exists. Show all your work. NO CAS!

Solution

a) $$\frac{4}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

$$4 = A(n+2) + B \cdot n$$

Let $n=0$: $4 = 2A$, $A = 2$ | $n=-2$: $4 = -2B$, $B = -2$

Let $$a_i = \frac{2}{i} - \frac{2}{i+2}$$

b) $$s_n = \sum_{i=1}^{n} \left[\frac{2}{i} - \frac{2}{i+2}\right]$$

c) $$s_n = \underbrace{\left(2 - \cancel{\tfrac{2}{3}}\right)}_{i=1} + \underbrace{\left(1 - \cancel{\tfrac{1}{2}}\right)}_{i=2} + \underbrace{\left(\cancel{\tfrac{2}{3}} - \cancel{\tfrac{2}{5}}\right)}_{i=3} + \underbrace{\left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right)}_{i=4} + \underbrace{\left(\cancel{\tfrac{2}{5}} - \cancel{\tfrac{2}{7}}\right)}_{i=5} + \dots$$

$$+ \underbrace{\left(\cancel{\tfrac{2}{n-3}} - \cancel{\tfrac{2}{n-1}}\right)}_{i=n-3} + \underbrace{\left(\cancel{\tfrac{2}{n-2}} - \cancel{\tfrac{2}{n}}\right)}_{i=n-2} + \underbrace{\left(\cancel{\tfrac{2}{n-1}} - \tfrac{2}{n+1}\right)}_{i=n-1} + \underbrace{\left(\cancel{\tfrac{2}{n}} - \tfrac{2}{n+2}\right)}_{i=n}$$

d) $$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left[2 + 1 - \frac{2}{n+1} - \frac{2}{n+2}\right]$$

$$= 3 \quad \text{Converges!}$$

For problems 3-6, determine whether the series converges or diverges. State any tests that you use and show all steps.

3. $\sum_{n=1}^{\infty} \frac{1+3^n}{2^n}$ $\lim_{n\to\infty} \frac{1+3^n}{2^n} \overset{?}{=} 0$, Maybe, Maybe Not

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{3^n}{2^n} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$$

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, since its geometric with $|r| = |\frac{1}{2}| < 1$

But

$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ diverges, since its geometric with $|r| = |\frac{3}{2}| \geq 1$.

Hence, the series diverges by GST.

4. $\sum_{k=1}^{\infty}\left(\frac{4k^2-3}{7k^2+6}\right)^k$

$\lim_{K\to\infty}\left(\frac{4K^2-3}{7K^2+6}\right)^K \overset{?}{=} 0$, Maybe

Solution lets use the root test.

$$\lim_{K\to\infty}\sqrt[K]{\left|\left(\frac{4K^2-3}{7K^2+6}\right)^K\right|} = \lim_{K\to\infty}\frac{4K^2-3}{7K^2+6}$$

$$\overset{DTE}{=} \lim_{K\to\infty}\frac{4K^2}{7K^2}$$

$$= \frac{4}{7} < 1$$

Hence, by the "Root Test" the series converges.

5. $\sum_{n=0}^{\infty} \frac{1+\sin^2 n}{5^n}$ $\lim_{n\to\infty} \frac{1+\sin^2 n}{5^n} \overset{?}{=} 0$, Yes.

Solution

$$\sum_{n=0}^{\infty}\left(\frac{1}{5^n}+\frac{\sin^2 n}{5^n}\right)$$

$$=\sum_{n=0}^{\infty}\frac{1}{5^n}+\sum_{n=0}^{\infty}\frac{\sin^2 n}{5^n}$$

① $\sum_{n=0}^{\infty}\frac{1}{5^n}$ converges, since its geometric with $|r|=|\frac{1}{5}|<1$

② $\sum_{n=0}^{\infty}\frac{\sin^2 n}{5^n}$ converges, since $0\le\sin^2 n\le 1$

and $\frac{\sin^2 n}{5^n}\le\frac{1}{5^n}$. So the series converges by the "Direct Comparison Test" (DCT)

Hence the given series converges.

6. $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$ $\lim_{n\to\infty} \frac{(2n)!}{n^{2n}} \stackrel{?}{=} 0$, I think so.

Solution: Let use the ratio test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{[2(n+1)]!}{(n+1)^{2(n+1)}} \cdot \frac{n^{2n}}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)!}{(n+1)^{2n+2}} \cdot \frac{n^{2n}}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)\cdot\cancel{(2n)!}\cdot n^{2n}}{(n+1)^{2n}\cdot(n+1)^{2}\cdot\cancel{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{n^{2n}}{(n+1)^{2n}} \cdot \frac{(2n+2)(2n+1)}{(n+1)^{2}}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{2n} \cdot \lim_{n\to\infty} \frac{4n^2+6n+2}{n^2+2n+1}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\right]^{-2} \cdot \lim_{n\to\infty} \frac{4n^2+6n+2}{n^2+2n+1}$$

$$= e^{-2}\cdot 4 = \frac{4}{e^2} < 1$$

Hence, by the "Ratio Test" the series converges.

7. Classify the series below as absolutely convergent, conditionally convergent, or divergent. State any tests you use and show all steps.

a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$

b) $\sum_{k=1}^{\infty} (-1)^k \frac{k^2}{\sqrt{k^6+1}}$

c) $\sum_{n=2}^{\infty} \frac{(-1)^n(2n-1)}{n^3-1}$

a) Let's first check for absolute convergence.
Let $b_n = |a_n| = \frac{1}{\sqrt{n+4}}$. So we get

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}}$$

Let's compare it with $\frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$

$$\lim_{n\to\infty} \left[\frac{\frac{1}{\sqrt{n+4}}}{\frac{1}{\sqrt{n}}}\right] = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+4}} = 1$$

Hence, since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the p-series Test, our series $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}}$ diverges by the "Limit Comparison Test." Therefore our original series is <u>Not</u> absolutely convergent.
Next we will check for conditionally convergent.

a) <u>Continued</u>

Let $b_n = \frac{1}{\sqrt{n+4}}$

① $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n+4}} = 0$

② Clearly $b_{n+1} \leq b_n$ since $\frac{1}{\sqrt{n+5}} \leq \frac{1}{\sqrt{n+4}}$.

Hence, by the "Alternating Series Test" the series $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$ is conditionally convergent.

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b) First we will check for absolute convergence. Let $b_n = |a_n| = \frac{K^2}{\sqrt{K^6+1}}$.

We get $\sum\limits_{K=1}^{\infty} \frac{K^2}{\sqrt{K^6+1}}$

Using the "Limit Comparison Test" with $\frac{1}{K}$

$$\lim_{K\to\infty}\left[\frac{\frac{K^2}{\sqrt{K^6+1}}}{\frac{1}{K}}\right] = \lim_{K\to\infty} \frac{K^3}{\sqrt{K^6+1}}$$

$$\stackrel{DTE}{=} \lim_{K\to\infty} \frac{K^3}{\sqrt{K^6}}$$

$$= \lim_{K\to\infty} \frac{K^3}{K^3}$$

$$= 1$$

b) <u>continued</u>

Since the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges by the "Limit Comparison Test", our series is not absolutely convergent.

Next will check for conditional convergence.

$$\text{let} \quad b_n = \frac{k^2}{\sqrt{k^6+1}}$$

① $$\lim_{k\to\infty} \frac{k^2}{\sqrt{k^6+1}} \overset{DTE}{=} \lim_{k\to\infty} \frac{k^2}{\sqrt{k^6}} = \lim_{k\to\infty} \frac{k^2}{k^3} = 0$$

② It is <u>not clear</u> that $b_{k+1} \leq b_k$

that is

$$\frac{(k+1)^3}{\sqrt{(k+1)^6+1}} \overset{?}{\leq} \frac{k^3}{\sqrt{k^6+1}}$$

So we let $f(x) = \dfrac{x^2}{\sqrt{x^6+1}}$

$$f'(x) = \frac{(x^6+1)^{\frac{1}{2}} \cdot 2x - x^2 \cdot \cancel{\tfrac{1}{2}}(x^6+1)^{-\frac{1}{2}} \cdot \overset{3}{\cancel{6}}x^5}{x^6+1}$$

$$= \frac{x(x^6+1)^{-\frac{1}{2}}\left(2(x^6+1) - 3x^6\right)}{(x^6+1)^1}$$

b) continued

$$f'(x) = \frac{x(2-x^6)}{(x^6+1)^{3/2}} < 0 \text{ for } x \geq 2$$

Hence,

$$b_{k+1} \leq b_k \text{ for } k \geq 2$$

Thus the series $\sum_{k=1}^{\infty} (-1)^k \frac{k^2}{\sqrt{k^6+1}}$ is conditionally convergent by AST!

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c) First we check for absolute convergence.

Let $b_n = |a_n| = \frac{2n-1}{n^3-1}$

We will use the "Limit Comparison Test", with $\frac{1}{n^2}$

$$\lim_{n\to\infty} \left[\frac{\frac{2n-1}{n^3-1}}{\frac{1}{n^2}}\right] = \lim_{n\to\infty} \frac{2n^3-n^2}{n^3-1} \overset{DTE}{=} \lim_{n\to\infty} \frac{2n^3}{n^3} = 2$$

Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges by the p-series Test, our series $\sum_{n=2}^{\infty} \frac{2n-1}{n^3-1}$ converges by "LCT."

So our original series is absolutely convergent and therefore converges by "ACT."

A calculator may be used to help solve problem 8.

8. Solve the problems below. CAS may be used to help solve these problem.

a) Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^6}$. Using the **Remainder Estimate for the Integral Test**, find the smallest n such that the approximation is accurate to five decimal places ($R_n \leq 0.000001 = 1 \times 10^{-6}$). Using the n you found, determine with your calculator an approximation for the given infinite series to four decimal places of accuracy.

b) Consider the infinite series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^6}$. Using the **Alternating Series Estimation Theorem**, find the smallest n such that the approximation is accurate to five decimal places ($R_n \leq 0.0000001 = 1 \times 10^{-6}$). Using the n you found, determine with your calculator an approximation for the given infinite series to four decimal places of accuracy.

a) Clearly the series converges by the Integral Test.

Let $\int_n^{\infty} \frac{1}{x^6}\,dx \leq 1 \times 10^{-6}$, by "REIT".

By CAS we get

$$\frac{1}{5n^5} \leq 1 \times 10^{-6}$$

Solve for n we get

$$n \geq 11.487$$

Let $n = 12$, smallest n.

$$\sum_{n=1}^{12} \frac{1}{n^6} \approx 1.01734$$

b) Clearly the series converges by AST.

Let $b^{n+1} \leq 1 \times 10^{-6}$, by "ASET"

$$\frac{1}{(n+1)^6} \leq 1 \times 10^{-6}$$

By CAS $n \geq 9$

Let $n = 9$, smallest n.

$$\sum_{n=1}^{9} (-1)^n \cdot \frac{1}{n^6} \approx -0.98555$$