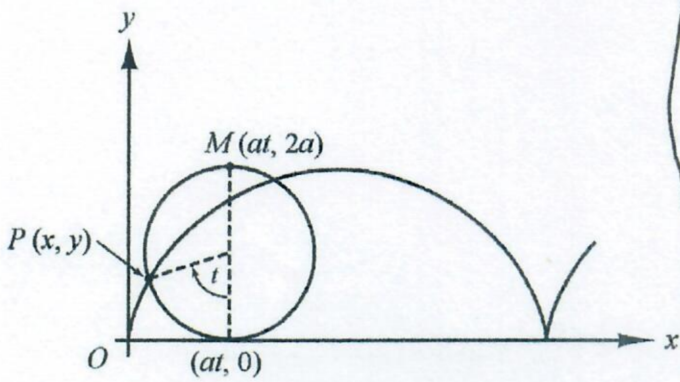
Math 1B

Test 2 Review

For problems 1 through 4 a calculator may NOT be used.

1. Recall that if a circle rolls along a straight line without slipping, then a point on the circle traces a curve called a cycloid, see figure below. If the circle has radius a, then the cycloid is given parametrically by:

$$x(t) = at - a\sin(t)$$
$$y(t) = a - a\cos(t)$$



(a) $\frac{dx}{dt} = a - a\cos t$

$\frac{dy}{dt} = a\sin t$

$S = 2\pi\int_0^{2\pi} (a - a\cos t)\cdot \sqrt{(a-a\cos t)^2 + (a\sin t)^2}\,dt$

a) Set up but **do not evaluate** the integral necessary to calculate the surface area of the solid formed by revolving one arch ($0 \le t \le 2\pi$) of the cycloid about the x-axis.

b) Find $\frac{dy}{dx}$ for the cycloid.

c) Let M be the point on the circle that is furthest from the x-axis. Show that the line through M and the point P of the cycloid is tangent to the cycloid at P.

(b) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\sin t}{a - a\cos t}$

$= \frac{\sin t}{1 - \cos t}$

(c) Slope of line $\overleftrightarrow{PM}$ is

$m = \frac{2a - y}{at - x}$

but $x = at - a\sin t$

$y = a - a\cos t$

so $m = \frac{2a - (a - a\cos t)}{at - (at - a\sin t)}$

$m = \frac{a + a\cos t}{a\sin t}$

$m = \frac{1 + \cos t}{\sin t}$ →

(1c) continued

Does $\dfrac{\sin t}{1-\cos t} \stackrel{?}{=} \dfrac{1+\cos t}{\sin t}$

Lets prove a trigonometric identity

$$\frac{\sin t}{1-\cos t} \cdot \frac{1+\cos t}{1+\cos t} = \frac{\sin t \cdot (1+\cos t)}{1-\cos^2 t}$$

$$= \frac{\sin t\,(1+\cos t)}{\sin^2 t}$$

$$= \frac{1+\cos t}{\sin t}$$

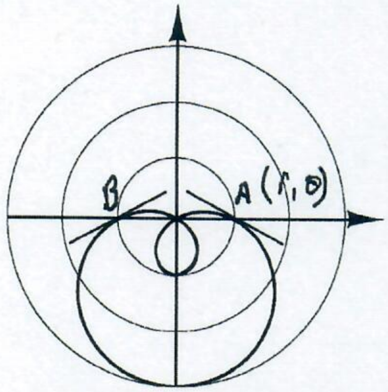
Yes! They are the same !!!

2. Consider the polar curve $r = 1 - 2\sin\theta$, the drawning is below.

a) Find $\frac{dy}{dx}$

b) Find the polar coordinates of the points of tangency in the figure below.

c) Find the slope of the tangent lines at the points you found in part (b).



(a) $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

$y = r \cdot \sin\theta$

$\frac{dy}{d\theta} = r \cdot \cos\theta + r' \cdot \sin\theta$

$x = r \cdot \cos\theta$

$\frac{dx}{d\theta} = -r \cdot \sin\theta + r' \cdot \cos\theta$

$r = 1 - 2\sin\theta$

$\frac{dr}{d\theta} = -2\cos\theta$

So $\frac{dy}{dx} = \frac{(1-2\sin\theta)\cdot\cos\theta - 2\cos\theta\cdot\sin\theta}{-(1-2\sin\theta)\cdot\sin\theta - 2\cos^2\theta}$

$= \frac{\cos\theta - 2\sin\theta\cdot\cos\theta - 2\sin\theta\cos\theta}{-\sin\theta + 2\sin^2\theta - 2\cos^2\theta}$

$\frac{dy}{dx} = \frac{\cos\theta - 2\sin(2\theta)}{-\sin\theta - 2\cos(2\theta)}$

(b) Points: $A(1, 0)$

Since $r = 1 - 2\sin(0)$

$r = 1$

$B(1, \pi)$

Since $r = 1 - 2\sin(\pi)$

$r = 1$

(c) Point: $A(1, 0)$

$\left.\frac{dy}{dx}\right|_{\theta=0} = \frac{1 - 2\cdot 0}{0 - 2} = -\frac{1}{2}$

Point: $B(1, \pi)$

$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{-1 - 0}{0 - 2} = \frac{1}{2}$

3. Consider the curvers $x^2 - y^2 = 1$ and $x^2 + 2y^2 = 4$.

a) Identify the two conic sections.

b) Determine the coordinates of the point of intersection of the curves in the first quadrant.

c) Show that the curves are orthogonal at the point found in part (a). That is, show that the tangent lines are perpendicular at the point of intersection.

(a) $x^2 - y^2 = 1$ is a hyperbola

$x^2 + 2y^2 = 4$ is an ellipse

(b) $x^2 = 1 + y^2$ substitute into $x^2 + 2y^2 = 4$

$1 + y^2 + 2y^2 = 4$

$3y^2 = 3$

$y^2 = 1$

$y = \pm 1$

$x^2 = 1 + (\pm 1)^2$

$x^2 = 2$

$x = \pm\sqrt{2}$

Points $(\sqrt{2}, 1), (\sqrt{2}, -1)$ $(-\sqrt{2}, 1)$ and $(-\sqrt{2}, -1)$

(c) $x^2 - y^2 = 1$

$2x - 2y \cdot y' = 0$

Hyperbola: $y' = \frac{x}{y}$

$x^2 + 2y^2 = 4$

$2x + 4y \cdot y' = 0$

Ellipse: $y' = -\frac{x}{2y}$

$y' = \frac{x}{y}$ & $y' = -\frac{x}{2y}$

① Take $(\sqrt{2}, 1)$

$y' = \sqrt{2}$ & $y' = -\frac{\sqrt{2}}{2} = -\frac{1}{\sqrt{2}}$

orthogonal.

② Take $(\sqrt{2}, -1)$

$y' = -\sqrt{2}$ & $y' = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

orthogonal.

③ Take $(-\sqrt{2}, 1)$

$y' = -\sqrt{2}$ & $y' = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

orthogonal.

④ Take $(-\sqrt{2}, -1)$,

$y' = \sqrt{2}$ & $y' = -\frac{\sqrt{2}}{2} = -\frac{1}{\sqrt{2}}$

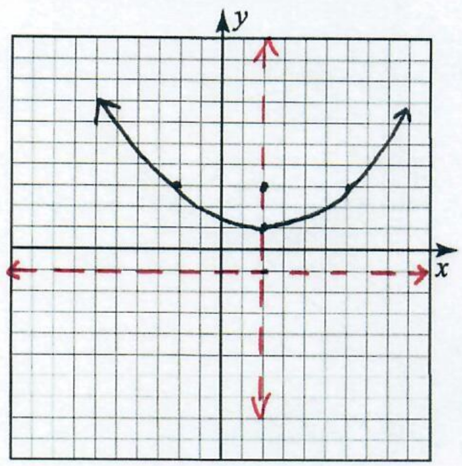
orthogonal.

4. Consider the parabola with vertex at (2, 1) and focus at (2, 3).

a) Write the equation.

b) Find the equation of the directrix.

c) Sketch the graph.



a) The parabola opens up

$$y = \frac{1}{4p} x^2$$

let $p = 2$

$$y = \frac{1}{8} x^2$$

Vertex: $(2, 1)$

$$y - 1 = \frac{1}{8}(x-2)^2$$

$$y = \frac{1}{8}(x-2)^2 + 1$$

$$y = \frac{1}{8}(x^2 - 4x + 4) + 1$$

$$y = \frac{1}{8}x^2 - \frac{1}{2}x + \frac{1}{2} + 1$$

$$y = \frac{1}{8}x^2 - \frac{1}{2}x + \frac{3}{2}$$

b) $y = -1$ is the directrix

A calculator may be used to help solve problems 5 through 7.

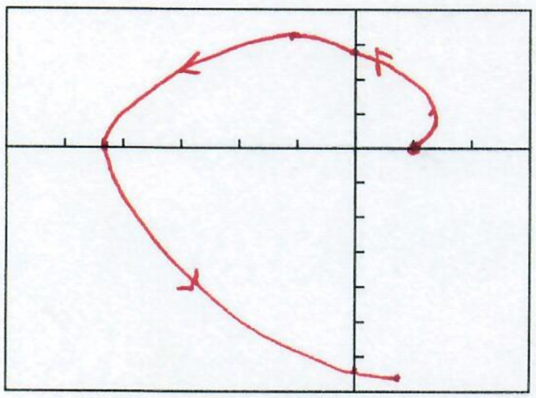
5. Consider a curve that is described parametrically by

$$\begin{cases} x(t) = \cos(t) + t\sin(t) \\ y(t) = \sin(t) - t\cos(t) \end{cases}$$

a) Use your calculator to obtain the graph of the curve in the window $-6 \le x \le 3$, $-7 \le y \le 4$, and $0 \le t \le 2\pi$. Sketch the graph below, and indicate with arrows how the curve was generated.

b) Write an integral that will find the length of the curve over the interval $0 \le t \le 2\pi$.

c) Use your calculator to find the value of your integral in part (a).



$$\frac{dx}{dt} = -\sin(t) + t\cdot\cos t + \sin t$$

$$= t\cos t$$

$$\frac{dy}{dt} = \cos t + t\sin t - \cos t$$

$$= t\sin t$$

(b) $L = \int_0^{2\pi} \sqrt{(t\cos t)^2 + (t\sin t)^2}\, dt$

(c) $2\pi^2$ units

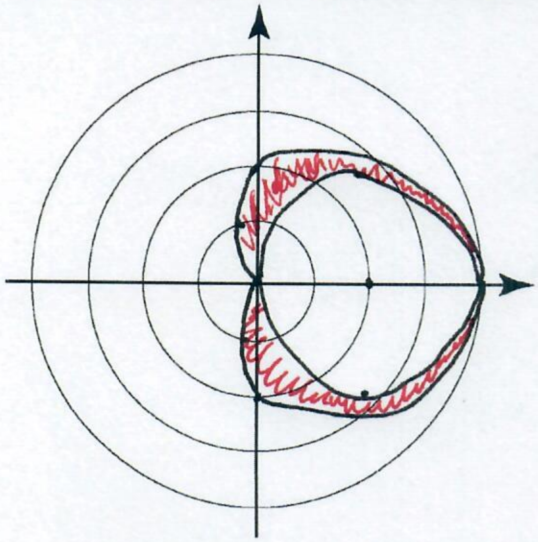
6. Consider the graphs of the cardioid $r = 2 + 2\cos\theta$ and the circle $r = 4\cos\theta$.

a) Graph the curves in the polar plane below.

b) Determine the integral that will find the area of the region inside the cardioid and outside the circle.

c) Use your calculator to find the value of the integral you found in part (b).

(a)

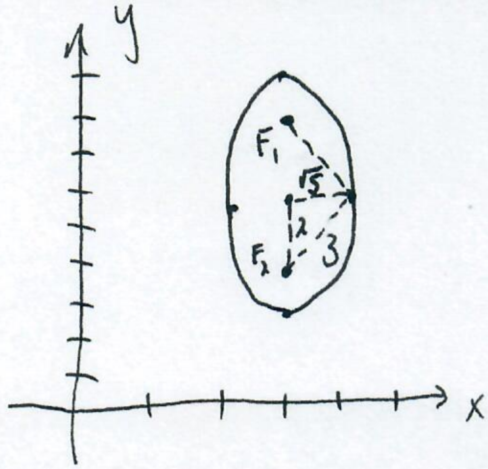


The cardioid is generated from 0 to 2π.
Whereas, the circle is generated from 0 to π.

(b) $A = 2 \cdot \left[\frac{1}{2} \int_0^{\pi} (2+2\cos\theta)^2 d\theta - \frac{1}{2} \int_0^{\frac{\pi}{2}} (4\cos\theta)^2 d\theta \right]$

(c) $A = 2\pi$ units^2

7. Write an equation of the ellipse such that for any point on the ellipse, the sum of the distances from the points (3, 4) and (3, 8) is 6.



Center: (3, 6)

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$a = \sqrt{5}$, $b = 3$

So the equation is:

$$\frac{(x-3)^2}{5} + \frac{(y-6)^2}{9} = 1$$

$$9(x^2-6x+9) + 5(y^2-12y+36) = 45$$

$$9x^2-54x+81 + 5y^2-60y+180 = 45$$

$$9x^2+5y^2-54x-60y+216=0$$