Math 1B

Final Review

A calculator may NOT be used to solve problems 1 - 5.

1. Match the equations below on the left with the graph of the surface on the right by writing the letter that corresponds to the surface in front of the equation. Note that one surface does not have a corresponding equation.

II $\quad \dfrac{x^2}{9} + \dfrac{y^2}{16} + \dfrac{z^2}{9} = 1$

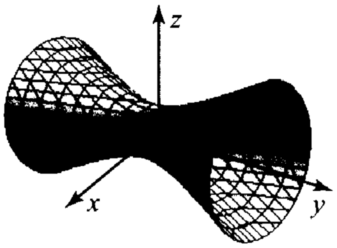
V $\quad 15x^2 - 4y^2 + 15z^2 = -4$

I $\quad 4x^2 - y^2 + 4z^2 = 4$

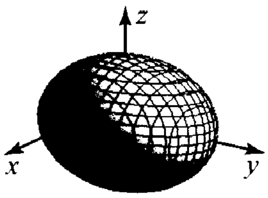
III $\quad y^2 = 4x^2 + 9z^2$

IV $\quad 4x^2 - 4y + z^2 = 0$

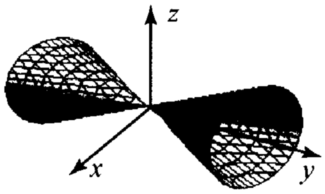
I



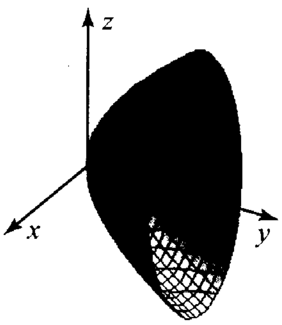
II



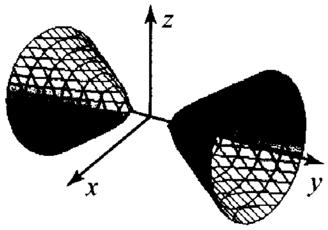
III



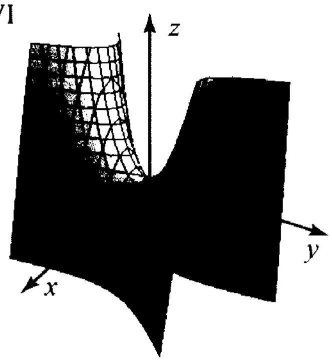
IV



V



VI



2. Determine the interval of convergence for the following power series.

a) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{e^n}{n+1}\right)x^n.$

b) $\displaystyle\sum_{n=1}^{\infty}\frac{2^n(x-4)^n}{n}.$

(*Note*: Be sure to check the endpoints.)

a) $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^{n+1}\cdot x^{n+1}}{n+2}\cdot\frac{n+1}{e^n\cdot x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{e\cdot x\cdot(n+1)}{n+2}\right|$$

$$= e\cdot|x|\cdot\lim_{n\to\infty}\frac{n+1}{n+2}$$

$$= e\cdot|x|\cdot 1 < 1$$

$$|x| < \frac{1}{e}$$

Radius of convergence is $R=\frac{1}{e}$. So we get the interval $-\frac{1}{e} < x < \frac{1}{e}$. Now we must check endpoints.

Let $x = -\frac{1}{e}$

$$\sum_{n=0}^{\infty}\left(\frac{e^n}{n+1}\right)\cdot\left(-\frac{1}{e}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$$

This series <u>converges</u> by "A.S.T."

2a Continued

Next we test $x = \frac{1}{e}$.

$$\sum_{n=0}^{\infty} \left(\frac{e^n}{n+1}\right) \cdot \left(\frac{1}{e}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

This series diverges by "LCT."

Hence the interval of convergence is $x \in [-\frac{1}{e}, \frac{1}{e})$.

2b] $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{2^{n+1}(x-4)^{n+1}}{n+1} \cdot \frac{n}{2^n (x-4)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2 \cdot (x-4) \cdot n}{n+1} \right|$$

$$= 2|x-4| \cdot \lim_{n\to\infty} \frac{n}{n+1}$$

$$= 2|x-4| \cdot 1$$

$$2 \cdot |x-4| < 1$$

$$|x-4| < \frac{1}{2}$$

The radius of convergence is $R = \frac{1}{2}$. So we get

$$-\frac{1}{2} < x-4 < \frac{1}{2}$$

$$\frac{7}{2} < x < \frac{9}{2}$$

2b) continued

Now we check endpoints, so let

$x = \frac{7}{2}$

$$\sum_{n=1}^{\infty} \frac{2^n\left(\frac{7}{2}-4\right)^n}{n} = \sum_{n=1}^{\infty} \frac{2^n\left(-\frac{1}{2}\right)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

This series converges by A.S.T.

Next let $x = \frac{9}{2}$

$$\sum_{n=1}^{\infty} \frac{2^n\left(\frac{9}{2}-4\right)^n}{n} = \sum_{n=1}^{\infty} \frac{2^n\left(\frac{1}{2}\right)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

This series diverges since it is the harmonic series. Hence, the interval of convergence is $x \in \left[\frac{7}{2}, \frac{9}{2}\right)$

3. Consider the function $f(x) = \pi^x$. (*Hint*: Recall that $\frac{d}{dx}[a^x] = a^x \cdot \ln a$.)

a) Derive the formula for the coefficients $\frac{f^{(n)}(0)}{n!}$ for a Maclaurin series.

b) Find a Maclaurin series for $f(x)$, writing your answer using summation notation.

Solution:

a)

n	$f^{(n)}(x)$	$f^{n}(0)$
0	π^x	1
1	$\pi^x \cdot \ln \pi$	$\ln \pi$
2	$\pi^x \cdot (\ln \pi)^2$	$(\ln \pi)^2$
3	$\pi^x (\ln \pi)^3$	$(\ln \pi)^3$
$\vdots$	$\vdots$	$\vdots$
n	$\pi^x (\ln \pi)^n$	$(\ln \pi)^n$

So $$\frac{f^{(n)}(0)}{n!} = \frac{(\ln \pi)^n}{n!}$$

b)

Hence, the Maclaurin Series is

$$f(x) = \sum_{n=0}^{\infty} \frac{(\ln \pi)^n}{n!} x^n$$

Because it is so much fun to do, lets check the radius and interval of convergence!

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(\ln \pi)^{n+1} \cdot x^{n+1}}{(n+1)!} \cdot \frac{n!}{(\ln \pi)^n \cdot x^n} \right|$$

#36] continued

$$= \lim_{n\to\infty} \left| \frac{\ln \pi}{n+1} \cdot x \right|$$

$$= \ln \pi \cdot |x| \cdot \lim_{n\to\infty} \frac{1}{n+1}$$

$$= \ln \pi \cdot |x| \cdot 0 < 1$$

Therefore the radius of convergence is $R = \infty$, and the interval of convergence is $x \in (-\infty, \infty)$

4. Determine whether the series $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$ converges or diverges. State any tests that you use and show all steps.

Lets use the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(2n+2)!}{(n+1)^{2n+2}} \cdot \frac{n^{2n}}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)\cancel{(2n)!} \cdot n^{2n}}{(n+1)^{2n}(n+1)^2 \cdot \cancel{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{2(n+1)\cdot(2n+1)\cdot n^{2n}}{(n+1)^2 \cdot (n+1)^{2n}}$$

$$= \lim_{n\to\infty} \frac{2\cancel{(n+1)}(2n+1)}{(n+1)^{\cancel{2}}} \cdot \frac{n^{2n}}{(n+1)^{2n}}$$

$$= \lim_{n\to\infty} \frac{4n+2}{n+1} \cdot \left(\frac{n}{n+1}\right)^{2n}$$

$$= \lim_{n\to\infty} \frac{4n+2}{n+1} \cdot \left(\frac{n+1}{n}\right)^{-2n}$$

$$= \lim_{n\to\infty} \frac{4n+2}{n+1} \cdot \left[\left(1+\frac{1}{n}\right)^{n}\right]^{-2}$$

$$= 4 \cdot e^{-2}$$

$$= \frac{4}{e^2} < 1$$

Hence, by The Ratio Test the series converges!

5. Consider the two planes $2x - y + 3z = 4$ and $x + 5y - 2z = 1$.

a) Find the angle between the two planes.

b) Determine the symmetric equations for the line of intersection between the two planes.

a) Let $\vec{n}_1 = \langle 2, -1, 3 \rangle$ and $\vec{n}_2 = \langle 1, 5 -2 \rangle$

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|}$$

$$\cos\theta = \frac{2 - 5 - 6}{\sqrt{14} \cdot \sqrt{30}}$$

$$\cos\theta = \frac{-9}{\sqrt{14} \cdot \sqrt{30}}$$

$$\theta = \cos^{-1}\left[\frac{-9}{\sqrt{14} \cdot \sqrt{30}}\right]$$ Remember no calculator!

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b) First find direction vector

$$\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 3 \\ 1 & 5 & -2 \end{vmatrix} = -13\vec{i} - (-7)\vec{j} + 11\vec{k}$$

$$= \langle -13, 7, 11 \rangle$$

Next we need to find a point on the line. Let $z = 0$

$$5 \cdot \begin{cases} 2x - y = 4 \\ x + 5y = 1 \end{cases} \Rightarrow \begin{cases} 10x - \cancel{5y} = 20 \\ x + \cancel{5y} = 1 \end{cases}$$

$$11x = 21$$

$$x = \frac{21}{11}$$

#56] Continued

$$\frac{21}{11} + 5y = 1$$

$$5y = -\frac{10}{11}$$

$$y = -\frac{10}{55}$$

$$y = -\frac{2}{11}$$

A point on the line is $\left(\frac{21}{11}, -\frac{2}{11}, 0\right)$

Hence the symmetric Equations are

$$\frac{x - \frac{21}{11}}{-13} = \frac{y + \frac{2}{11}}{7} = \frac{z}{11}$$

A calculator may be used to help solve problems 6 - 9.

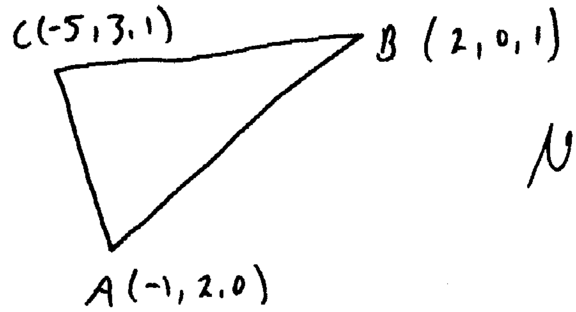
6. Consider the points in space $A(-1, 2, 0)$, $B(2, 0, 1)$, and $C(-5, 3, 1)$.

   a) Solve the triangle. (Find the lengths of all the sides and the measures of all the angles to the nearest degree.)

   b) Find the area of the triangle $\triangle ABC$.

   c) Find the equation of the plane passing through the points.

a)



Note: $\vec{BA} = \langle -3, 2, -1 \rangle$
(For later use.)

First $\vec{AB} = \langle 3, -2, 1 \rangle \quad |\vec{AB}| = \sqrt{9+4+1} = \sqrt{14}$

$\vec{AC} = \langle -4, 1, 1 \rangle \quad |\vec{AC}| = \sqrt{16+1+1} = 3\sqrt{2}$

$\vec{BC} = \langle -7, 3, 0 \rangle \quad |\vec{BC}| = \sqrt{49+9} = \sqrt{58}$

So side lengths are $AB = \sqrt{14}$, $AC = 3\sqrt{2}$, and $BC = \sqrt{58}$

$$\cos \angle A = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}| \cdot |\vec{AC}|} = \frac{-12-2+1}{\sqrt{14} \cdot 3\sqrt{2}}$$

$$\angle A \approx 145^\circ$$

$$\cos \angle B = \frac{\vec{BA} \cdot \vec{BC}}{|\vec{BA}| \cdot |\vec{BC}|} = \frac{21+6+0}{\sqrt{14} \cdot \sqrt{58}}$$

$\angle B \approx 19^\circ$ so $\angle C = 180^\circ - \angle A - \angle B = 16^\circ$

#6] Continued

b) Area of ΔABC is

$$A = \frac{1}{2} \left| \vec{AB} \times \vec{AC} \right|$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 1 \\ -4 & 1 & 1 \end{vmatrix} = \langle -3, -7, -5 \rangle$$

$$A = \frac{1}{2} \left| \langle -3, -7, -5 \rangle \right| = \frac{1}{2} \sqrt{9 + 49 + 25}$$

$$= \frac{1}{2} \sqrt{83} \text{ units}^2$$

c) let $\vec{n} = \vec{AB} \times \vec{AC} = \langle -3, -7, -5 \rangle$ and pick the point $(2, 0, 1)$ we get

$$-3(x-2) - 7(y-0) - 5(z-1) = 0$$

$$-3x - 7y - 5z + 11 = 0$$

7.) The Maclaurin series for $\sin x$ is given by $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

a) Use the Maclaurin series for $\sin x$ to find the power series for $f(x) = \frac{\sin(x^2)}{x}$. Write your answer using summation notation.

b) Use the results of part (a) to obtain $\int_0^1 \frac{\sin(x^2)}{x}\,dx$ to four decimal place accuracy.

a) $$f(x) = \frac{1}{x} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$= \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n (x^{4n+2})}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n+1)!}$$

b) $$\int_0^1 \frac{\sin(x^2)}{x}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n+1)!}\,dx$$

$$\left. \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(4n+2)(2n+1)!} \right]_0^1$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+2)(2n+1)!}$$

#7 Continued

Using the error test for alternating series:

Let $b_n = \dfrac{1}{(4n+2)(2n+1)!}$

So $b_{n+1} = \dfrac{1}{(4n+6)(2n+3)!}$

n	b_{n+1}
0	$.028$
1	$.83 \times 10^{-3}$
2	$.14 \times 10^{-4} < .5 \times 10^{-4}$

$$\sum_{n=0}^{2} \frac{(-1)^n}{(4n+2)(2n+1)!} \approx 0.4730$$

8. A bee is flying through space along a curve given by the position vector $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$.

a) Find $\mathbf{T}(t)$

b) When $t = \frac{\pi}{2}$ the bee flies off along a tangent line to the curve. Find the parametric equations of its line of flight.

c) Show that the curvature of $\mathbf{r}(t)$ is constant for all t.

a) $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$

$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1}$
$= \sqrt{2}$

$\vec{T}(t) = \frac{1}{\sqrt{2}} \cdot \langle -\sin t, \cos t, 1 \rangle$

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b) $\vec{r}\left(\frac{\pi}{2}\right) = \langle \cos\frac{\pi}{2}, \sin\frac{\pi}{2}, \frac{\pi}{2} \rangle$
$= \langle 0, 1, \frac{\pi}{2} \rangle$ $\quad (0, 1, \frac{\pi}{2})$ point of Tangency

Let the direction vector be $\vec{T}\left(\frac{\pi}{2}\right)$

$\vec{V} = \frac{1}{\sqrt{2}} \langle -\sin\frac{\pi}{2}, \cos\frac{\pi}{2}, 1 \rangle$

$= \frac{1}{\sqrt{2}} \langle -1, 0, 1 \rangle = \langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle$

Parametric ~~Eqautions~~ for the line of flight is

$x = 0 - \frac{1}{\sqrt{2}} \cdot t, \quad y = 1 + 0t, \quad z = \frac{\pi}{2} + \frac{1}{\sqrt{2}} t$

$x = -\frac{1}{\sqrt{2}} \cdot t, \quad y = 1, \quad z = \frac{\pi}{2} + \frac{1}{\sqrt{2}} t$

8c] continued,

$$K = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

$$\vec{T}'(t) = \frac{1}{\sqrt{2}} \cdot \langle -\cos t, -\sin t, 0 \rangle$$

$$|\vec{T}'(t)| = \frac{1}{\sqrt{2}} \cdot \sqrt{(-\cos t)^2 + (-\sin t)^2 + 0^2}$$

$$= \frac{1}{\sqrt{2}} \sqrt{\cos^2 t + \sin^2 t}$$

$$= \frac{1}{\sqrt{2}}$$

$$\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$$

From part (a) $|\vec{r}'(t)| = \sqrt{2}$

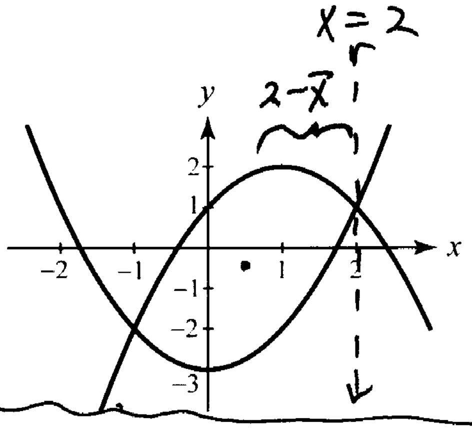
$$K = \frac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \frac{1}{2}$$ "A constant"!

9. Consider the lamina determined by the region bounded by the graphs of $y = x^2 - 3$ and $y = -x^2 + 2x + 1$.

a) Find the area of the lamina.

b) Find the centroid of the lamina.

c) Using the Theorem of Pappus, determine the volume of the solid when the lamina is rotated about the line $x = 2$.



a) $A = \int_{-1}^{2} [(-x^2 + 2x + 1) - (x^2 - 3)]\, dx$

$A \overset{CAS}{=} 9$

b) $\bar{x} = \frac{1}{A} \int_a^b x\, [f(x) - g(x)]\, dx$

$= \frac{1}{9} \int_{-1}^{2} x\, [(-x^2 + 2x + 1) - (x^2 - 3)]\, dx$

$\overset{CAS}{=} \frac{1}{2}$

$\bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} [(f(x))^2 - (g(x))^2]\, dx$

$= \frac{1}{9} \int_{-1}^{2} \frac{1}{2} [(-x^2 + 2x + 1)^2 - (x^2 - 3)^2]\, dx$

$\overset{CAS}{=} -\frac{1}{2}$ Centroid: $\left(\frac{1}{2}, -\frac{1}{2}\right)$

c) $V = A \cdot 2\pi \cdot (2 - \bar{x})$

$\overset{CAS}{=} 27\pi$ units^3

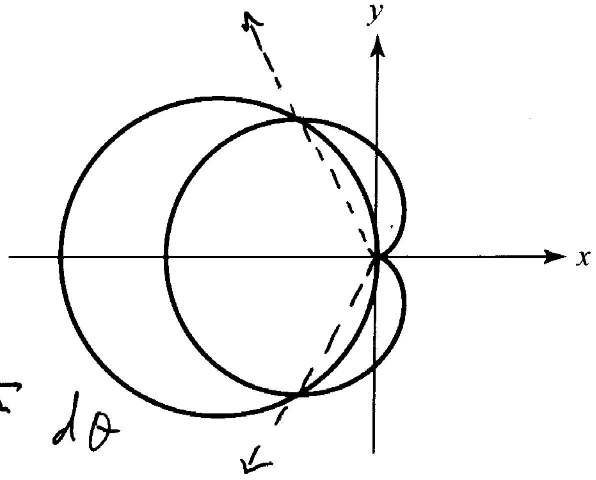
10. Consider the polar curves, $r = 2 - 2\cos\theta$ and $r = -6\cos\theta$.

a) Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ for $r = 2 - 2\cos\theta$

b) Find the points where the two curves intersect. Show all your work.

c) Write and evaluate an integral expression that will calculate the area of the region common to the two regions bounded by the given polar curves.

a) $L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$



$\frac{dr}{d\theta} = 2\sin\theta$

$L = \int_0^{2\pi} \sqrt{(2 - 2\cos\theta)^2 + (2\sin\theta)^2}\, d\theta$

$L \overset{CAS}{=} 16$

b) $-6\cos\theta = 2 - 2\cos\theta$

$-4\cos\theta = 2$

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}$ or $\theta = \frac{4\pi}{3}$

Polar points are $\left(3, \frac{2\pi}{3}\right)$ and $\left(3, \frac{4\pi}{3}\right)$

c) $A = 2\left[\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{2}(-6\cos\theta)^2\, d\theta + \int_{\frac{2\pi}{3}}^{\pi} \frac{1}{2}(2 - 2\cos\theta)^2\, d\theta\right]$

$A \overset{CAS}{=} 5\pi$ units^2