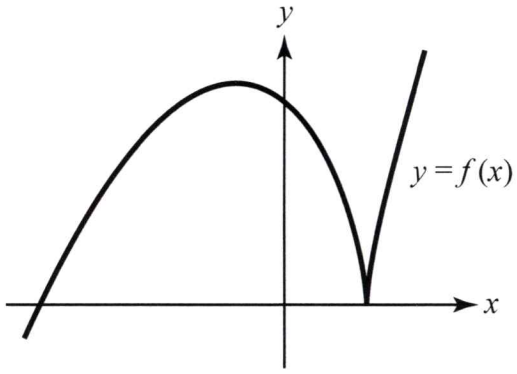
Math 1A

Test 3 - Review

No calculator for problems 1-3. Show all work.

1. Consider the function $f(x)$ whose domain is all real numbers, the graph of $f(x)$ is shown below.



The first derivative of f is given by $f'(x) = \dfrac{2x+9}{(x-7)^{\frac{1}{3}}}$.

a) Find the critical numbers for f.

b) Find the intervals of increasing and decreasing.

c) Determine the x-coordinates of all extrema, if any. Identify them as local or absolute.

a) Let $f'(x) = 0$

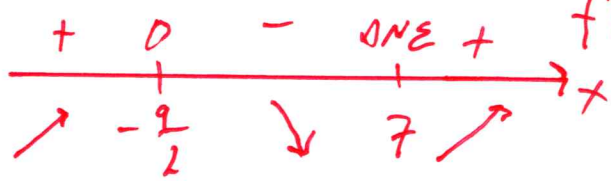
$$\frac{2x+9}{(x-7)^{1/3}} = 0$$

$$x = -\frac{9}{2}$$

$f'(x)$ does not exist at $x = 7$

So, $x = -\frac{9}{2}$ and $x = 7$ are critical numbers since they are both in the domain

b)



By I D T,

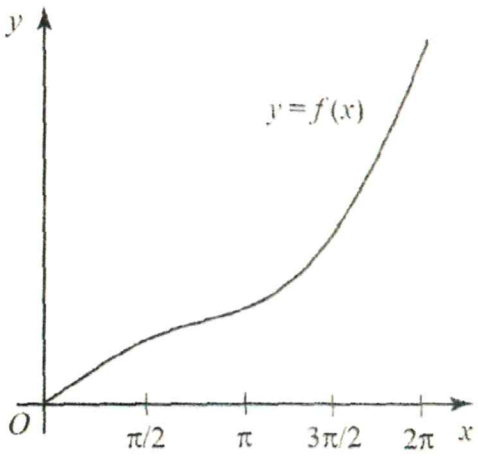
Increasing: $x \in (-\infty, -\frac{9}{2}) \cup (7, +\infty)$

Decreasing: $x \in (-\frac{9}{2}, 7)$

#1. continued

c) By FDT, a local maximum occurs at $x = -\frac{9}{2}$ and a local minimum occurs at $x = 7$. Because of end behavior $f(x)$ has not absolute extrema.

2. Consider the function $y = f(x)$ defined on the interval $[0, 2\pi]$, the graph of $y = f(x)$ is show below. (15 points)



The second derivative of $f(x)$ is given by

$$f''(x) = 1 - 2\sin x.$$

a) Find the x-values for which $f''(x) = 0$ for the given domain $[0, 2\pi]$.

b) Find the intervals of concavity.

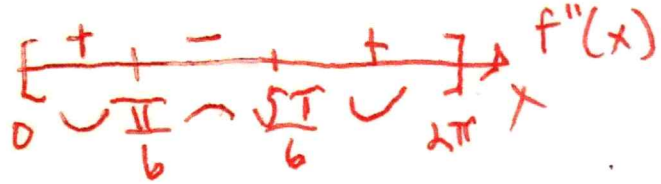
c) Find the x-coordinates of the inflection points, if any.

a) Let $1 - 2\sin x = 0$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$

b)



By CT,

Concave up: $x \in (0, \frac{\pi}{6}) \cup (\frac{5\pi}{6}, 2\pi)$

Concave down: $x \in (\frac{\pi}{6}, \frac{5\pi}{6})$

c) By DIP, the x-coordinates of inflection points are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$

3. Evaluate each integral and derivative, show all steps.

a) If $f''(x) = e^x - 2\sin x$ with $f(0) = 3$ and $f(\pi/2) = 0$, find $f(x)$.

b) $\int \frac{\cos x}{1 + \sin^2 x}\, dx$

c) $\int_{1/16}^{1/4} \frac{\sqrt{t} - 1}{t}\, dt$

d) $\int 3x\sqrt{3x^2 + 1}\, dx$

e) $\int_0^1 x \cdot e^{-x^2}\, dx$

f) If $g(x) = \int_{\ln x}^{5} 3t \cdot \sin t\, dt$, then find $\frac{d}{dx}[g(x)]$.

a) $f'(x) = \int (e^x - 2\sin x)\, dx$

$f'(x) = e^x + 2\cos x + C$

$f(x) = \int (e^x + 2\cos x + C)\, dx$

$f(x) = e^x + 2\sin x + Cx + D$

$f(0) = 3$

We get $3 = e^0 + 2\sin(0) + C(0) + D$

$3 = 1 + 0 + 0 + D$

$D = 2$

$f\left(\frac{\pi}{2}\right) = 0$

$0 = e^{\pi/2} + 2(1) + C\left(\frac{\pi}{2}\right) + 2$

$\frac{\pi}{2} C = -e^{\pi/2} - 4$

$C = \frac{-e^{\pi/2} - 8}{\pi}$

$f(x) = e^x + 2\sin x + \frac{-e^{\frac{\pi}{2}} - 8}{\pi} x + 2$

b) $\int \frac{\cos x}{1 + \sin^2 x}\, dx$ Let $u = \sin x$

$du = \cos x\, dx$

$= \int \frac{1}{1 + u^2}\, du$

$= \tan^{-1}(u) + C = \tan^{-1}(\sin x) + C$

Problem 3 continued

c) $\int_{1/16}^{1/4} \left(\frac{\sqrt{t}}{t} - \frac{1}{t}\right) dt$

$= \int_{1/16}^{1/4} \left(t^{-1/2} - \frac{1}{t}\right) dt$

$= \left[2\sqrt{t} - \ln|t|\right]_{1/16}^{1/4}$

$= \left(2 \cdot \frac{1}{2} - \ln\left(\frac{1}{4}\right)^*\right) - \left(2 \cdot \frac{1}{4} - \ln\left(\frac{1}{16}\right)\right)$

$= (1 + \ln 4) - \left(\frac{1}{2} + \ln 16\right)$

$= \frac{1}{2} + \ln\left(\frac{4}{16}\right)$

$= \frac{1}{2} + \ln\left(\frac{1}{4}\right)$

$= \frac{1}{2} - \ln(4)$

*Note:
$\ln\left(\frac{1}{4}\right) = \ln(4^{-1})$
$= -\ln 4$

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d) $\int 3x\sqrt{3x^2+1}\, dx$ Let $u = 3x^2+1$

$= 3\int x\sqrt{3x^2+1}\, dx$ $du = 6x\, dx$

$= 3 \cdot \frac{1}{6}\int u^{\frac{1}{2}} du$ $\frac{1}{6} du = x\, dx$

Problem 3 continued

d) $\frac{1}{2} \cdot \left[\frac{2}{3} u^{3/2}\right] + C$

$= \frac{1}{3} u^{3/2} + C \qquad u = 3x^2 + 1$

$= \frac{1}{3}\left(3x^2+1\right)^{3/2} + C$

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e) $\int_0^1 x \cdot e^{-x^2} dx$

Let $u = -x^2$

$du = -2x\,dx$

$-\frac{1}{2} du = x\,dx$

New Limits

$u(0) = -(0)^2 = 0$

$u(1) = -(1)^2 = -1$

$= -\frac{1}{2}\int_0^{-1} e^u du$

$= \frac{1}{2}\int_{-1}^{0} e^u du$

$= \frac{1}{2}\left[e^u\right]_{-1}^{0}$

$= \frac{1}{2}\left[e^0 - e^{-1}\right]$

$= \frac{1}{2}\left[1 - \frac{1}{e}\right]$

$= \frac{1}{2} - \frac{1}{2e}$

$= \frac{e-1}{2e}$

Problem 3 continued

f) $g(x) = \int_{\ln x}^{5} 3t \cdot \sin t \, dt$

$$g(x) = -\int_{5}^{\ln x} 3t \cdot \sin t \, dt$$

$$= -3 \cdot \ln x \cdot \sin(\ln x) \cdot \frac{d}{dx}[\ln x]$$

$$= -3 \cdot \ln x \cdot \sin(\ln x) \cdot \frac{1}{x}$$

$$= \frac{-3 \cdot \ln x \cdot \sin(\ln x)}{x}$$

A calculator may be used for problems 4-6.

4. A particle moves along the x-axis so that its velocity at ant time $t \geq 0$ is given by $v(t) = t \cdot \sin\left(\frac{1}{5}t^2\right)$, velocity is measured in feet per second.

   a) Write an expression for the position function $s(t)$ of the particle if at time $t = 0$, the position of the particle is $s(0) = 12$ feet.

   b) Using a definite integral, find the displacement of the particle from $t = 0$ to $t = 5$ seconds.

   c) Using a definite integral, find the total distance traveled by the particle from $t = 0$ to $t = 5$ seconds.

   d) Write an expression for the acceleration of the particle in terms of t.

   e) Is the particle speeding up or slowing down at $t = 7.191$ seconds. Justify your anwser.

a) $s(t) = \int v(t)\,dt = \int t \cdot \sin\left(\frac{1}{5}t^2\right) dt$

$\overset{CAS}{=} -\frac{5}{2} \cdot \cos\left(\frac{1}{5}t^2\right) + C$

$s(0) = 12\text{ft}$, we get

$12 = -\frac{5}{2} \cdot \cos(0) + C$

$12 = -\frac{5}{2} + C$

$C = \frac{29}{2} = 14.5$

$s(t) = -\frac{5}{2}\cos\left(\frac{1}{5}t^2\right) + \frac{29}{2}$

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b) $\text{Displacement} = \int_0^5 t \cdot \sin\left(\frac{1}{5}t^2\right) dt$

$\overset{CAS}{=} -\frac{5}{2}\cos(5) + \frac{5}{2}$

$\approx 1.791\text{ feet}$

Problem 4 continued

c) Total distance $= \int_0^5 \left| t \cdot \sin\left(\frac{1}{5}t^3\right) \right| dt$

$\overset{CAS}{\approx} 8.209$ ft

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d) $a(t) = v'(t)$

$\overset{CAS}{=} \sin\left(\frac{1}{5}t^3\right) + \frac{3}{5}t^3 \cdot \cos\left(\frac{1}{5}t^3\right)$

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e) $v(7.191) \overset{CAS}{\approx} -5.709$ ft/sec

$a(7.191) \overset{CAS}{\approx} -13.369$ ft/sec^2

The particle is speeding up since velocity and acceleration have the same sign, both negative

5. A community group wants to construct a rectangular garden plot alongside a straight stone wall of a historic building. The wall will act as one side of the garden, so no fencing is needed along that side. The remaining three sides will be enclosed using 500 feet of decorative metal fencing. Additionally, some of the metal fencing will be placed parallel to the stone wall to split the garden into two rectangular sections (one for vegetables, one for herbs). Find the dimensions of the garden that will maximize the total enclosed area.

a) State your objective.

b) Write a function of a single variable that describes the area of rectangular garden plot.

c) Determine the domain of the function you found in part (b).

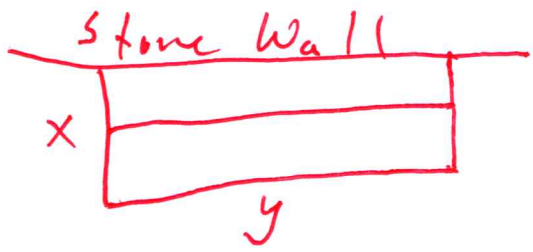
d) Find the derivative of the function you found in part (b), and test any critical numbers for absolute extrema. State the theorems you used.

e) Determine the absolute maximum area of rectangular garden plot.

a) Maximize the total area enclosed by the rectangular plot.

Let x = width (ft)

y = length (ft)



Constraint Equation $2x + 2y = 500$

$2y = 500 - 2x$

$y = 250 - x$

b) $A = \ell \cdot w = x \cdot y$

$= x(250 - x)$

$A(x) = 250x - x^2$

c) Domain

$x \geq 0$

$y \geq 0$

$250 - x \geq 0$

$-x \geq -250$

$x \leq 250$

$x \in [0, 250]$

Problem 5 continued

d) $A'(x) = 250 - 2x$

Let $A'(x) = 0$

$250 - 2x = 0$

$-2x = -250$

$x = 125$ is the critical number

By EVT,

x	A(x)
0	0
125	15625
250	0

$A(0) = 0$

$A(250) = 0$

$A(125) = 15625$

$y = 250 - x$

$y = 250 - 125$

$y = 125$

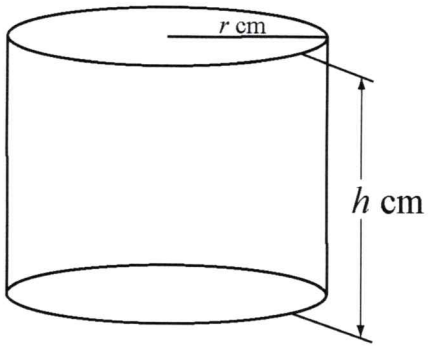
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e) The absolute maximum area of the plot is 15625 sq ft with dimensions 125 ft × 125 ft

6. A beverage company is designing a standard cylindrical aluminum can that must hold a fixed volume of 356 cm^3 (approximately 12 fl oz). The top and bottom lids of the can are made from thicker, reinforced aluminum that costs \$0.03 per cm^2, while the curved side (the lateral surface) is made from standard aluminum costing \$0.01 per cm^2. Find the radius r and height h that will minimize the production cost of each can.

$V = \pi r^2 h$

$A = 2\pi r^2 + 2\pi rh$



$356 = \pi r^2 h$

$h = \frac{356}{\pi r^2}$

a) State your objective.

b) Write a function of a single variable that describes the cost of the aluminum can.

c) Determine the domain of the function you found in part (b).

d) Find the derivative of the function you found in part (b), and test any critical numbers for absolute extrema. State the theorems you used.

e) Determine the absolute minimum cost of the aluminum can.

a) Minimize the cost of producing an aluminum can.

b) Cost of Top and Bottom

$2\pi r^2 (0.03) = 0.06\pi r^2$

Cost of lateral side

$2\pi rh (0.01) = 2\pi r \left(\frac{356}{\pi r^2}\right)(0.01)$

$= .02 \left(\frac{356}{r}\right)$

$= \frac{7.12}{r}$

$C(r) = 0.06\pi r^2 + \frac{7.12}{r}$

$C(r) = \frac{0.06\pi r^3 + 7.12}{r}$

Problem 6 continued

c) $r > 0$

Domain: $r \in (0, +\infty)$

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d) $C'(r) \overset{CAS}{=} \dfrac{0.12\pi r^3 - 7.12}{r^2}$

Let $C'(r) = 0$

By CAS $r \doteq 2.663$ cm

By IDT, FDT and OLE the absolute minimum cost will occur at $r = 2.663$ cm

$h = \dfrac{356}{\pi(2.663)^2} \overset{CAS}{\simeq} 15.978$ cm

Cost $= C(2.663)$

$\overset{CAS}{=} 4.011$

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e) The absolute minimum cost of producing one can is $\$4.01$