Math 1A

Test 2 Review

A calculator can not be use for problems 1 through 3.

1. Consider the function $h(x) = \sqrt{4 + 5f(x)}$, where $f(2) = 1$ and $f'(2) = 3$.

   a) Find $h(2)$.

   b) Determine $h'(x)$.

   c) Find the slope of the tangent line to $h(x)$ at $x = 2$.

   d) Find the equations of the tangent line and normal line to $h(x)$ at $x = 2$.

a) $h(2) = \sqrt{4 + 5f(2)}$

$= \sqrt{4 + 5(1)}$

$= \sqrt{9}$

$= 3$

---

b) $h'(x) = \frac{1}{2}(4 + 5f(x))^{-\frac{1}{2}} \cdot 5 \cdot f'(x)$

$= \frac{5 \cdot f'(x)}{2\sqrt{4 + 5 \cdot f(x)}}$

---

c) $h'(2) = \frac{5 \cdot f'(2)}{2\sqrt{4 + 5 \cdot f(2)}}$

$= \frac{5 \cdot 3}{2\sqrt{9}}$

$= \frac{15}{6}$

$= \frac{5}{2}$

Problem 1 continued

1d) Tangent Line:

① $m = \frac{5}{2}$

② POT: $(2, 3)$

③ $y - 3 = \frac{5}{2}(x - 2)$

$y = \frac{5}{2}x - 2$

Normal Line

$y - 3 = -\frac{2}{5}(x - 2)$

$y = -\frac{2}{5}x + \frac{19}{5}$

2. Consider the function $x\ln(y) + ye^{x-1} = 2$.

a) Find $\frac{dy}{dx}$.

b) Find the slope of the tangent line at (1, 1).

c) Find the equations of the tangent and normal lines to the point (1, 1).

a)

$$\frac{d}{dx}[x\cdot\ln(y)] + \frac{d}{dx}[y\cdot e^{x-1}] = 0$$

$$x\cdot\frac{1}{y}\cdot y' + \ln(y) + y\cdot e^{x-1} + e^{x-1}\cdot y' = 0$$

$$\left(\frac{x}{y} + e^{x-1}\right)\cdot y' = -\ln y - y\cdot e^{x-1}$$

$$y' = -\frac{\ln y + y\cdot e^{x-1}}{\frac{x}{y} + e^{x-1}}\cdot\frac{y}{y}$$

$$y' = -\frac{y\cdot\ln y + y^2\cdot e^{x-1}}{x + y\cdot e^{x-1}}$$

b)

$$m|_{(1,1)} = -\frac{1\cdot\ln(1) + 1^2\cdot e^0}{1 + 1\cdot e^0} = -\frac{1\cdot 0 + 1\cdot 1}{1+1\cdot 1}$$

$$= -\frac{1}{2}$$

c) Tangent Line

$$y-1 = -\frac{1}{2}(x-1)$$

$$y = -\frac{1}{2}x + \frac{3}{2}$$

Normal Line

$$y-1 = 2(x-1)$$

$$y = 2x - 1$$

3. Find $\frac{dy}{dx}$ for each function below. (Show all symbolic work.)

a) $y = \dfrac{1}{(3x^2+5x-7)^{-30}}$

b) $y = \sqrt[4]{5-2x^3}$

c) $y = x^3 \cdot \sin\left(\sqrt{x+1}\right)$

d) $y = x^2 \cdot \sinh(3x) + \cosh^4(x)$

e) $y = \sin^{-1} x + x\sqrt{1-x^2}$

f) $y = \dfrac{(x^2+1)^3 \cdot \sqrt{2x-5}}{(3x-2)^4}$

g) $y = (\ln(x))^{x^2}$

a) $y = (3x^2+5x-7)^{30}$

$\frac{dy}{dx} = 30\cdot(3x^2+5x-7)\cdot(6x+5)$

b) $\frac{dy}{dx} = \frac{1}{4}(5-2x^3)^{-\frac{3}{4}}\cdot(-6x^2)$

$\frac{dy}{dx} = -\frac{3x^2}{2(5-2x^3)^{3/4}}$

c) $\frac{dy}{dx} = x^3\cdot\cos(\sqrt{x+1})\cdot\frac{1}{2\sqrt{x+1}} + \sin(\sqrt{x+1})\cdot 3x^2$

$= \frac{x^3\cdot\cos(\sqrt{x+1})}{2\sqrt{x+1}} + 3x^2\cdot\sin(\sqrt{x+1})$

d) $\frac{dy}{dx} = 3x^2\cdot\cosh(3x) + 2x\cdot\sinh(3x) + 4\cosh^3 x\cdot\sinh x$

e) $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + x\cdot\frac{-2x}{2\sqrt{1-x^2}} + \sqrt{1-x^2}$

$= \frac{1}{\sqrt{1-x^2}} + \frac{-x^2}{\sqrt{1-x^2}} + \frac{1-x^2}{\sqrt{1-x^2}}$ common denominator

$= \frac{2-2x^2}{\sqrt{1-x^2}}$

3e) continued

$$= \frac{2(1-x^2)}{\sqrt{1-x^2}}$$

$$= \frac{2(1-x^2)^1}{(1-x^2)^{1/2}}$$

$$= 2(1-x^2)^{1/2}$$

$$= 2\sqrt{1-x^2}$$

---

f) Logarithmic Differentiation

$$\ln y = 3\cdot\ln(x^2+1) + \frac{1}{2}\ln(2x-5) - 4\cdot\ln(3x-2)$$

$$\frac{1}{y}\cdot y' = \frac{3}{x^2+1}\cdot 2x + \frac{1}{2}\cdot\frac{2}{2x-5}\cdot 4\cdot\frac{1}{3x-2}\cdot 3$$

$$y' = \frac{(x^2+1)^3\cdot\sqrt{2x-5}}{(3x-2)^4}\cdot\left(\frac{6x}{x^2+1} + \frac{1}{2x-5} - \frac{12}{3x-2}\right)$$

---

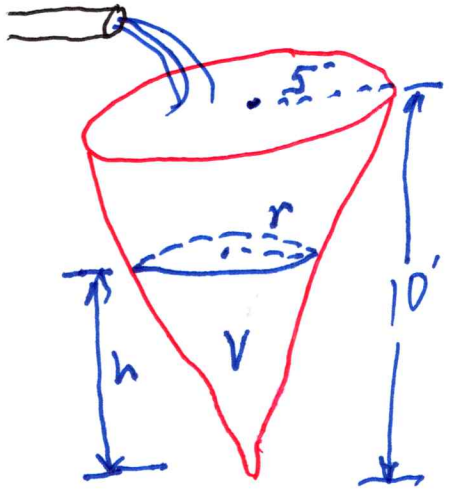
g) $\ln y = x^2\cdot\ln(\ln x)$

$$\frac{1}{y}\cdot\frac{dy}{dx} = x^2\cdot\frac{1}{\ln x}\cdot\frac{1}{x} + 2x\cdot\ln(\ln x)$$

$$\frac{dy}{dx} = (\ln(x))^{x^2}\cdot\left(\frac{x}{\ln x} + 2x\cdot\ln(\ln(x))\right)$$

A calculator may be used to help solve problems 4 through 6.

4. Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 feet and a base radius of 5 feet. How fast is the water level rising when the water is 6 feet deep? (Recall: $V = \frac{1}{3}\pi r^2 h$)



I. Introduction

V = Volume of cone (ft^3)
h = height of cone (ft)
r = radius of water level

II. Known Derivative

$\frac{dV}{dt} = 9\ ft^3/min$

III. Unknown Derivative

$\frac{dh}{dt}\Big|_{h=6\ ft}$

IV. Equation

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 \cdot h$

$V = \frac{1}{3}\pi \cdot \frac{h^3}{4}$

$V = \frac{1}{12}\pi h^3$



V. Differential Equation

$\frac{dV}{dt} = \frac{\pi}{4} h^2 \cdot \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{4}{\pi h^2} \cdot \frac{dV}{dt}$

$\frac{r}{5} = \frac{h}{10}$

$r = \frac{h}{2}$

VI. Snap Shot

$\frac{dV}{dt} = 9, \quad h = 6$

$\frac{dh}{dt} = \frac{4}{\pi (6)^2} \cdot 9$

$\frac{dh}{dt} = \frac{1}{\pi}\ ft/min$

$\approx 0.318\ ft/min$

VII. Conclusion

The water level is rising at a rate of $\frac{1}{\pi}$ ft/min when the water level is at 6 ft.

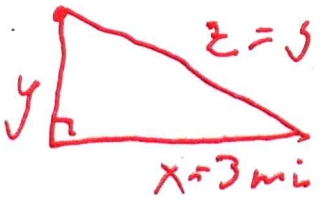
5. A rocket that is launched vertically is tracked by a radar station located on the ground 3 miles from the launch site. What is the vertical speed of the rocket at the instant that its distance from the radar station is 5 miles and this distance is increasing at the rate of 5000 mph?

I. Introduction

Let: x = Horizontal distance from launch site to radar
y = Vertical height of rocket
z = Distance from the radar station to rocket.

II. Known Derivative

$\frac{dz}{dt} = 5000 \text{ mph}$



III. Unknown Derivative

$\frac{dy}{dx} \Big| y = 4 \text{ miles}$

IV. Equation

$x^2 + y^2 = z^2 \quad \Big| \quad 3^2 + y^2 = z^2$

$x = 3$

V. Differential Equation

$\frac{d}{dt}(9 + y^2) = \frac{d}{dt} z^2 \quad \Big| \quad \frac{dy}{dx} = \frac{z}{y} \cdot \frac{dz}{dt}$

$2y \frac{dy}{dt} = 2z \cdot \frac{dz}{dt}$

VI. Snap Shot

$x = 3 \text{ mi}$ and $z = 4 \text{ mi}$, $\frac{dz}{dt} = 5000 \text{ mph}$

$9 + y^2 = 25 \quad \Big| \quad \frac{dy}{dt} = \frac{5}{4} \cdot 5000$

$y^2 = 16$

$y = 4$

$\frac{dy}{dt} = 6250 \text{ mph}$

VII. Conclusion

The vertical speed of the rocket is 6250 mph when the rocket is 5 miles away from the radar station.

6. A particle moves along a horizontal coordinate line so that its position at any time $t \geq 0$ (in seconds) is given by the function:

$$s(t) = t^3 - 9t^2 + 24t + 5$$

where s is measured in feet.

a) Find the velocity $v(t)$ and the acceleration $a(t)$ of the particle at any time t.

b) Find the position, velocity, and acceleration of the particle at $t = 3.147$ seconds.

c) Is the particle speeding up or slowing down at this instant? Explain your reasoning.

d) At what time(s) is the particle at rest?

e) During what time intervals is the particle moving to the left?

f) Moving to the right?

g) Find the total distance traveled by the particle during the first 5 seconds (from $t = 0$ to $t = 5$).

a) $v(t) = 3t^2 - 18t + 24$ $\quad a(t) = 6t - 18$

b) $s(3.147) \overset{CAS}{\approx} 22.502$ ft $\quad v(3.147) \approx -2.934$ ft/sec

$a(3.147) \approx 0.882$ ft/sec^2

c) The particle is slowing down since velocity ($v < 0$) and acceleration ($a > 0$) are opposite signs at $t = 3.147$ seconds

d) Let $v(t) = 0$

$3t^2 - 18t + 24 = 0$

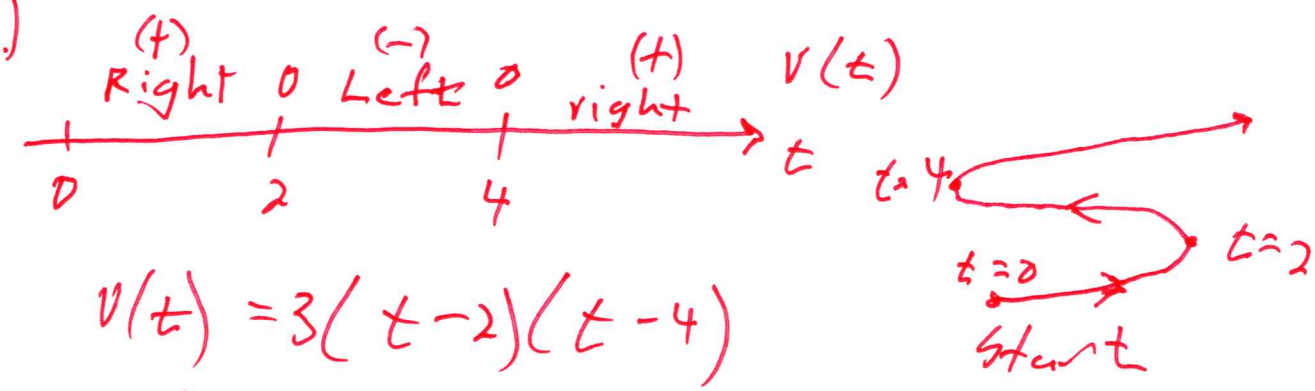
$3(t^2 - 6t + 8) = 0$

$3(t-2)(t-4) = 0$

The particle is stopped at $t = 2$ and $t = 4$ seconds

#6 continued

e.) & f.)



$v(t) = 3(t-2)(t-4)$

e.) Moving Left : $t \in (2, 4)$ seconds

Moving right : $t \in (0, 2) \cup (4, +\infty)$ seconds

---

g) Total Distance

$$|s(2) - s(0)| + |s(4) - s(2)| + |s(5) - s(4)|$$

$\overset{cAS}{=}$ $20 + 4 + 4$

$= 28$ feet total distance from $t = 0$ to $t = 5$ seconds.