2.6 Limits at Infinity; Horizontal Asymptotes



Euclid of Alexandria
325 – 265 B.C.

Euclid was a Greek mathematician best known for his treatise on geometry: *The Elements*. This influenced the development of Western mathematics for more than 2000 years and counting.

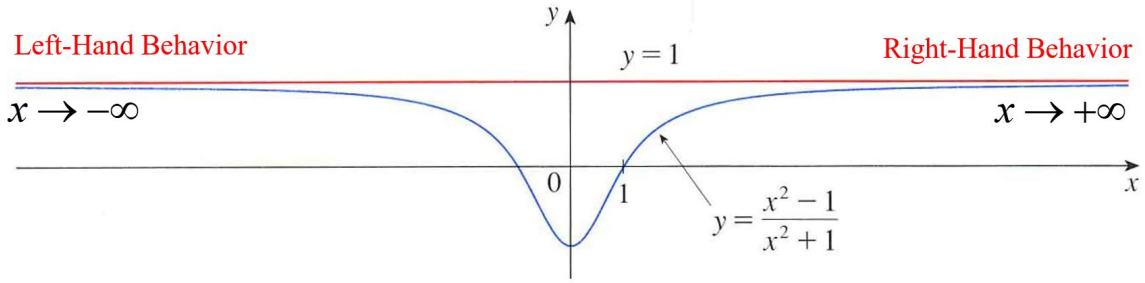
Let's begin by investigating the behavior of the function f defined by

x	$f(x)$
0	-1
± 1	0
± 2	0.600000
± 3	0.800000
± 4	0.882353
± 5	0.923077
± 10	0.980198
± 50	0.999200
± 100	0.999800
± 1000	0.999998

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

So, we say, $\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$ and $\lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1$

These limits shows that $f(x)$ has a horizontal asymptote at $y = 1$.



1 Intuitive Definition of a Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Another notation for $\lim_{x \to \infty} f(x) = L$ is

$$f(x) \to L \quad \text{as} \quad x \to \infty$$

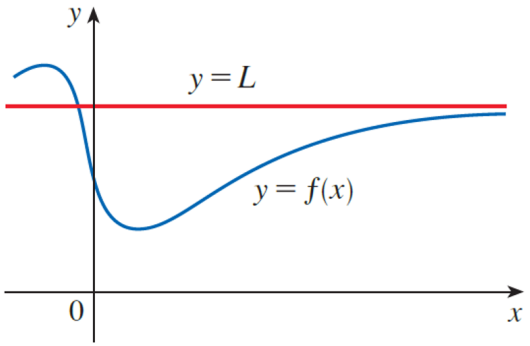
The symbol ∞ does not represent a number. Nonetheless, the expression $\lim_{x \to \infty} f(x) = L$ is often read as

"the limit of $f(x)$, as x approaches infinity, is L"

or "the limit of $f(x)$, as x becomes infinite, is L"

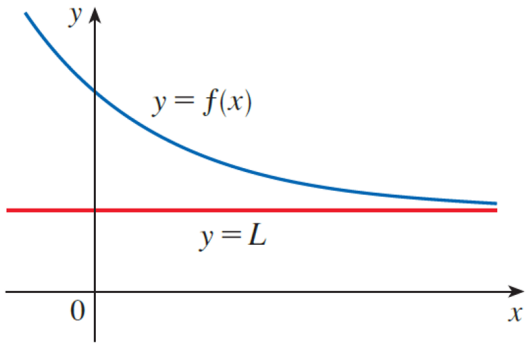
or "the limit of $f(x)$, as x increases without bound, is L"

We take limits to infinity to analyze **end-behavior** of functions.



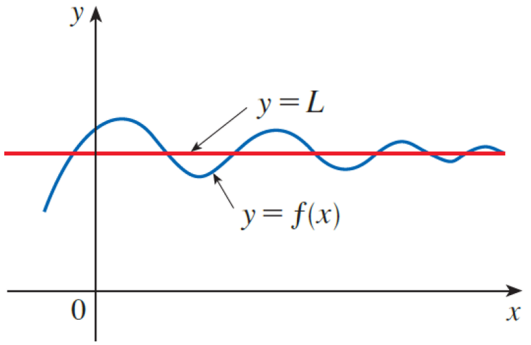
$\lim_{x \to \infty} f(x) = L$

Right-Hand Behavior



$\lim_{x \to \infty} f(x) = L$

Right-Hand Behavior



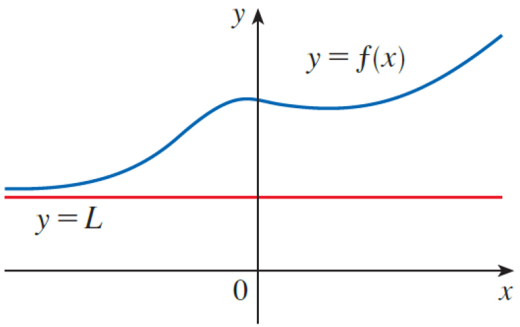
$\lim_{x \to \infty} f(x) = L$

Right-Hand Behavior

2 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

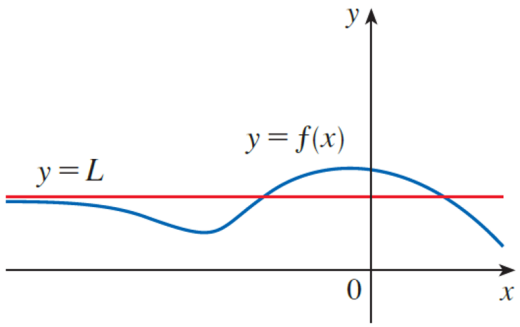
$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.



$$\lim_{x \to -\infty} f(x) = L$$

Left-Hand Behavior



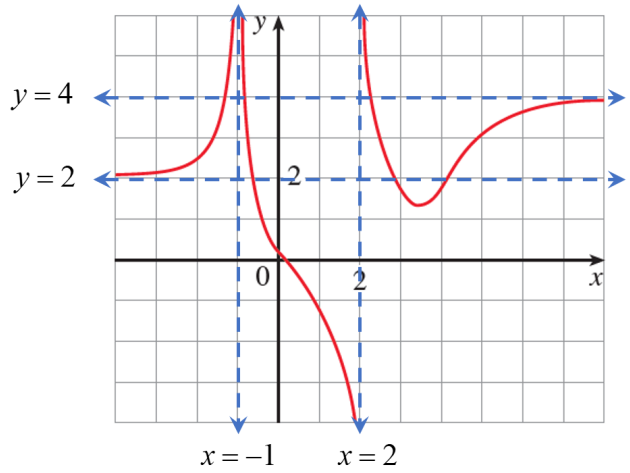
$$\lim_{x \to -\infty} f(x) = L$$

Left-Hand Behavior

3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

Example Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown in the figure.



Solution

$$\lim_{x \to -1} f(x) = \infty$$

$$\lim_{x \to 2^-} f(x) = -\infty \qquad \text{and} \qquad \lim_{x \to 2^+} f(x) = \infty$$

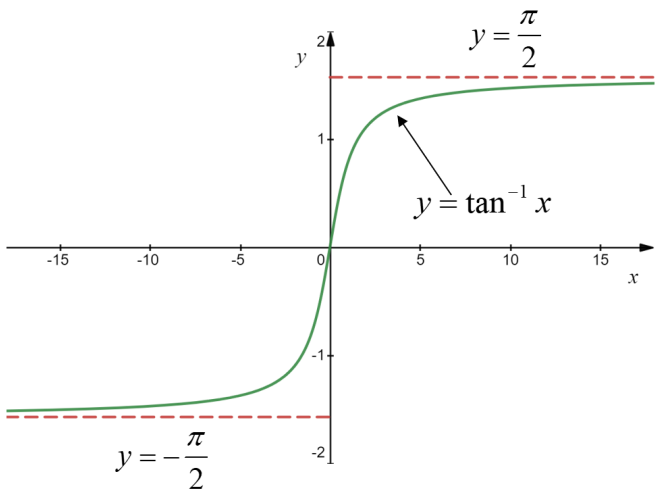
Thus both of the lines $x = -1$ and $x = 2$ are vertical asymptotes.

$$\lim_{x \to \infty} f(x) = 4 \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = 2$$

This means that both $y = 4$ and $y = 2$ are horizontal asymptotes.

Example Find $\lim_{x \to \pm\infty} \tan^{-1} x$

Solution



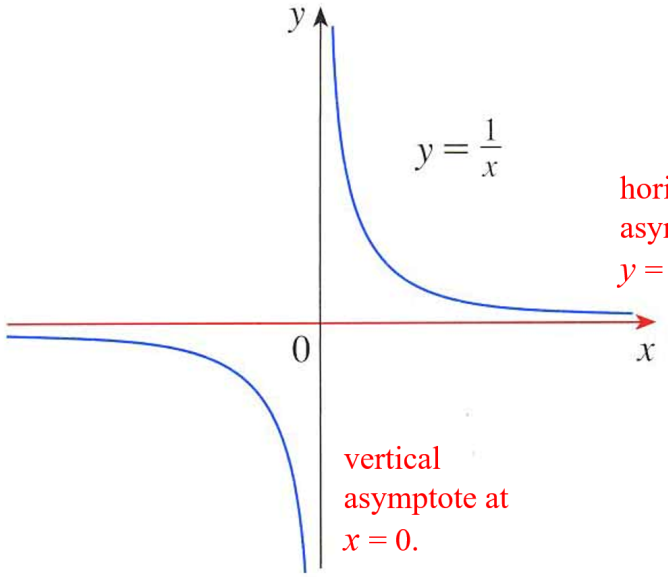
So, $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes for $y = \tan^{-1} x$.

$$\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2} \qquad \lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$$

Recall $f(x) = \frac{1}{x}$

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$ $\lim_{x \to 0^+} \frac{1}{x} = +\infty$

$\lim_{x \to +\infty} \frac{1}{x} = 0$ $\lim_{x \to -\infty} \frac{1}{x} = 0$



horizontal asymptote at $y = 0$.

vertical asymptote at $x = 0$.

In 2.3 these limits showed $x = 0$ is a vertical asymptote.

In these limits, as the denominator gets larger, the value of the fraction gets closer to zero. So, by definition, $y = 0$ is a horizontal asymptote.

Evaluating Limits at Infinity

5 Theorem If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

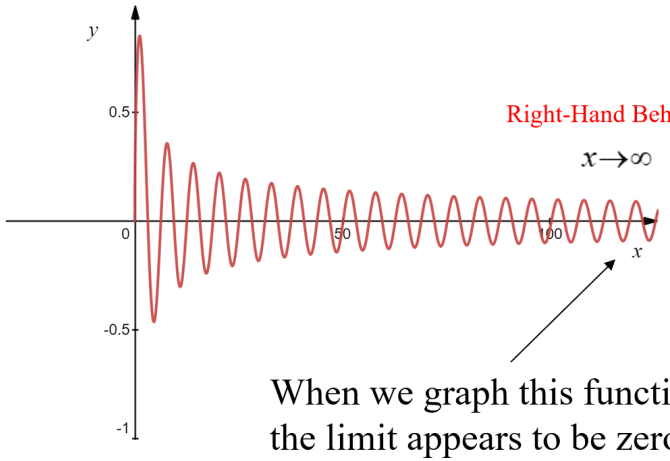
Example Find: $\lim_{x\to\infty} \dfrac{\sin x}{\sqrt{x}}$

Solution

We know that $-1 \le \sin x \le 1$.

Since $x > 0$ it follows that $\dfrac{1}{\sqrt{x}} > 0$,

So, $\dfrac{-1}{\sqrt{x}} \le \dfrac{\sin x}{\sqrt{x}} \le \dfrac{1}{\sqrt{x}}$ (1)



Right-Hand Behavior

$x \to \infty$

When we graph this function, the limit appears to be zero.

As $x \to \infty$, $\sqrt{x}$ becomes arbitrarily large, which means that

$$\lim_{x\to\infty} \frac{-1}{\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0. \quad (2)$$

Theorem 5

$$\frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}}$$

So from inequality (1) we get,

$$\lim_{x\to\infty} \frac{-1}{\sqrt{x}} \le \lim_{x\to\infty} \frac{\sin x}{\sqrt{x}} \le \lim_{x\to\infty} \frac{1}{\sqrt{x}}$$

5 Theorem If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

Example Find: $\lim_{x\to\infty}\dfrac{\sin x}{\sqrt{x}}$

Solution

$$\lim_{x\to\infty}\frac{-1}{\sqrt{x}} \le \lim_{x\to\infty}\frac{\sin x}{\sqrt{x}} \le \lim_{x\to\infty}\frac{1}{\sqrt{x}} \qquad (2)$$

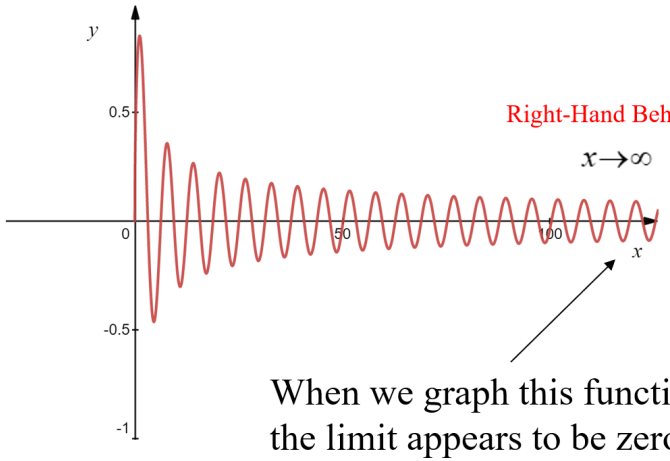
($\lim_{x\to\infty}\frac{-1}{\sqrt{x}}$ and $\lim_{x\to\infty}\frac{1}{\sqrt{x}}$ each $\to 0$)

Applying the limits in (2) we get,

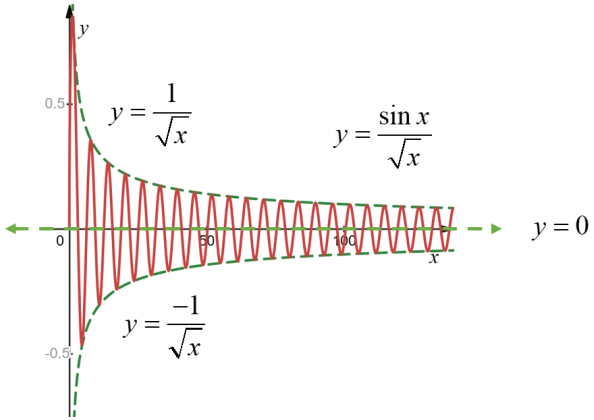
$$0 \le \lim_{x\to\infty}\frac{\sin x}{\sqrt{x}} \le 0.$$

Thus, by the Squeeze Theorem:

$\lim_{x\to\infty}\dfrac{\sin x}{\sqrt{x}} = 0$, and $y = 0$ is a horizontal asymptote.



When we graph this function, the limit appears to be zero.



4 Definition Infinite Limits at Infinity

If $f(x)$ becomes arbitrarily large as x becomes arbitrarily large, then we write

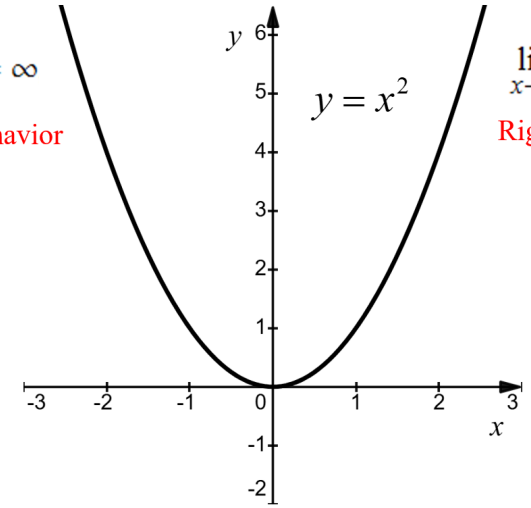
$$\lim_{x\to\infty} f(x) = \infty.$$

The limits $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$, and $\lim_{x\to-\infty} f(x) = -\infty$ are defined similarly.

For polynomials

$n > 0$ even :

$\lim_{x\to\pm\infty} x^n = \infty$



$\lim_{x\to-\infty} f(x) = \infty$

Left-Hand Behavior

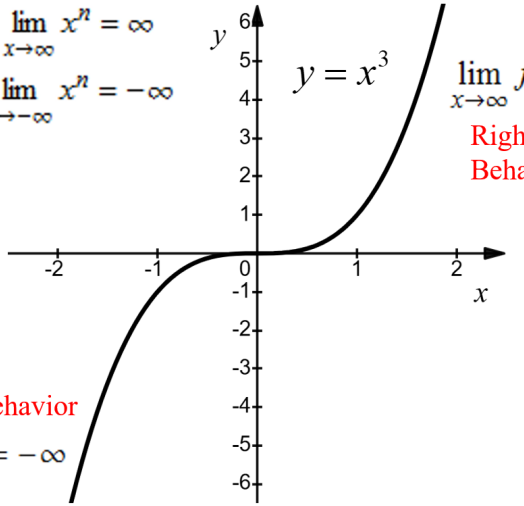
$\lim_{x\to\infty} f(x) = \infty$

Right-Hand Behavior

$n > 0$ odd :

$\lim_{x\to\infty} x^n = \infty$

$\lim_{x\to-\infty} x^n = -\infty$



$\lim_{x\to\infty} f(x) = \infty$

Right-Hand Behavior

Left-Hand Behavior

$\lim_{x\to-\infty} f(x) = -\infty$

Example Evaluate the following limit and indicate which properties of limits are used at each stage.

$$\lim_{x\to\infty}\frac{3x^2-x-2}{5x^2+4x+1}$$

Solution

$$\lim_{x\to\infty}\frac{3x^2-x-2}{5x^2+4x+1}=\lim_{x\to\infty}\frac{\dfrac{3x^2-x-2}{x^2}}{\dfrac{5x^2+4x+1}{x^2}}$$

$$\frac{+\infty}{+\infty}$$

"Indeterminate Form"

$$=\lim_{x\to\infty}\frac{3-\dfrac{1}{x}-\dfrac{2}{x^2}}{5+\dfrac{4}{x}+\dfrac{1}{x^2}}$$

$$=\frac{\lim\limits_{x\to\infty}\left(3-\dfrac{1}{x}-\dfrac{2}{x^2}\right)}{\lim\limits_{x\to\infty}\left(5+\dfrac{4}{x}+\dfrac{1}{x^2}\right)}$$

$$=\frac{\lim\limits_{x\to\infty}3-\lim\limits_{x\to\infty}\dfrac{1}{x}-2\lim\limits_{x\to\infty}\dfrac{1}{x^2}}{\lim\limits_{x\to\infty}5+4\lim\limits_{x\to\infty}\dfrac{1}{x}+\lim\limits_{x\to\infty}\dfrac{1}{x^2}}$$

($\lim_{x\to\infty}\frac{1}{x}\to 0$, $\lim_{x\to\infty}\frac{1}{x^2}\to 0$)

(by 1, 2, and 3)

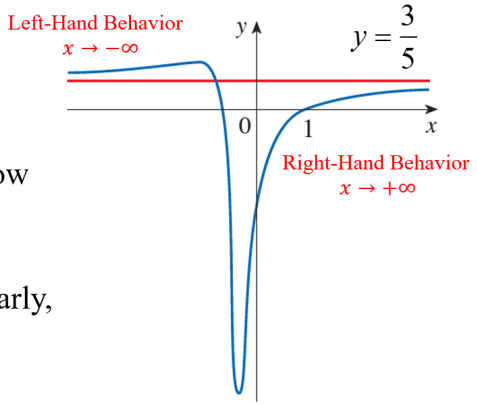
(by Theorem 5)

$$=\frac{3-0-0}{5+0+0}$$

(by 8 and Theorem 5)

$$=\frac{3}{5}$$

The results of these calculations show that the graph of the given rational function approaches the horizontal asymptote $y=\frac{3}{5}$ as $x\to+\infty$. Similarly, $y=\frac{3}{5}$ as $x\to-\infty$.



Theorem **Limits at Infinity of Powers and Polynomials**

Let n be a positive integer and let p be the polynomial

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, where $a_n \neq 0$.

1. $\lim_{x \to \pm\infty} x^n = \infty$ when n is even.

2. $\lim_{x \to \infty} x^n = \infty$ and $\lim_{x \to -\infty} x^n = -\infty$ when n is odd.

3. $\lim_{x \to \pm\infty} \frac{1}{x^n} = \lim_{x \to \pm\infty} x^{-n} = 0.$

4. $\lim_{x \to \pm\infty} p(x) = \lim_{x \to \pm\infty} a_n x^n = \pm\infty$, depending on the degree of the polynomial and the sign of the leading coefficient a_n. Dominating Term Effect (DTE)

Example Evaluate the limits as $x \to \pm\infty$ of the following function

$$q(x) = -2x^3 + 3x^2 - 12$$

Solution

Theorem Limits at Infinity of Powers and Polynomials (DTE)

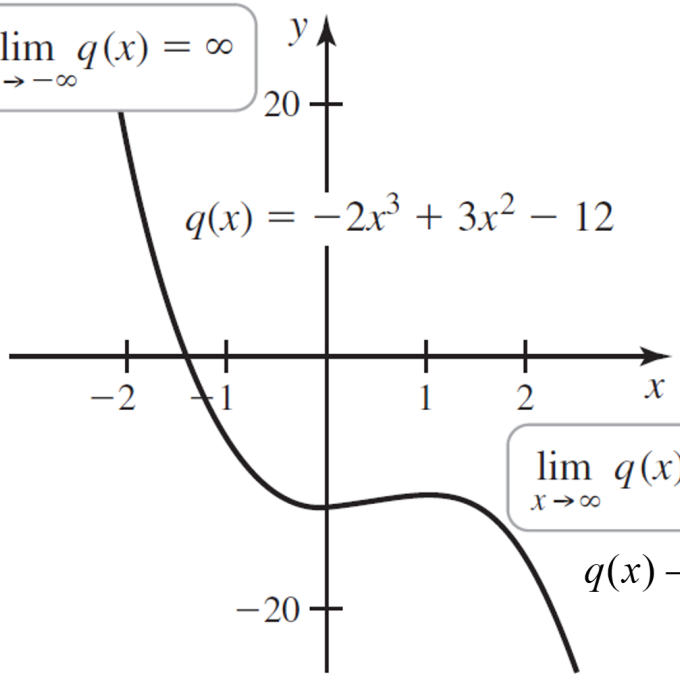
$$\lim_{x\to\infty}\left(\underbrace{-2x^3}_{\text{Dominating term}} + 3x^2 - 12\right) \overset{\text{DTE}}{=} \lim_{x\to\infty}\left(-2\underbrace{x^3}_{\to\infty}\right) = -\infty$$

$$\lim_{x\to-\infty}\left(\overbrace{-2x^3}^{\text{Dominating term}} + 3x^2 - 12\right) \overset{\text{DTE}}{=} \lim_{x\to-\infty}\left(-2\underbrace{x^3}_{\to-\infty}\right) = \infty$$

$q(x) \to \infty$ as $x \to -\infty$

Left-Hand Behavior

$\lim_{x \to -\infty} q(x) = \infty$



$\lim_{x \to \infty} q(x) = -\infty$

Right-Hand Behavior

$q(x) \to -\infty$ as $x \to \infty$

Algebraic Limits

Method I

$$\lim_{x\to\infty}\frac{3x^2-x-2}{5x^2+4x+1}=\lim_{x\to\infty}\frac{\frac{3x^2-x-2}{x^2}}{\frac{5x^2+4x+1}{x^2}}=\lim_{x\to\infty}\frac{3-\frac{1}{x}-\frac{2}{x^2}}{5+\frac{4}{x}+\frac{1}{x^2}}$$

$$\frac{\infty}{\infty}$$

$$=\frac{\lim_{x\to\infty}\left(3-\frac{1}{x}-\frac{2}{x^2}\right)}{\lim_{x\to\infty}\left(5+\frac{4}{x}+\frac{1}{x^2}\right)}$$

$$=\frac{\lim_{x\to\infty}3-\lim_{x\to\infty}\frac{1}{x}-2\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty}5+4\lim_{x\to\infty}\frac{1}{x}+\lim_{x\to\infty}\frac{1}{x^2}}$$

$$=\frac{3-0-0}{5+0+0}$$ Theorem 5

$$=\frac{3}{5}$$

Method II

$$\lim_{x\to\infty}\frac{3x^2-x-2}{5x^2+4x+1}$$

$$\overset{\text{DTE}}{=}\lim_{x\to\infty}\frac{3x^2}{5x^2}$$ Theorem Limits at Infinity of Powers and Polynomials (DTE)

$$=\lim_{x\to\infty}\frac{3}{5}$$

$$=\frac{3}{5}$$

Which method do you like better?

DTE can only be used with polynomials!

Example Find $\lim_{x\to+\infty} \dfrac{2x^5+x^4-x^2+1}{3x^2-5x+7}$ $\quad \dfrac{\infty}{\infty}$

Solution

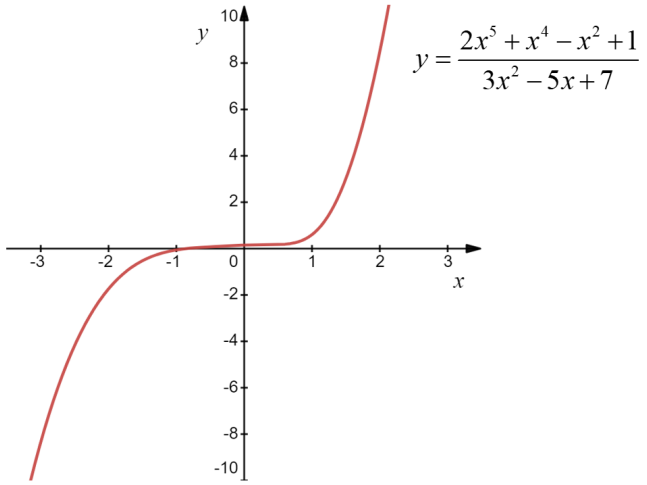
Dominating term effect for rational functions (DTE) we get,

$$\lim_{x\to+\infty} \frac{2x^5+x^4-x^2+1}{3x^2-5x+7} \overset{\text{DTE}}{=} \lim_{x\to+\infty} \frac{2x^5}{3x^2}$$

$$= \lim_{x\to+\infty} \frac{2}{3}x^3$$

$$= +\infty$$

Dominant terms in numerator and denominator



Theorem **End Behavior and Asymptotes of Rational Functions**

Suppose $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function, where

$$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_2 x^2 + a_1 x + a_0 \text{ and}$$
$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_2 x^2 + b_1 x + b_0$$

and $a_m \neq 0$ and $b_n \neq 0$.

a. If $m < n$, then $\lim_{x \to \pm\infty} f(x) = 0$, and $y = 0$ is a horizontal asymptote of f.

b. If $m = n$, then $\lim_{x \to \pm\infty} f(x) = a_m / b_n$, and $y = a_m / b_n$ is a horizontal asymptote of f.

c. If $m > n$, then $\lim_{x \to \pm\infty} f(x) = \infty$ or $-\infty$, and f has no horizontal asymptote.

d. If $m = n + 1$, then $\lim_{x \to \pm\infty} f(x) = \infty$ or $-\infty$, f has no horizontal asymptote, but f has a slant asymptote.

e. Assuming that $f(x)$ is in reduced form (p and q share no common factors), vertical asymptotes occur at the zeros of q.

Recall:

$$\sqrt{x^2} = |x|$$

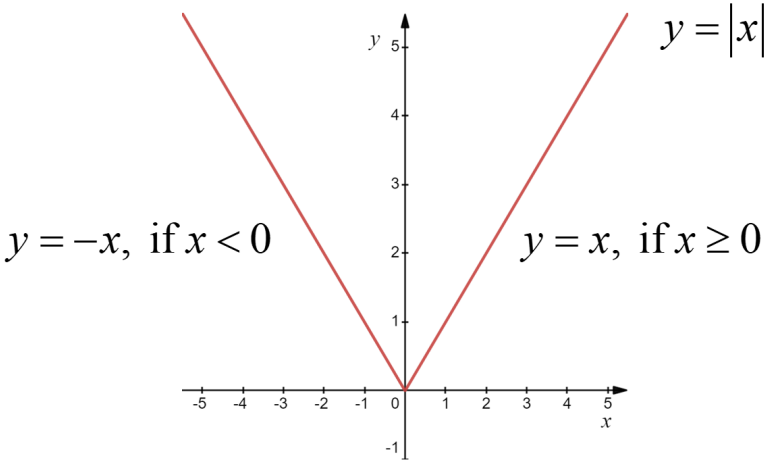
$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Piecewise Function

Example

$$|13| = 13$$

$$|-23| = -(-23) = 23$$



In general, we get the following 2 cases for roots of powers of the form $\sqrt[n]{x^{k\cdot n}}$.

Assume $x \in \mathbb{R},\ n \geq 2,\ k \geq 1,\ n$ and $k \in \mathbb{Z}^+$.

Cases 1: n is even

$$\sqrt[n]{x^{k\cdot n}} = \left|x^k\right| = \begin{cases} x^k & \text{if } x \geq 0 \\ x^k & \text{if } x < 0,\ k \text{ even} \\ -x^k & \text{if } x < 0,\ k \text{ odd} \end{cases}$$

Cases 2: n is odd

$$\sqrt[n]{x^{k\cdot n}} = x^k$$

The sign of x^k we follow the rules of a polynomial function.

Example Determine the end behavior of $f(x) = \dfrac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$.

Solution

Right-Hand Behavior

$$\lim_{x\to+\infty} \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}} = \frac{\lim\limits_{x\to+\infty}\left(10x^3 - 3x^2 + 8\right)}{\sqrt{\lim\limits_{x\to+\infty}\left(25x^6 + x^4 + 2\right)}}$$

$\dfrac{\infty}{\infty}$

$$\overset{\text{DTE}}{=} \lim_{x\to+\infty} \frac{10x^3}{\sqrt{25x^6}}$$

$$= \lim_{x\to+\infty} \frac{10x^3}{5|x^3|}$$

Note: $|x^3| = x^3$, since $x > 0$, $x \to +\infty$

$$= \lim_{x\to+\infty} \frac{10x^3}{5x^3}$$

$$= \lim_{x\to+\infty} 2$$

$$= 2$$

Left-Hand Behavior

$$\lim_{x\to-\infty} \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}} = \frac{\lim\limits_{x\to+\infty}\left(10x^3 - 3x^2 + 8\right)}{\sqrt{\lim\limits_{x\to+\infty}\left(25x^6 + x^4 + 2\right)}}$$

$\dfrac{-\infty}{\infty}$

$$\overset{\text{DTE}}{=} \lim_{x\to-\infty} \frac{10x^3}{\sqrt{25x^6}}$$

$$= \lim_{x\to-\infty} \frac{10x^3}{5|x^3|}$$

Note: $|x^3| = -x^3$, since $x < 0$, $x \to -\infty$

$$= \lim_{x\to-\infty} \frac{10x^3}{-5x^3}$$

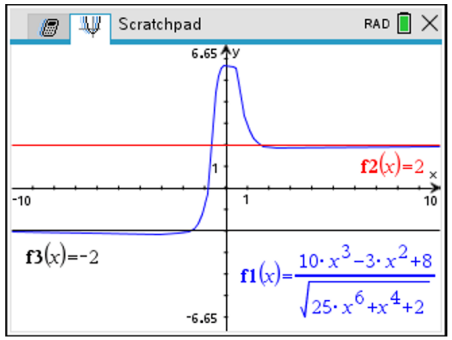
$$= \lim_{x\to-\infty} (-2)$$

$$= -2$$

Example Determine the end behavior of $f(x) = \dfrac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$.

Solution

So $y = 2$ and $y = -2$ are horizontal asymptotes.



Example Find the limit or show that it does not exist.

$$\lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)$$

Solution

$$\lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)=\lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)\left[\frac{\sqrt{4x^2+3x}-2x}{\sqrt{4x^2+3x}-2x}\right]$$

$\infty-\infty$

"Indeterminate Form"

$$=\lim_{x\to-\infty}\frac{(4x^2+3x)-(2x)^2}{\sqrt{4x^2+3x}-2x}$$

$$=\lim_{x\to-\infty}\frac{3x}{\sqrt{4x^2+3x}-2x}$$

$$\overset{\text{DTE}}{=}\lim_{x\to-\infty}\frac{3x}{\sqrt{4x^2}-2x}$$

$$=\lim_{x\to-\infty}\frac{3x}{2|x|-2x}$$

Example Find the limit or show that it does not exist.

$$\lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right) \qquad \infty-\infty$$

Solution

$$\lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)=\lim_{x\to-\infty}\frac{3x}{2|x|-2x}$$

Note: $|x|=-x$, since $x<0$ $x\to-\infty$

$$=\lim_{x\to-\infty}\frac{3x}{-2x-2x}$$

$$=\lim_{x\to-\infty}\frac{3\cancel{x}}{-4\cancel{x}}$$

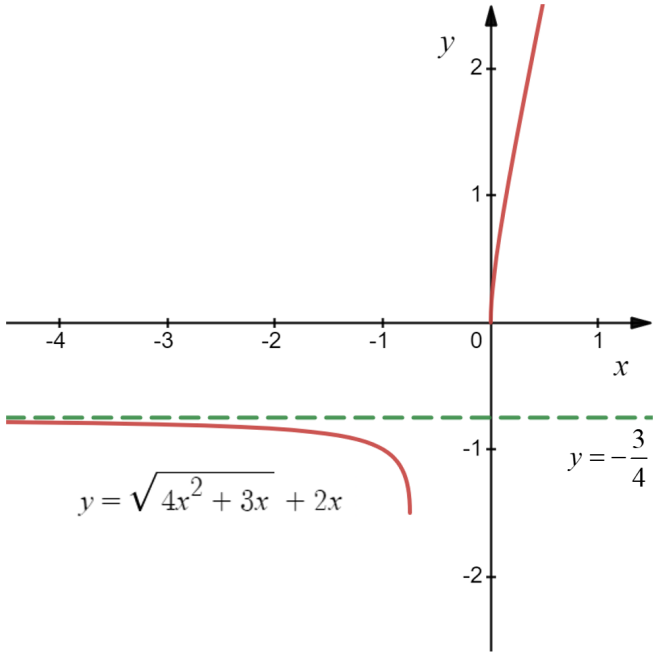
$$=\lim_{x\to-\infty}\left(-\frac{3}{4}\right)$$

$$=-\frac{3}{4}$$

So $y=-\frac{3}{4}$ is horizontal asymptotes.

$$\lim_{x\to+\infty}\left(\sqrt{4x^2+3x}+2x\right)=+\infty+\infty$$

$$=+\infty$$



End behavior of transcendental functions

Example Determine the end behavior of the following transcendental functions.

a. $f(x) = e^x$ and $f(x) = e^{-x}$

b. $g(x) = \ln x$

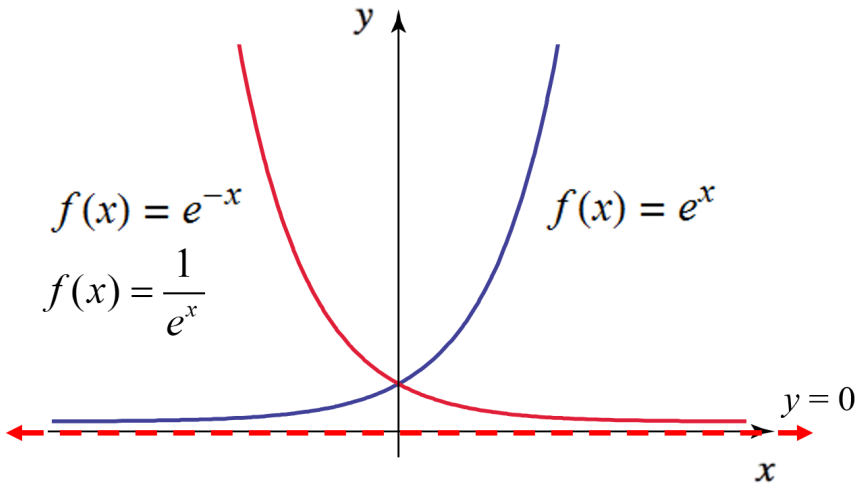
c. $f(x) = \cos x$

Solution

a. $\lim_{x \to \infty} e^x = \infty$ and $\lim_{x \to -\infty} e^x = 0$

$\lim_{x \to \infty} e^{-x} = 0$ and $\lim_{x \to -\infty} e^{-x} = \infty$

Hence, $y = 0$ is a horizontal asymptote.



b. The domain of $\ln x$ is $\{x : x > 0\}$, so we evaluate $\lim_{x \to 0^+} \ln x$ and $\lim_{x \to \infty} \ln x$ to determine end behavior. For the first limit, recall that $\ln x$ is the inverse of e^x

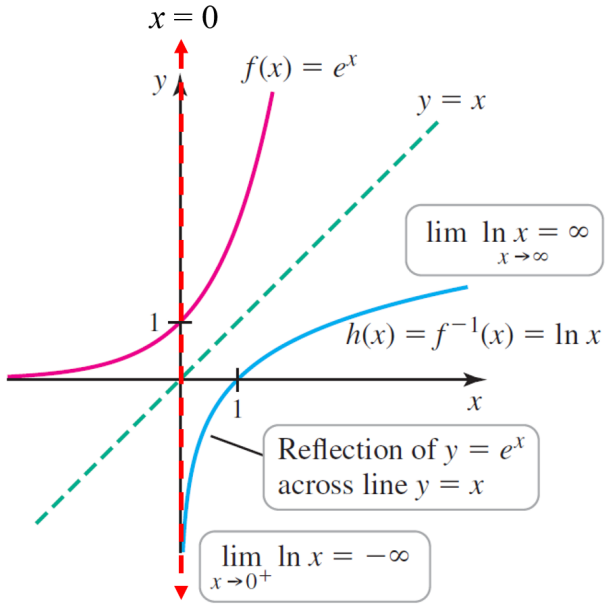
$$\lim_{x \to \infty} \ln x = \infty$$

There is no left-hand behavior because of the domain restriction.

Hence, there is no horizontal asymptote.

$$\lim_{x \to 0^+} \ln x = -\infty$$

So, $x = 0$ is a vertical asymptote.



c. The cosine function oscillates between -1 and 1 as x approaches infinity (Figure 2.45). Therefore, $\lim_{x\to\infty} \cos x$ does not exist.

For the same reason, $\lim_{x\to-\infty} \cos x$ does not exist.

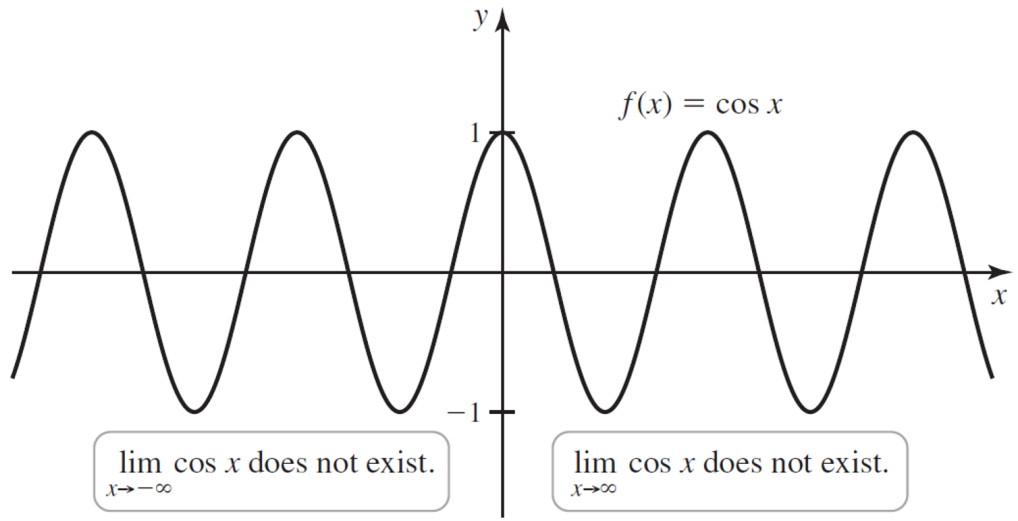


Figure 2.45

Similarly, $\lim_{x\to\pm\infty} \sin x$, $\lim_{x\to\pm\infty} \tan x$, $\lim_{x\to\pm\infty} \sec x$, $\lim_{x\to\pm\infty} \csc x$, $\lim_{x\to\pm\infty} \cot x$, do not exist.

In summary,

Theorem **End Behavior of e^x, e^{-x}, and $\ln x$**

The end behavior for e^x and e^{-x} on $(-\infty, \infty)$ and $\ln x$ on $(0, \infty)$ is given by the following limits:

$$\lim_{x\to\infty} e^x = \infty \quad \text{and} \quad \lim_{x\to-\infty} e^x = 0,$$

$$\lim_{x\to\infty} e^{-x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} e^{-x} = \infty,$$

$$\lim_{x\to0^+} \ln x = -\infty \quad \text{and} \quad \lim_{x\to\infty} \ln x = \infty.$$

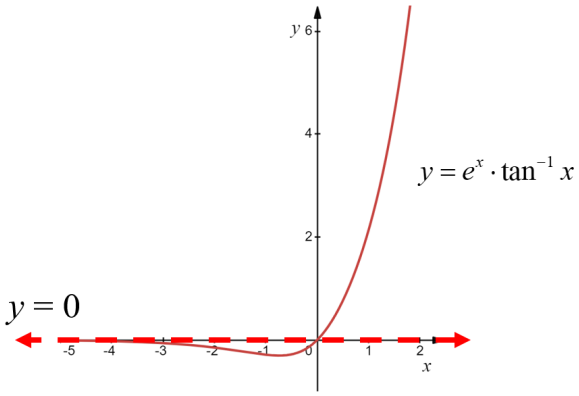
Example Find $\lim_{x\to-\infty} \left(e^x \cdot \tan^{-1} x\right)$

Solution

$$\lim_{x\to-\infty} \left(e^x \cdot \tan^{-1} x\right) = \lim_{x\to-\infty} e^x \cdot \lim_{x\to-\infty} \tan^{-1} x$$

$$= 0 \cdot \left(-\frac{\pi}{2}\right)$$

$$= 0$$



Example Compute the following limits.

(a) $\lim_{x\to-\infty} \dfrac{\sqrt[3]{x^6+x}}{\sqrt{4x^2+3}}$ (b) $\lim_{x\to\infty} \dfrac{t-t\sqrt{t}}{2t^{3/2}+3t-5}$ (c) $\lim_{x\to\pm\infty} \dfrac{3-5e^x}{1+2e^x}$

Solution

(a) $\lim_{x\to-\infty} \dfrac{\sqrt[3]{x^6+x}}{\sqrt{4x^2+3}} = \dfrac{\sqrt[3]{\lim_{x\to-\infty}\left(x^6+x\right)}}{\sqrt{\lim_{x\to-\infty}\left(4x^2+3\right)}}$

$\dfrac{\infty}{\infty}$

$\overset{\text{DTE}}{=} \lim_{x\to-\infty} \dfrac{\sqrt[3]{x^6}}{\sqrt{4x^2}}$

$= \lim_{x\to-\infty} \dfrac{x^2}{2|x|}$

$= \lim_{x\to-\infty} \dfrac{x^2}{-2x}$

$= -\lim_{x\to-\infty} \dfrac{x}{2}$

Note:

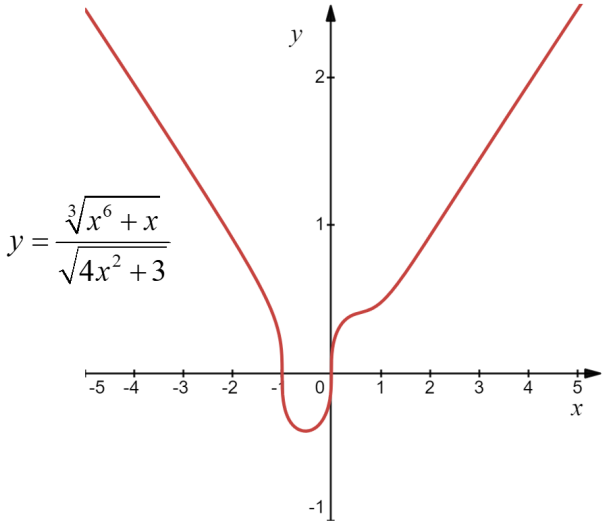
$|x| = -x,$

since $x < 0$

$x \to -\infty$

$= -(-\infty)$

$= \infty$



Example Compute the following limits.

(a) $\lim_{x\to-\infty}\dfrac{\sqrt[3]{x^6+x}}{\sqrt{4x^2+3}}$

(b) $\lim_{t\to\infty}\dfrac{t-t\sqrt{t}}{2t^{3/2}+3t-5}$

(c) $\lim_{x\to\pm\infty}\dfrac{3-5e^x}{1+2e^x}$

Solution

(b) $\lim_{t\to\infty}\dfrac{t-t\sqrt{t}}{2t^{3/2}+3t-5}=\lim_{t\to\infty}\dfrac{t-t^{3/2}}{2t^{3/2}+3t-5}\cdot\dfrac{\frac{1}{t^{3/2}}}{\frac{1}{t^{3/2}}}$

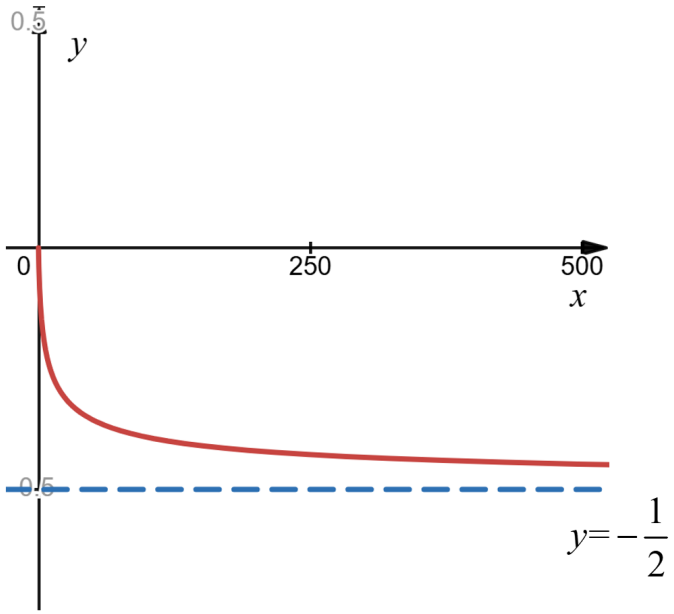
$\dfrac{-\infty}{\infty}$

$$=\lim_{t\to\infty}\frac{\frac{1}{t^{1/2}}-1}{2+\frac{3}{t^{1/2}}-\frac{5}{t^{3/2}}}$$

($\frac{1}{t^{1/2}}\to 0$, $\frac{3}{t^{1/2}}\to 0$, $\frac{5}{t^{3/2}}\to 0$)

$$=\lim_{t\to\infty}\left(\frac{-1}{2}\right)$$

$$=-\frac{1}{2}$$



Example Compute the following limits.

(a) $\lim_{x\to-\infty}\dfrac{\sqrt[3]{x^6+x}}{\sqrt{4x^2+3}}$

(b) $\lim_{x\to\infty}\dfrac{t-t\sqrt{t}}{2t^{3/2}+3t-5}$

(c) $\lim_{x\to\pm\infty}\dfrac{3-5e^x}{1+2e^x}$

Solution

(c) $\lim_{x\to\infty}\dfrac{3-5e^x}{1+2e^x}=\lim_{x\to\infty}\dfrac{3-5e^x}{1+2e^x}\cdot\dfrac{\frac{1}{e^x}}{\frac{1}{e^x}}$

$\dfrac{-\infty}{\infty}$

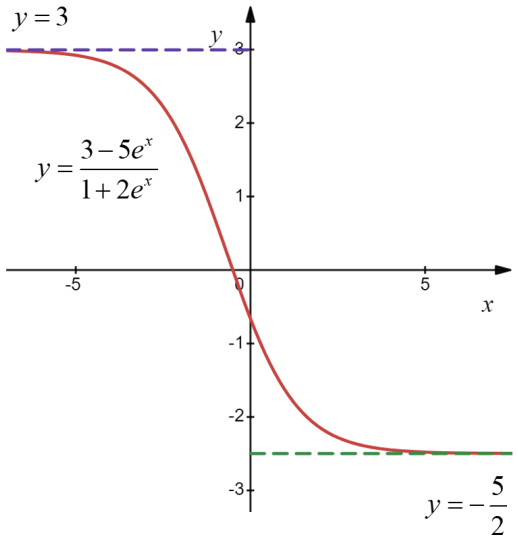
$=\lim_{x\to\infty}\dfrac{\frac{3}{e^x}-5}{\frac{1}{e^x}+2}$ (numerator $\to 0$, denominator $\to 0$)

$=\lim_{x\to\infty}\left(\dfrac{-5}{2}\right)$

$=-\dfrac{5}{2}$

$\lim_{x\to-\infty}\dfrac{3-5e^x}{1+2e^x}=\lim_{x\to-\infty}\left(\dfrac{3}{1}\right)$ (numerator $\to 0$, denominator $\to 0$)

$=3$



In summary,

Indeterminant Forms

The following expressions have no meaning and are thus called indeterminant forms. We will study these along with other indeterminate forms further using L'Hospital's Theorem in Chapter 4.4.

(i) $\frac{0}{0}$ (iii) $\infty - \infty$

(ii) $\frac{\pm\infty}{\pm\infty}$ (iv) $\pm\infty \cdot 0$

More to come!!

Meaningful Expressions Involving 0 and ∞

Let k be a fixed nonzero real number.

(i) $\infty + \infty = \infty$

(ii) $\frac{\infty}{k} = \pm\infty$ depending on the sign of k

(iii) $k \cdot \infty = \pm\infty$ depending on the sign of k

(iv) $\infty \cdot \infty = \infty$

(v) $\frac{0}{\pm\infty} = 0$

Bring on the homework!!