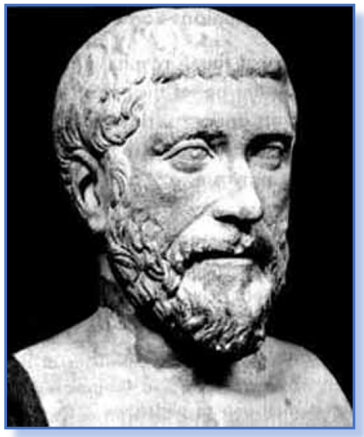
2.3 Calculating Limits Using the Limit Laws



Pythagoras of Samos
580 – 500 B.C.

Pythagoras was a Greek philosopher who made important developments in mathematics, astronomy, and the theory of music. The theorem now known as Pythagoras's theorem was known to the Babylonians 1000 years earlier, but he may have been the first to prove it.

In Section 2.2 we used calculators and graphs to guess the values of limits, but such methods don't always lead to the correct answers. In this section we use the following theorems of limits, called the *Limit Laws* (theorems), to calculate limits. Sometimes called analytic limits, or symbolic limits.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$
4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

These five laws can be stated verbally as follows:

Sum Law **1.** The limit of a sum is the sum of the limits.

Difference Law **2.** The limit of a difference is the difference of the limits.

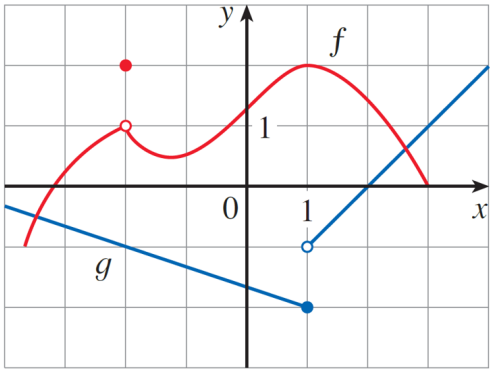
Constant Multiple Law **3.** The limit of a constant times a function is the constant times the limit of the function.

Product Law **4.** The limit of a product is the product of the limits.

Quotient Law **5.** The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

These laws are also true for directional/one-side limits.

Example Use the Limit Laws and the graphs of f and g in the figure 1 to evaluate the following limits, if they exist.



(a) $\lim_{x \to -2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 1} [f(x)g(x)]$ (c) $\lim_{x \to 2} \dfrac{f(x)}{g(x)}$

Solution

(a) From the graphs of f and g we see that

$$\lim_{x \to -2} f(x) = 1 \quad \text{and} \quad \lim_{x \to -2} g(x) = -1$$

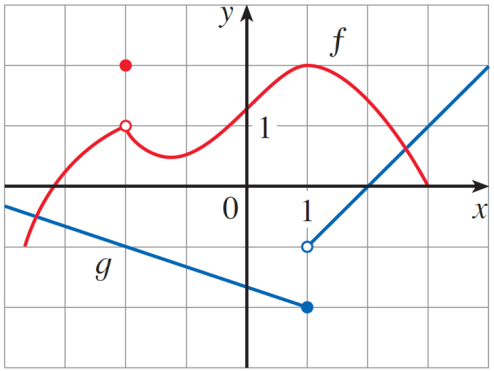
Therefore, we have

$$\lim_{x \to -2} [f(x) + 5g(x)] = \lim_{x \to -2} f(x) + \lim_{x \to -2} [5g(x)] \quad \text{(by Law 1)}$$

$$= \lim_{x \to -2} f(x) + 5 \lim_{x \to -2} g(x) \quad \text{(by Law 3)}$$

$$= 1 + 5(-1) = -4$$

Example Use the Limit Laws and the graphs of f and g in the figure 1 to evaluate the following limits, if they exist.



(a) $\lim_{x \to -2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 1} [f(x)g(x)]$ (c) $\lim_{x \to 2} \dfrac{f(x)}{g(x)}$

Solution

(b) We see that $\lim_{x \to 1} f(x) = 2$. But $\lim_{x \to 1} g(x)$ does not exist because the left and right limits are different:

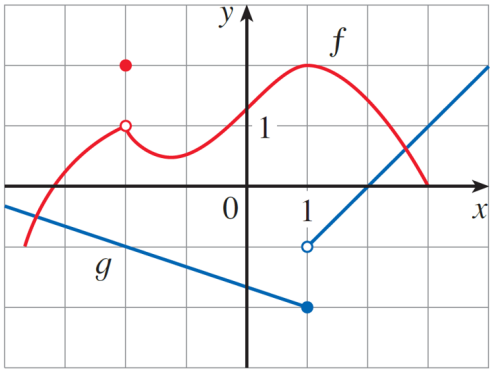
$$\lim_{x \to 1^-} g(x) = -2 \qquad \lim_{x \to 1^+} g(x) = -1$$

So we can't use Law 4 for the desired limit. But we *can* use Law 4 for the one-sided limits:

$$\lim_{x \to 1^-} [f(x)g(x)] = 2 \cdot (-2) = -4 \qquad \lim_{x \to 1^+} [f(x)g(x)] = 2 \cdot (-1) = -2$$

The left and right limits aren't equal, so $\lim_{x \to 1} [f(x)g(x)]$ does not exist.

Example Use the Limit Laws and the graphs of f and g in the figure 1 to evaluate the following limits, if they exist.



(a) $\lim_{x \to -2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 1} [f(x)g(x)]$ (c) $\lim_{x \to 2} \frac{f(x)}{g(x)}$

Solution

(c) The graphs show that

$$\lim_{x \to 2} f(x) \approx 1.4 \quad \text{and} \quad \lim_{x \to 2} g(x) = 0$$

Because the limit of the denominator is 0, we can't use Law 5. The given limit does not exist because the denominator approaches 0 while the numerator approaches a nonzero number. □

If we use the Product Law repeatedly with $g(x) = f(x)$, we obtain the following law.

Power Law

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

A similar property holds for roots:

Root Law

7. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim_{x \to a} f(x) > 0$.]

In applying these seven limit laws, we need to use two special limits:

8. $\lim_{x \to a} c = c$

9. $\lim_{x \to a} x = a$

If we now put $f(x) = x$ in Law 6 and use Law 9, we get a useful special limit for power functions.

10. $\lim_{x \to a} x^n = a^n$ where n is a positive integer

If we put $f(x) = x$ in Law 7 and use Law 9, we get a similar special limit for roots.

11. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Example Evaluate $\lim_{x\to 2} \dfrac{\sqrt{2x^3+9}+3x-1}{4x+1}$.

Solution

$$\lim_{x\to 2} \frac{\sqrt{2x^3+9}+3x-1}{4x+1} = \frac{\lim_{x\to 2}\left(\sqrt{2x^3+9}+3x-1\right)}{\lim_{x\to 2}(4x+1)} \qquad \text{Limit Law 5}$$

$$= \frac{\sqrt{\lim_{x\to 2}(2x^3+9)}+\lim_{x\to 2}(3x-1)}{\lim_{x\to 2}(4x+1)} \qquad \text{Limit Laws 1 and 7}$$

$$= \frac{\sqrt{(2(2)^3+9)}+(3(2)-1)}{(4(2)+1)} \qquad \text{Direct Substitution Property}$$

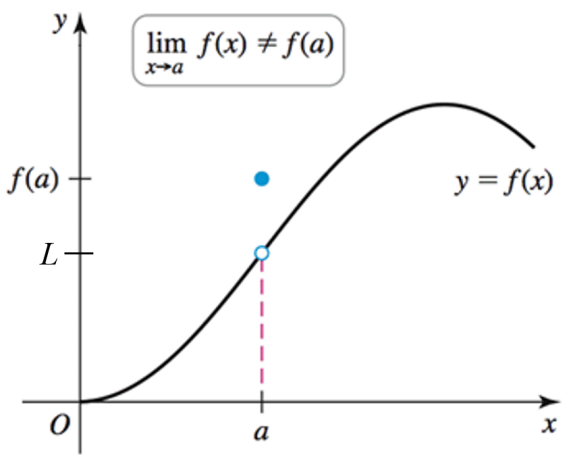
$$= \frac{\sqrt{25}+5}{9} = \frac{10}{9}. \qquad \text{Simplifying}$$

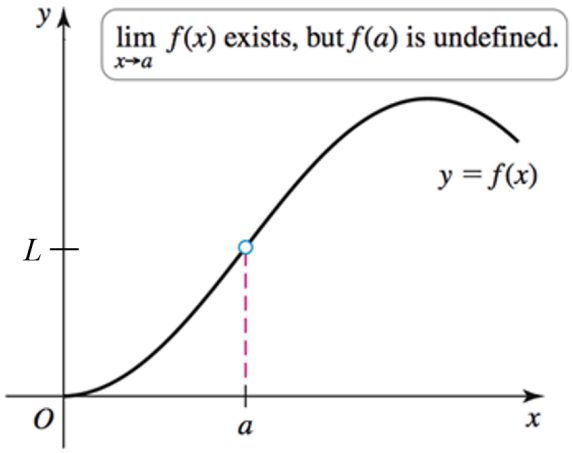
The function in the Example above is made up with polynomials, the use of the Limit Laws proves that the **Direct Substitution Property** always works for such functions. We state this fact as follows.

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

Functions that have the **Direct Substitution Property** are called *continuous at a* and will be studied in Section 2.5. However, not all limits can be evaluated initially by direct substitution, as the following examples show.





Example Evaluate the limit, if it exists. Describe what functional behavior occurs at the limit point.

(a) $\lim_{x\to -5} \dfrac{2x^2+9x-5}{x^2-25}$ (b) $\lim_{h\to 0} \dfrac{(h-3)^2-9}{h}$ (c) $\lim_{h\to 0} \dfrac{\sqrt{9+h}-3}{h}$

Solution

(a) $\lim_{x\to -5} \dfrac{2x^2+9x-5}{x^2-25} = \lim_{x\to -5} \dfrac{(2x-1)\cancel{(x+5)}}{(x-5)\cancel{(x+5)}}$

Direct Subtitution gives us $\dfrac{0}{0}$.

What do we do?

Clever math!

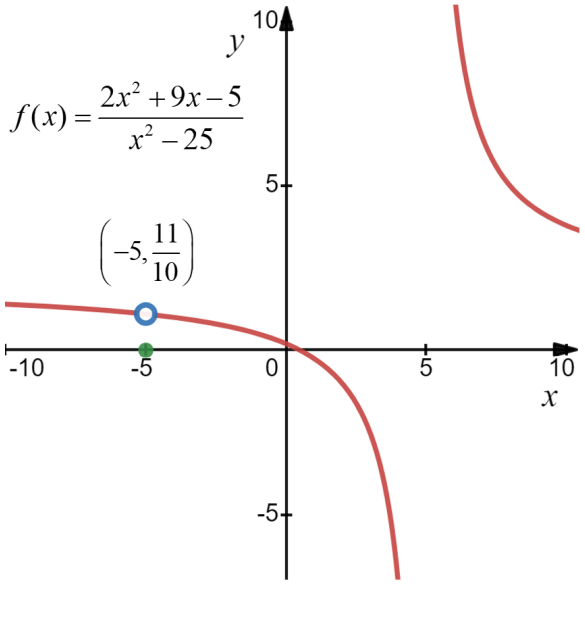
$= \lim_{x\to -5} \dfrac{2x-1}{x-5}$

$= \dfrac{2(-5)-1}{-5-5}$ Direct Substitution

$= \dfrac{-11}{-10}$

$= \dfrac{11}{10}$

There is a hole in the graph at $\left(-5, \dfrac{11}{10}\right)$.



Example Evaluate the limit, if it exists. Describe what functional behavior occurs at the limit point.

(a) $\lim\limits_{x\to -5} \dfrac{2x^2+9x-5}{x^2-25}$ (b) $\lim\limits_{h\to 0} \dfrac{(h-3)^2-9}{h}$ (c) $\lim\limits_{h\to 0} \dfrac{\sqrt{9+h}-3}{h}$

Solution

(b) $\lim\limits_{h\to 0} \dfrac{(h-3)^2-9}{h} = \lim\limits_{h\to 0} \dfrac{h^2-6h+9-9}{h}$

Direct Subtitution gives us $\dfrac{0}{0}$.

$$= \lim_{h\to 0} \frac{h^2-6h}{h}$$

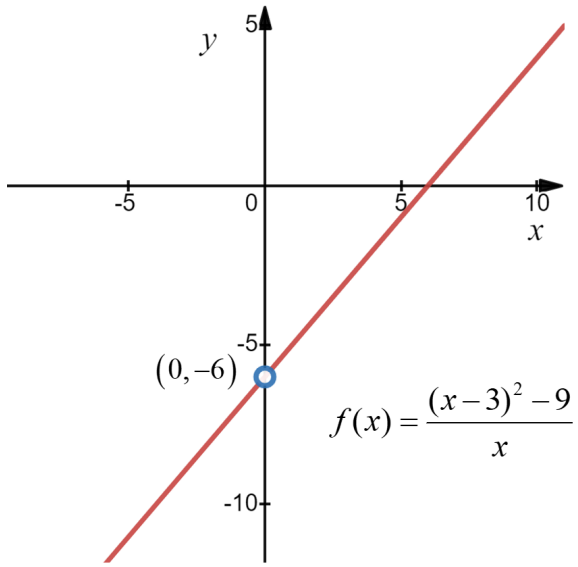
$$= \lim_{h\to 0} \frac{\cancel{h}(h-6)}{\cancel{h}}$$

$$= \lim_{h\to 0} (h-6)$$

$$= 0-6 \qquad \text{Direct Substitution}$$

$$= -6$$

There is a hole in the graph at $(0,-6)$.



Example Evaluate the limit, if it exists. Describe what functional behavior occurs at the limit point.

(a) $\lim\limits_{x\to -5} \dfrac{2x^2+9x-5}{x^2-25}$ (b) $\lim\limits_{h\to 0} \dfrac{(h-3)^2-9}{h}$ (c) $\lim\limits_{h\to 0} \dfrac{\sqrt{9+h}-3}{h}$

Solution

(b) $\lim\limits_{h\to 0} \dfrac{(h-3)^2-9}{h} = \lim\limits_{h\to 0} \dfrac{h^2-6h+9-9}{h}$

Direct Subtitution gives us $\dfrac{0}{0}$.

$$= \lim_{h\to 0} \frac{h^2-6h}{h}$$

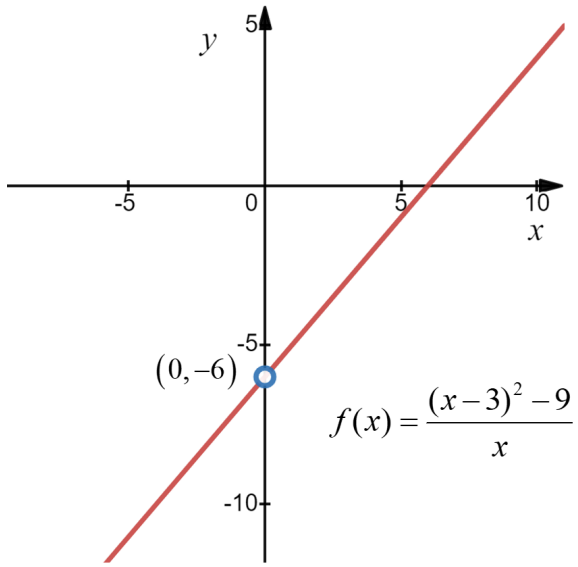
$$= \lim_{h\to 0} \frac{\cancel{h}(h-6)}{\cancel{h}}$$

$$= \lim_{h\to 0} (h-6)$$

$$= 0-6 \qquad \text{Direct Substitution}$$

$$= -6$$

There is a hole in the graph at $(0,-6)$.



Using One-Sided Limits

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem is a reminder of what we discovered in Section 2.2. It says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

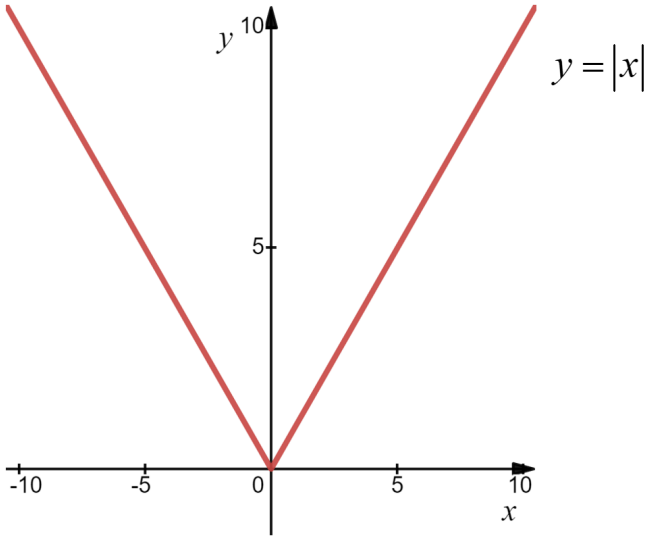
1 Theorem $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

Example Determine the limit $\lim_{x \to 0} \dfrac{|x|}{x}$.

Solution Recall that

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$



Example Determine the limit $\lim_{x \to 0} \frac{|x|}{x}$. Direct Subtitution gives us $\frac{0}{0}$.

Solution Using the facts that $|x| = x$ when $x > 0$ and $|x| = -x$ when $x < 0$, we have

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} (-1) = -1$$

Since the right- and left-hand limits are different, it follows from Theorem 1 that $\lim_{x \to 0} |x|/x$ does not exist. The graph of the function $f(x) = |x|/x$ is shown in Figure 4 and supports the one-sided limits that we found.

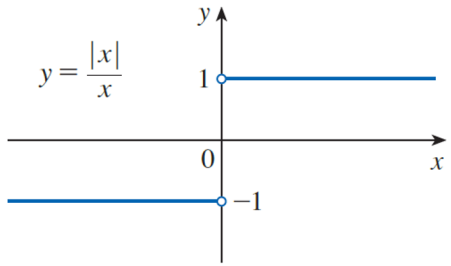


FIGURE 4

Example

(a) Estimate the value of

$$\lim_{x \to 0} \frac{x}{\sqrt{1 + 3x} - 1}$$ Direct Subtitution gives us $\frac{0}{0}$.

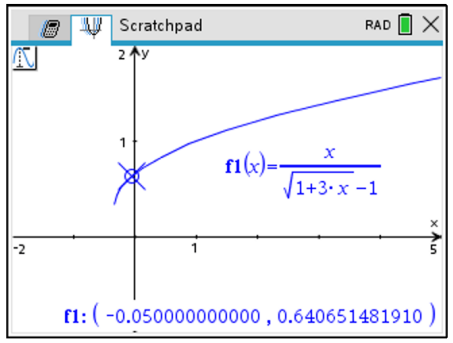
by graphing the function $f(x) = x/\left(\sqrt{1 + 3x} - 1\right)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

Solution

(a)

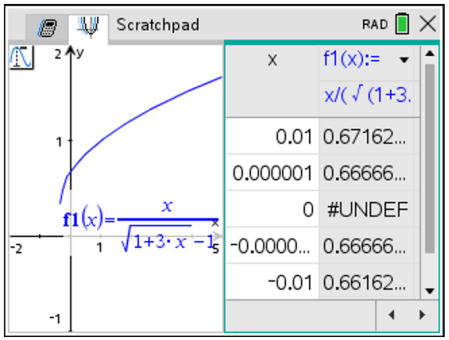


$$\lim_{x \to 0} \frac{x}{\sqrt{1 + 3x} - 1} \approx 0.665$$

$$\lim_{x \to 0} \frac{x}{\sqrt{1 + 3x} - 1} \approx 0.667$$

What's your guess for the exact value of the limit?

(b)



Example

(a) Estimate the value of

$$\lim_{x \to 0} \frac{x}{\sqrt{1 + 3x} - 1}$$

by graphing the function $f(x) = x/\left(\sqrt{1 + 3x} - 1\right)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

Solution

(c) $$\lim_{x \to 0} \frac{x}{\sqrt{1+3x} - 1} = \lim_{x \to 0} \left(\frac{x}{\sqrt{1+3x} - 1} \cdot \frac{\sqrt{1+3x} + 1}{\sqrt{1+3x} + 1} \right)$$

Direct Subtitution gives us $\frac{0}{0}$.

$$= \lim_{x \to 0} \frac{x\left(\sqrt{1+3x} + 1\right)}{(1+3x) - 1}$$

$$= \lim_{x \to 0} \frac{\cancel{x}\left(\sqrt{1+3x} + 1\right)}{3\cancel{x}}$$

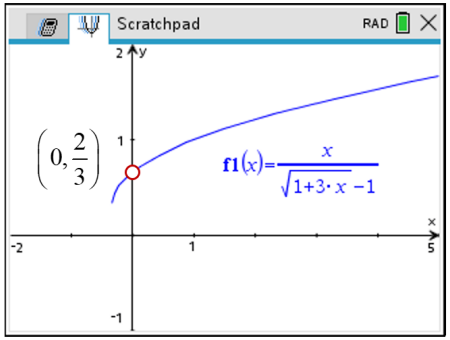
$$= \frac{1}{3} \lim_{x \to 0} \left(\sqrt{1+3x} + 1\right)$$

$$= \frac{1}{3} \left[\sqrt{\lim_{x \to 0}(1+3x)} + \lim_{x \to 0} 1 \right]$$

$$= \frac{1}{3}\left(\sqrt{1+3 \cdot 0} + 1\right)$$

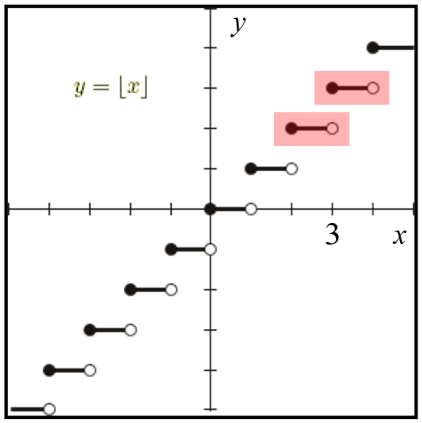
$$= \frac{1}{3}(1+1)$$

$$= \frac{2}{3}$$



Greatest Integer Function

Floor Function



$$y = \lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\}$$

Also known as the greatest integer function $y = [\![x]\!]$.

$[\![4]\!] = 4 \quad [\![4.8]\!] = 4 \quad [\![\pi]\!] = 3 \quad [\![-2.6]\!] = -3$

Example Show that $\lim_{x \to 3} [\![x]\!]$ does not exist.

Solution

$\lim_{x \to 3^+} [\![x]\!] = \lim_{x \to 3^+} 3 = 3 \qquad [\![x]\!] = 3 \text{ for } 3 \leqslant x < 4$

$\lim_{x \to 3^-} [\![x]\!] = \lim_{x \to 3^-} 2 = 2 \qquad [\![x]\!] = 2 \text{ for } 2 \leqslant x < 3$

Because these one-sided limits are not equal,

$\lim_{x \to 3} [\![x]\!]$ does not exist by Theorem 1.

Example Find the limit, if it exists. If the limit does not exist, explain why.

(a) $\lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

(b) Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x - 2)^2 & \text{if } x \geqslant 1 \end{cases}$$

(a) Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$.
(b) Does $\lim_{x \to 1} f(x)$ exist?
(c) Sketch the graph of f.

(c) $\lim_{h \to 0} \dfrac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}$

Solution

(a) $\lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right) = \lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{-x} \right)$ Since $|x| = -x$ for $x < 0$

$$= \lim_{x \to 0^-} \frac{2}{x}$$

which does not exist since the denominator approaches 0 and the numerator does not. We will talk more about this later.

Example Find the limit, if it exists. If the limit does not exist, explain why.

(b) Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x-2)^2 & \text{if } x \geqslant 1 \end{cases}$$

(a) Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$.
(b) Does $\lim_{x \to 1} f(x)$ exist?
(c) Sketch the graph of f.

Solution

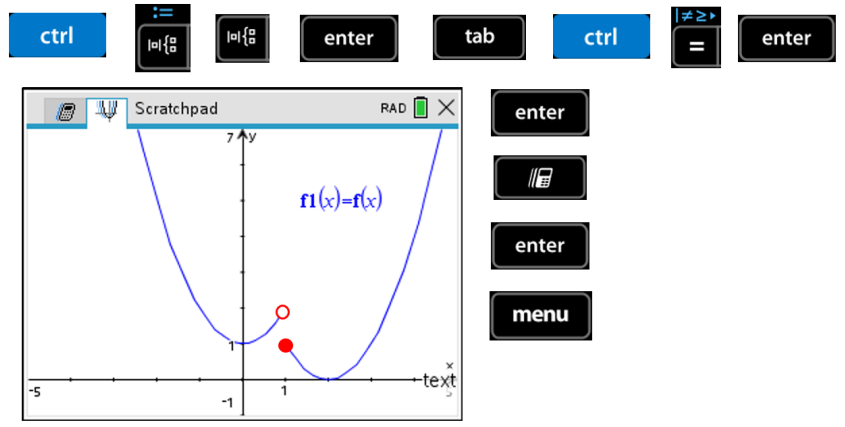
(a) Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$.

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (x^2 + 1) \\ &= 1^2 + 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1+} (x-2)^2 \\ &= (-1)^2 \\ &= 1 \end{aligned}$$

(b) Since the right-hand and left-hand limits of f at $x = 1$ are not equal, $\lim_{x \to 1} f(x)$ does not exist.

(c)



Example Find the limit, if it exists. If the limit does not exist, explain why.

(c) $\lim_{h\to 0} \dfrac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}$ Direct Subtitution gives us $\dfrac{0}{0}$.

Solution

$$\lim_{h\to 0} \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h} = \lim_{h\to 0} \frac{\dfrac{x^2-(x+h)^2}{(x+h)^2x^2}}{h}$$

$$= \lim_{h\to 0} \frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2}$$

$$= \lim_{h\to 0} \frac{x^2-x^2-2xh-h^2}{hx^2(x+h)^2}$$

$$= \lim_{h\to 0} \frac{-2xh-h^2}{hx^2(x+h)^2}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}(2x+h)}{\cancel{h}x^2(x+h)^2}$$

$$\frac{\frac{a}{b}}{c} = \frac{a}{b} \div c$$

$$= \frac{a}{b} \cdot \frac{1}{c}$$

$$= \frac{a}{bc}$$

$$= \lim_{h\to 0} \frac{-(2x+h)}{x^2(x+h)^2}$$

$$= \frac{-(2x+0)}{x^2(x+0)^2}$$

$$= \frac{-2x}{x^2 \cdot x^2}$$

$$= \frac{-2x}{x^4}$$

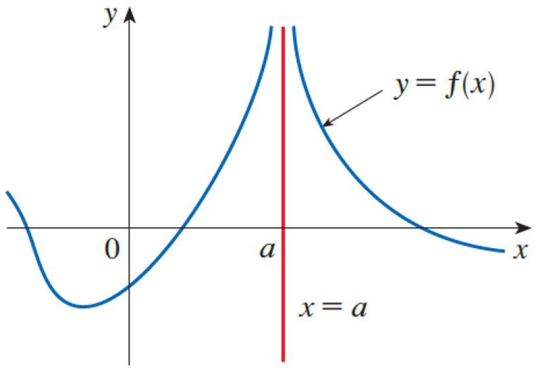
$$= -\frac{2}{x^3}$$

Recall the following definitions from 2.2

4 Intuitive Definition of an Infinite Limit Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

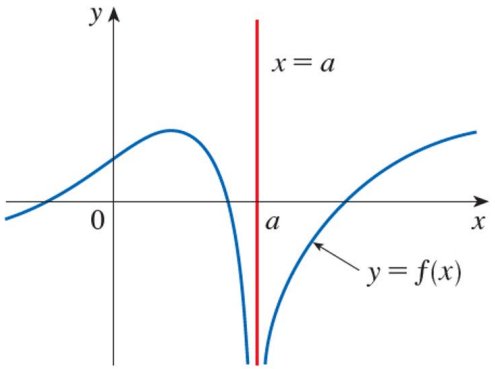


$$\lim_{x \to a} f(x) = \infty$$

5 Definition Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

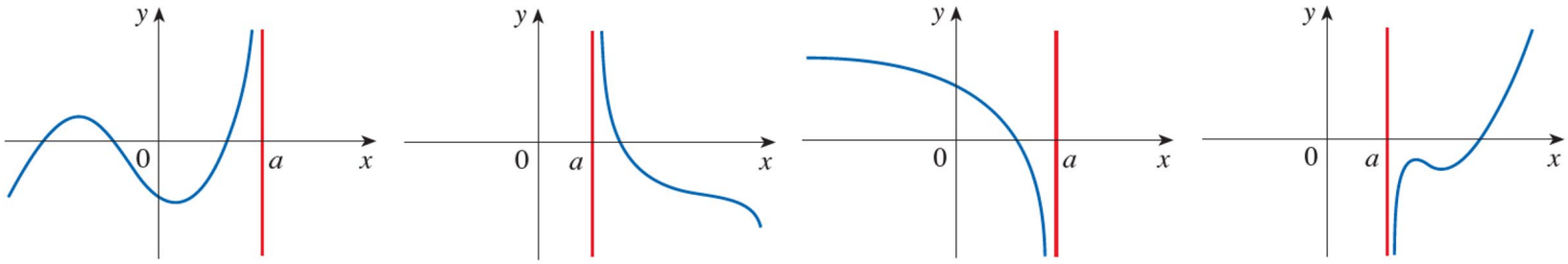


$$\lim_{x \to a} f(x) = -\infty$$

Similar definitions can be given for the one-sided infinite limits

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$



(a) $\lim_{x \to a^-} f(x) = \infty$ (b) $\lim_{x \to a^+} f(x) = \infty$ (c) $\lim_{x \to a^-} f(x) = -\infty$ (d) $\lim_{x \to a^+} f(x) = -\infty$

6 Definition The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Finding Infinite Limits Analytically

Many infinite limits are analyzed using a simple arithmetic property: The fraction $\frac{a}{b}$ grows arbitrarily large in magnitude if b approaches 0 while a remains nonzero and relatively constant. For example, consider the fraction $\frac{5+x}{x}$ for values of x approaching 0 from the right (Table 2.8).

Table 2.8

x	$\frac{5+x}{x}$
0.01	$\frac{5.01}{0.01} = 501$
0.001	$\frac{5.001}{0.001} = 5001$
0.0001	$\frac{5.0001}{0.0001} = 50{,}001$
$\downarrow$	$\downarrow$
0^+	∞

Numerator going to a nonzero constant.

Denominator going to zero.

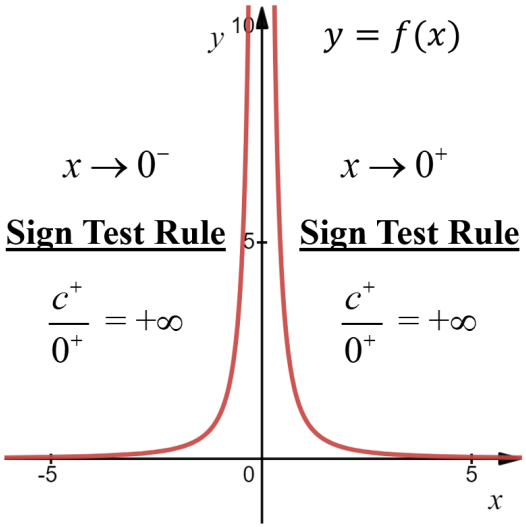
Sign Test Rule

$$\frac{c^+}{0^+} = +\infty \qquad \frac{c^+}{0^-} = -\infty$$

$$\frac{c^-}{0^+} = -\infty \qquad \frac{c^-}{0^-} = +\infty$$

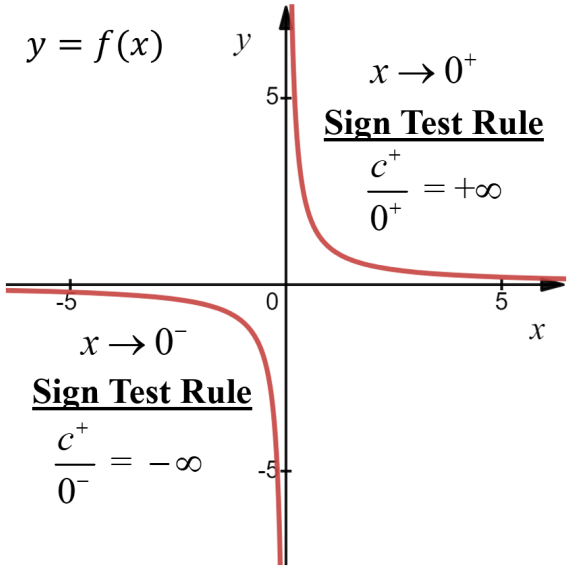
Limit Theorems at Infinity

13. $\lim_{x\to 0^-} \frac{c}{x^n} = +\infty$ and $\lim_{x\to 0^+} \frac{c}{x^n} = +\infty$ where $n \geq 2$ and n is even, c is a positive constant.



Limit Theorems at Infinity

14. $\lim\limits_{x\to 0^-} \dfrac{c}{x^n} = -\infty$ and $\lim\limits_{x\to 0^+} \dfrac{c}{x^n} = +\infty$ where $n \geq 1$ and n is odd, c is a positive constant.



Example Evaluate the limits below. What does each this limit tell us about the functional behavior of $f(x)$?

(a) $\lim\limits_{x\to -2^+} \dfrac{x^3-5x^2+6x}{x^4-4x^2}$ (b) $\lim\limits_{x\to 0^+} \dfrac{x^3-5x^2+6x}{x^4-4x^2}$ (c) $\lim\limits_{x\to 2} \dfrac{x^3-5x^2+6x}{x^4-4x^2}$

Solution

(a) $\lim\limits_{x\to -2^+} \dfrac{x^3-5x^2+6x}{x^4-4x^2} = \lim\limits_{x\to -2^+} \dfrac{x(x^2-5x+6)}{x^2(x^2-4)}$

$= \lim\limits_{x\to -2^+} \dfrac{\cancel{x(x-2)}(x-3)}{x^{\cancel{2}}\cancel{(x-2)}(x+2)}$ Direct Subtitution gives us $\dfrac{c}{0}$, $c \neq 0$.

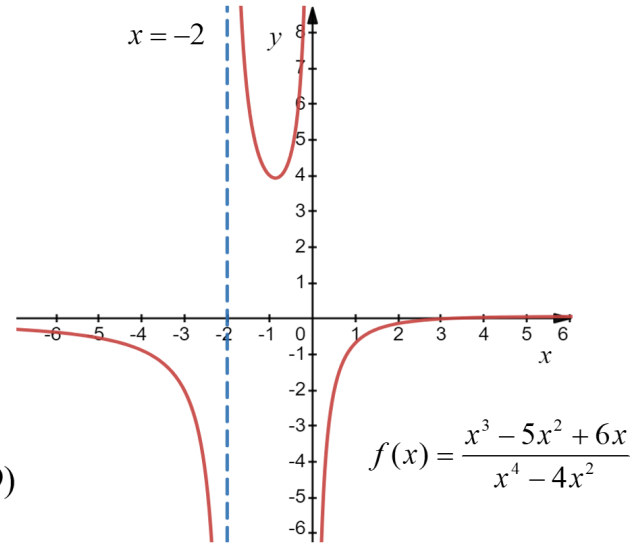
$= \lim\limits_{x\to -2^+} \dfrac{\overbrace{x-3}^{c^-}}{\underbrace{x(x+2)}_{0^-}}$

$= +\infty$

Sign Test Rule

$x \to -2^+$ (Let $x = -1.99$)

$\dfrac{c^-}{0^-} = +\infty$



This limit does not exist, but what it tells me is that the function has a vertical asymptote at $x = -2$.

Example Evaluate the limits below. What does this limit tell us about the functional behavior of $f(x)$?

(a) $\lim\limits_{x\to -2^+} \dfrac{x^3-5x^2+6x}{x^4-4x^2}$ (b) $\lim\limits_{x\to 0^+} \dfrac{x^3-5x^2+6x}{x^4-4x^2}$ (c) $\lim\limits_{x\to 2} \dfrac{x^3-5x^2+6x}{x^4-4x^2}$

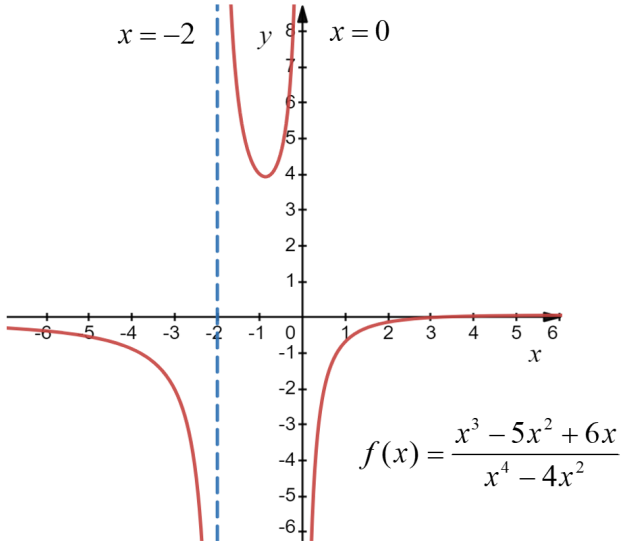
Solution

(b) $\lim\limits_{x\to 0^+} \dfrac{x^3-5x^2+6x}{x^4-4x^2} = \lim\limits_{x\to 0^+} \dfrac{x(x^2-5x+6)}{x^2(x^2-4)}$

$= \lim\limits_{x\to 0^+} \dfrac{\cancel{x(x-2)}(x-3)}{x^{\cancel{2}}\cancel{(x-2)}(x+2)}$ Direct Subtitution gives us $\dfrac{0}{0}$.

$= \lim\limits_{x\to 0^+} \dfrac{\overbrace{x-3}^{c^-}}{\underbrace{x(x+2)}_{0^+}}$ $x \to 0^+$ (Let $x = 0.09$)

$= -\infty$ $\dfrac{c^-}{0^+} = -\infty$



Again This limit does not exist, but what it tells me is that the function has a vertical asymptote at $x = 0$.

Example Evaluate the limits below. What does this limit tell us about the functional behavior of $f(x)$?

(a) $\lim\limits_{x\to -2^+} \dfrac{x^3-5x^2+6x}{x^4-4x^2}$ (b) $\lim\limits_{x\to 0^+} \dfrac{x^3-5x^2+6x}{x^4-4x^2}$ (c) $\lim\limits_{x\to 2} \dfrac{x^3-5x^2+6x}{x^4-4x^2}$

Solution

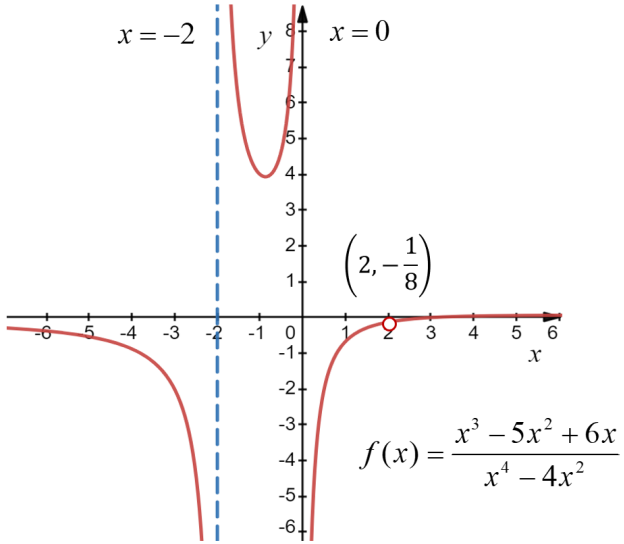
(c) $\lim\limits_{x\to 2} \dfrac{x^3-5x^2+6x}{x^4-4x^2} = \lim\limits_{x\to 2} \dfrac{x(x^2-5x+6)}{x^2(x^2-4)}$

$= \lim\limits_{x\to 2} \dfrac{\cancel{x(x-2)}(x-3)}{x^{\cancel{2}}\cancel{(x-2)}(x+2)}$

$= \lim\limits_{x\to 2} \dfrac{x-3}{x(x+2)}$

$= -\dfrac{1}{8}$

Direct Subtitution gives us $\dfrac{0}{0}$.



This limit tells me that the function has a hole in the graph at $\left(2, -\frac{1}{8}\right)$.

The Squeeze Theorem

The following two theorems describe how the limits of functions are related when the values of one function are greater than (or equal to) those of another.

2 Theorem If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

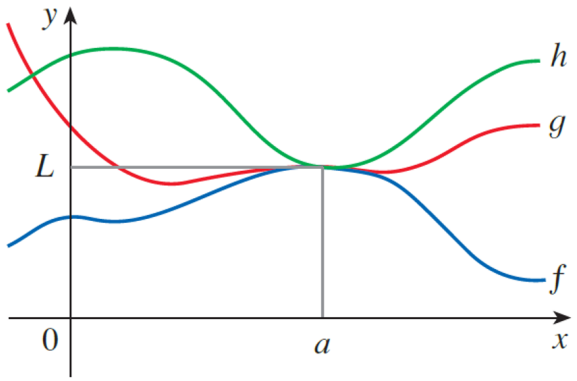
$$\lim_{x \to a} g(x) = L$$

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$



The Squeeze Theorem, sometimes called the Sandwich Theorem or the Pinching Theorem.

Example Consider the function $g(x) = x^2 \sin\left(\frac{1}{x}\right)$.

(a) Use the Squeeze Theorem to find $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$.

(b) Graph the function $g(x)$ and the two functions you found in part (a) that squeeze $g(x)$.

Solution

(a) We know that $\sin\left(\frac{1}{x}\right)$ oscillates between -1 and 1, so

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

Since $x^2 > 0$, we get

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

Taking the limit we get,

$$\lim_{x\to 0}\left(-x^2\right) \le \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) \le \lim_{x\to 0} x^2$$

$$0 \le \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) \le 0$$

Hence, by the Squeeze Theorem $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$

(b) Graph the function $g(x)$ and the two functions you found in part (a) that squeeze $g(x)$.

