We can also interpret the third derivative physically in the case where the function is the position function $s = s(t)$ of an object that moves along a straight line. Because $s''' = (s'')' = a'$, the third derivative of the position function is the derivative of the acceleration function and is called the **jerk**:

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

Thus the jerk j is the rate of change of acceleration. It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement.

The differentiation process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$. In general, the nth derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times. If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

EXAMPLE 7 If $f(x) = x^3 - x$, find $f'''(x)$ and $f^{(4)}(x)$.

SOLUTION In Example 6 we found that $f''(x) = 6x$. The graph of the second derivative has equation $y = 6x$ and so it is a straight line with slope 6. Since the derivative $f'''(x)$ is the slope of $f''(x)$, we have

$$f'''(x) = 6$$

for all values of x. So f''' is a constant function and its graph is a horizontal line. Therefore, for all values of x,

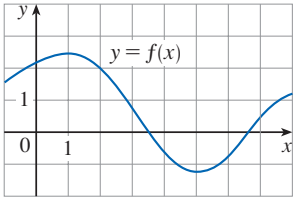
$$f^{(4)}(x) = 0$$ ■

We have seen that one application of second and third derivatives occurs in analyzing the motion of objects using acceleration and jerk. We will investigate another application of second derivatives in Section 4.3, where we show how knowledge of f'' gives us information about the shape of the graph of f. In Chapter 11 we will see how second and higher derivatives enable us to represent functions as sums of infinite series.

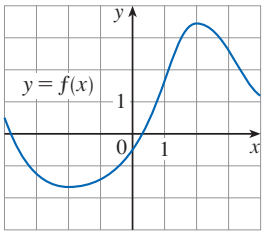
2.8 Exercises

1–2 Use the given graph to estimate the value of each derivative. Then sketch the graph of f'.

1. (a) $f'(0)$ (b) $f'(1)$ (c) $f'(2)$ (d) $f'(3)$
(e) $f'(4)$ (f) $f'(5)$ (g) $f'(6)$ (h) $f'(7)$

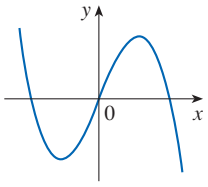


2. (a) $f'(-3)$ (b) $f'(-2)$ (c) $f'(-1)$
(d) $f'(0)$ (e) $f'(1)$ (f) $f'(2)$
(g) $f'(3)$

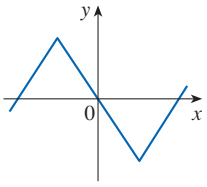


3. Match the graph of each function in (a)–(d) with the graph of its derivative in I–IV. Give reasons for your choices.

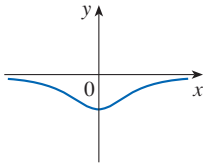
(a)



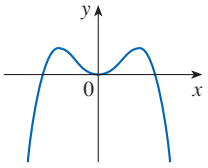
(b)



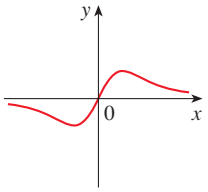
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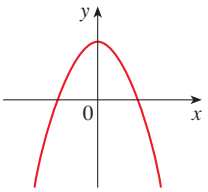
(d)



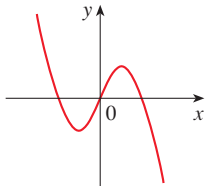
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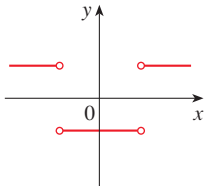
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III

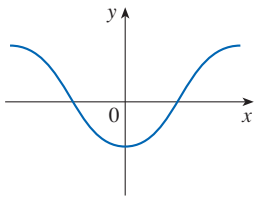


IV

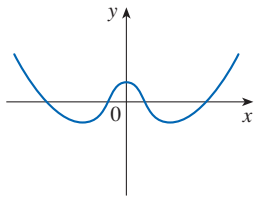


4–11 Trace or copy the graph of the given function f. (Assume that the axes have equal scales.) Then use the method of Example 1 to sketch the graph of f' below it.

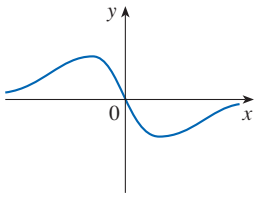
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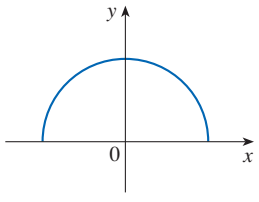
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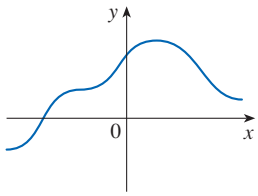
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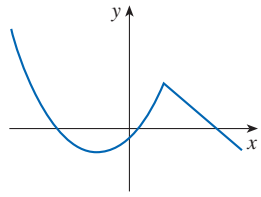
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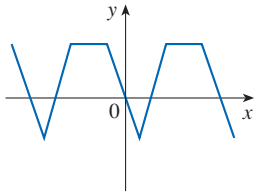
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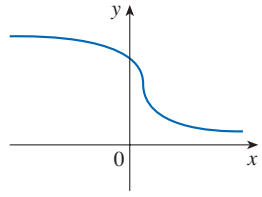
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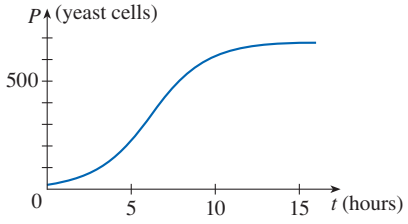
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11.



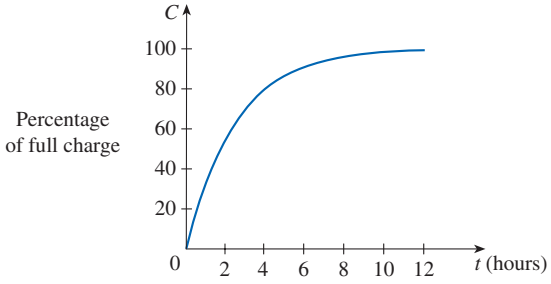
12. Shown is the graph of the population function $P(t)$ for yeast cells in a laboratory culture. Use the method of Example 1 to graph the derivative $P'(t)$. What does the graph of P' tell us about the yeast population?



13. A rechargeable battery is plugged into a charger. The graph shows $C(t)$, the percentage of full capacity that the battery reaches as a function of time t elapsed (in hours).

(a) What is the meaning of the derivative $C'(t)$?

(b) Sketch the graph of $C'(t)$. What does the graph tell you?

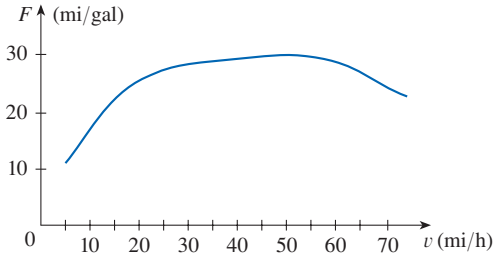


14. The graph (from the US Department of Energy) shows how driving speed affects gas mileage. Fuel economy F is measured in miles per gallon and speed v is measured in miles per hour.

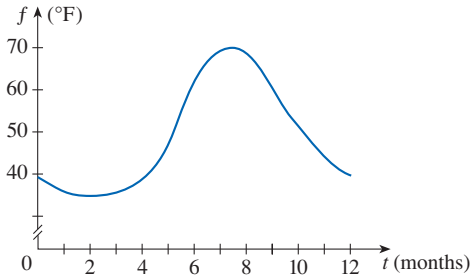
(a) What is the meaning of the derivative $F'(v)$?

(b) Sketch the graph of $F'(v)$.

(c) At what speed should you drive if you want to save on gas?



15. The graph shows how the average surface water temperature f of Lake Michigan varies over the course of a year (where t is measured in months with $t = 0$ corresponding to January 1). The average was calculated from data obtained over a 20-year period ending in 2011. Sketch the graph of the derivative function f'. When is $f'(t)$ largest?



16–18 Make a careful sketch of the graph of f and below it sketch the graph of f' in the same manner as in Exercises 4–11. Can you guess a formula for $f'(x)$ from its graph?

16. $f(x) = \sin x$ **17.** $f(x) = e^x$ **18.** $f(x) = \ln x$

19. Let $f(x) = x^2$.
(a) Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, and $f'(2)$ by zooming in on the graph of f.
(b) Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, and $f'(-2)$.
(c) Use the results from parts (a) and (b) to guess a formula for $f'(x)$.
(d) Use the definition of derivative to prove that your guess in part (c) is correct.

20. Let $f(x) = x^3$.
(a) Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, $f'(2)$, and $f'(3)$ by zooming in on the graph of f.
(b) Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, $f'(-2)$, and $f'(-3)$.
(c) Use the values from parts (a) and (b) to graph f'.
(d) Guess a formula for $f'(x)$.
(e) Use the definition of derivative to prove that your guess in part (d) is correct.

21–32 Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

21. $f(x) = 3x - 8$ **22.** $f(x) = mx + b$

23. $f(t) = 2.5t^2 + 6t$ **24.** $f(x) = 4 + 8x - 5x^2$

25. $A(p) = 4p^3 + 3p$ **26.** $F(t) = t^3 - 5t + 1$

27. $f(x) = \dfrac{1}{x^2 - 4}$ **28.** $F(v) = \dfrac{v}{v + 2}$

29. $g(u) = \dfrac{u + 1}{4u - 1}$ **30.** $f(x) = x^4$

31. $f(x) = \dfrac{1}{\sqrt{1 + x}}$ **32.** $g(x) = \dfrac{1}{1 + \sqrt{x}}$

33. (a) Sketch the graph of $f(x) = 1 + \sqrt{x + 3}$ by starting with the graph of $y = \sqrt{x}$ and using the transformations of Section 1.3.
(b) Use the graph from part (a) to sketch the graph of f'.
(c) Use the definition of a derivative to find $f'(x)$. What are the domains of f and f'?
(d) Graph f' and compare with your sketch in part (b).

34. (a) If $f(x) = x + 1/x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

35. (a) If $f(x) = x^4 + 2x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

36. The table gives the number $N(t)$, measured in thousands, of minimally invasive cosmetic surgery procedures performed in the United States for various years t.

t	$N(t)$ (thousands)
2000	5,500
2002	4,897
2004	7,470
2006	9,138
2008	10,897
2010	11,561
2012	13,035
2014	13,945

Source: American Society of Plastic Surgeons

(a) What is the meaning of $N'(t)$? What are its units?
(b) Construct a table of estimated values for $N'(t)$.
(c) Graph N and N'.
(d) How would it be possible to get more accurate values for $N'(t)$?

37. The table gives the height as time passes of a typical pine tree grown for lumber at a managed site.

Tree age (years)	14	21	28	35	42	49
Height (feet)	41	54	64	72	78	83

Source: Arkansas Forestry Commission

If $H(t)$ is the height of the tree after t years, construct a table of estimated values for H' and sketch its graph.

38. Water temperature affects the growth rate of brook trout. The table shows the amount of weight gained by brook trout after 24 days in various water temperatures.

Temperature (°C)	15.5	17.7	20.0	22.4	24.4
Weight gained (g)	37.2	31.0	19.8	9.7	−9.8

If $W(x)$ is the weight gain at temperature x, construct a table of estimated values for W' and sketch its graph. What are the units for $W'(x)$?

Source: Adapted from J. Chadwick Jr., "Temperature Effects on Growth and Stress Physiology of Brook Trout: Implications for Climate Change Impacts on an Iconic Cold-Water Fish." *Masters Theses*. Paper 897. 2012. scholarworks.umass.edu/theses/897.

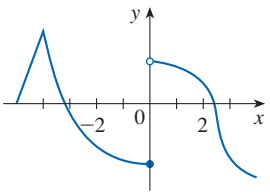
39. Let P represent the percentage of a city's electrical power that is produced by solar panels t years after January 1, 2020.
(a) What does dP/dt represent in this context?
(b) Interpret the statement

$$\left.\frac{dP}{dt}\right|_{t=2} = 3.5$$

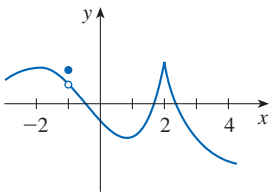
40. Suppose N is the number of people in the United States who travel by car to another state for a vacation in a year when the average price of gasoline is p dollars per gallon. Do you expect dN/dp to be positive or negative? Explain.

41–44 The graph of f is given. State, with reasons, the numbers at which f is *not* differentiable.

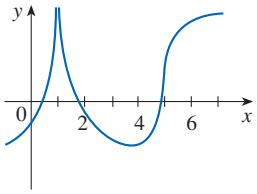
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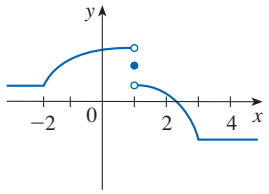
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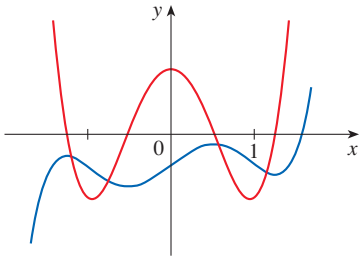


45. Graph the function $f(x) = x + \sqrt{|x|}$. Zoom in repeatedly, first toward the point $(-1, 0)$ and then toward the origin. What is different about the behavior of f in the vicinity of these two points? What do you conclude about the differentiability of f?

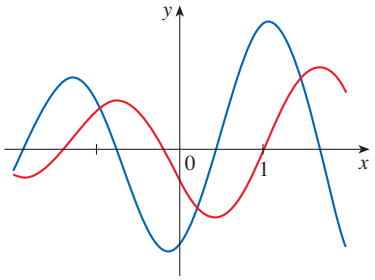
46. Zoom in toward the points $(1, 0)$, $(0, 1)$, and $(-1, 0)$ on the graph of the function $g(x) = (x^2 - 1)^{2/3}$. What do you notice? Account for what you see in terms of the differentiability of g.

47–48 The graphs of a function f and its derivative f' are shown. Which is bigger, $f'(-1)$ or $f''(1)$?

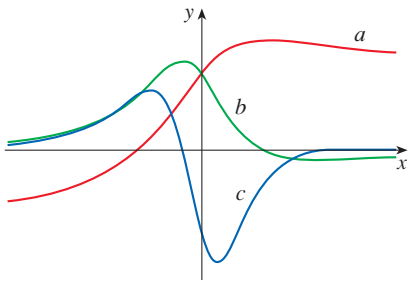
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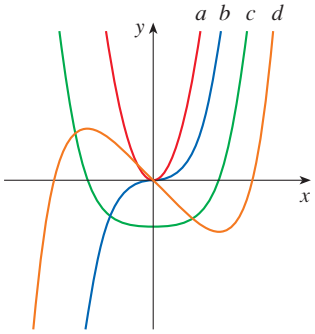
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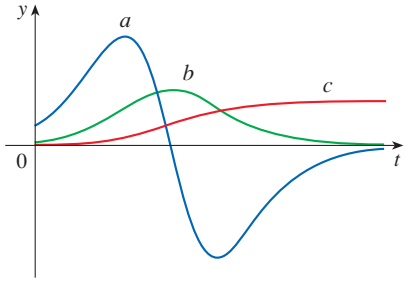
49. The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.



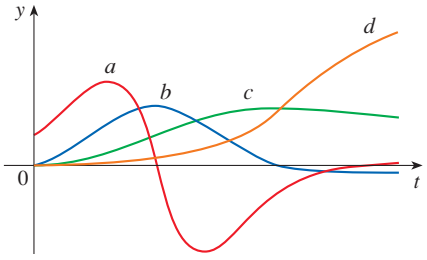
50. The figure shows graphs of f, f', f'', and f'''. Identify each curve, and explain your choices.



51. The figure shows the graphs of three functions. One is the position function of a car, one is the velocity of the car, and one is its acceleration. Identify each curve, and explain your choices.



52. The figure shows the graphs of four functions. One is the position function of a car, one is the velocity of the car, one is its acceleration, and one is its jerk. Identify each curve, and explain your choices.

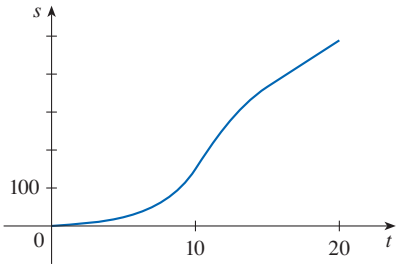


53–54 Use the definition of a derivative to find $f'(x)$ and $f''(x)$. Then graph f, f', and f'' on a common screen and check to see if your answers are reasonable.

53. $f(x) = 3x^2 + 2x + 1$ **54.** $f(x) = x^3 - 3x$

55. If $f(x) = 2x^2 - x^3$, find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$. Graph f, f', f'', and f''' on a common screen. Are the graphs consistent with the geometric interpretations of these derivatives?

56. (a) The graph of a position function of a car is shown, where s is measured in feet and t in seconds. Use it to graph the velocity and acceleration of the car. What is the acceleration at $t = 10$ seconds?



(b) Use the acceleration curve from part (a) to estimate the jerk at $t = 10$ seconds. What are the units for jerk?

57. Let $f(x) = \sqrt[3]{x}$.
(a) If $a \neq 0$, use Equation 2.7.5 to find $f'(a)$.
(b) Show that $f'(0)$ does not exist.
(c) Show that $y = \sqrt[3]{x}$ has a vertical tangent line at $(0, 0)$. (Recall the shape of the graph of f. See Figure 1.2.13.)

58. (a) If $g(x) = x^{2/3}$, show that $g'(0)$ does not exist.
(b) If $a \neq 0$, find $g'(a)$.
(c) Show that $y = x^{2/3}$ has a vertical tangent line at $(0, 0)$.
(d) Illustrate part (c) by graphing $y = x^{2/3}$.

59. Show that the function $f(x) = |x - 6|$ is not differentiable at 6. Find a formula for f' and sketch its graph.

60. Where is the greatest integer function $f(x) = [\![x]\!]$ not differentiable? Find a formula for f' and sketch its graph.

61. (a) Sketch the graph of the function $f(x) = x|x|$.
(b) For what values of x is f differentiable?
(c) Find a formula for f'.

62. (a) Sketch the graph of the function $g(x) = x + |x|$.
(b) For what values of x is g differentiable?
(c) Find a formula for g'.

63. Derivatives of Even and Odd Functions Recall that a function f is called *even* if $f(-x) = f(x)$ for all x in its domain and *odd* if $f(-x) = -f(x)$ for all such x. Prove each of the following.
(a) The derivative of an even function is an odd function.
(b) The derivative of an odd function is an even function.

64–65 Left- and Right-Hand Derivatives The *left-hand* and *right-hand derivatives* of f at a are defined by

$$f'_-(a) = \lim_{h \to 0^-} \frac{f(a + h) - f(a)}{h}$$

and

$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h}$$

if these limits exist. Then $f'(a)$ exists if and only if these one-sided derivatives exist and are equal.

64. Find $f'_-(0)$ and $f'_+(0)$ for the given function f. Is f differentiable at 0?

(a) $$f(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ x & \text{if } x > 0 \end{cases}$$

(b) $$f(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ x^2 & \text{if } x > 0 \end{cases}$$

65. Let

$$f(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ 5 - x & \text{if } 0 < x < 4 \\ \dfrac{1}{5 - x} & \text{if } x \geqslant 4 \end{cases}$$

(a) Find $f'_-(4)$ and $f'_+(4)$.
(b) Sketch the graph of f.
(c) Where is f discontinuous?
(d) Where is f not differentiable?

66. When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running. In Example 1.1.4 we sketched a possible graph of T as a function of the time t that has elapsed since the faucet was turned on.
(a) Describe how the rate of change of T with respect to t varies as t increases.
(b) Sketch a graph of the derivative of T.

67. Nick starts jogging and runs faster and faster for 3 mintues, then he walks for 5 minutes. He stops at an intersection for 2 minutes, runs fairly quickly for 5 minutes, then walks for 4 minutes.
(a) Sketch a possible graph of the distance s Nick has covered after t minutes.
(b) Sketch a graph of ds/dt.

68. Let ℓ be the tangent line to the parabola $y = x^2$ at the point (1, 1). The *angle of inclination* of ℓ is the angle ϕ that ℓ makes with the positive direction of the x-axis. Calculate ϕ correct to the nearest degree.

2 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. Explain what each of the following means and illustrate with a sketch.
(a) $\lim_{x \to a} f(x) = L$ (b) $\lim_{x \to a^+} f(x) = L$ (c) $\lim_{x \to a^-} f(x) = L$
(d) $\lim_{x \to a} f(x) = \infty$ (e) $\lim_{x \to \infty} f(x) = L$

2. Describe several ways in which a limit can fail to exist. Illustrate with sketches.

3. State the following Limit Laws.
(a) Sum Law
(b) Difference Law
(c) Constant Multiple Law
(d) Product Law
(e) Quotient Law
(f) Power Law
(g) Root Law

4. What does the Squeeze Theorem say?

5. (a) What does it mean to say that the line $x = a$ is a vertical asymptote of the curve $y = f(x)$? Draw curves to illustrate the various possibilities.
(b) What does it mean to say that the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$? Draw curves to illustrate the various possibilities.

6. Which of the following curves have vertical asymptotes? Which have horizontal asymptotes?
(a) $y = x^4$ (b) $y = \sin x$ (c) $y = \tan x$
(d) $y = \tan^{-1} x$ (e) $y = e^x$ (f) $y = \ln x$
(g) $y = 1/x$ (h) $y = \sqrt{x}$

7. (a) What does it mean for f to be continuous at a?
(b) What does it mean for f to be continuous on the interval $(-\infty, \infty)$? What can you say about the graph of such a function?

8. (a) Give examples of functions that are continuous on $[-1, 1]$.
(b) Give an example of a function that is not continuous on $[0, 1]$.

9. What does the Intermediate Value Theorem say?

10. Write an expression for the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.

11. Suppose an object moves along a straight line with position $f(t)$ at time t. Write an expression for the instantaneous velocity of the object at time $t = a$. How can you interpret this velocity in terms of the graph of f?