2.7 Exercises

1. A curve has equation $y = f(x)$.
(a) Write an expression for the slope of the secant line through the points $P(3, f(3))$ and $Q(x, f(x))$.
(b) Write an expression for the slope of the tangent line at P.

2. Graph the curve $y = e^x$ in the viewing rectangles $[-1, 1]$ by $[0, 2]$, $[-0.5, 0.5]$ by $[0.5, 1.5]$, and $[-0.1, 0.1]$ by $[0.9, 1.1]$. What do you notice about the curve as you zoom in toward the point $(0, 1)$?

3. (a) Find the slope of the tangent line to the parabola $y = x^2 + 3x$ at the point $(-1, -2)$
(i) using Definition 1 (ii) using Equation 2
(b) Find an equation of the tangent line in part (a).
(c) Graph the parabola and the tangent line. As a check on your work, zoom in toward the point $(-1, -2)$ until the parabola and the tangent line are indistinguishable.

4. (a) Find the slope of the tangent line to the curve $y = x^3 + 1$ at the point $(1, 2)$
(i) using Definition 1 (ii) using Equation 2
(b) Find an equation of the tangent line in part (a).
(c) Graph the curve and the tangent line in successively smaller viewing rectangles centered at $(1, 2)$ until the curve and the line appear to coincide.

5–8 Find an equation of the tangent line to the curve at the given point.

5. $y = 2x^2 - 5x + 1, \quad (3, 4)$

6. $y = x^2 - 2x^3, \quad (1, -1)$

7. $y = \dfrac{x + 2}{x - 3}, \quad (2, -4)$

8. $y = \sqrt{1 - 3x}, \quad (-1, 2)$

9. (a) Find the slope of the tangent to the curve $y = 3 + 4x^2 - 2x^3$ at the point where $x = a$.
(b) Find equations of the tangent lines at the points $(1, 5)$ and $(2, 3)$.
(c) Graph the curve and both tangents on a common screen.

10. (a) Find the slope of the tangent to the curve $y = 2\sqrt{x}$ at the point where $x = a$.
(b) Find equations of the tangent lines at the points $(1, 2)$ and $(9, 6)$.
(c) Graph the curve and both tangents on a common screen.

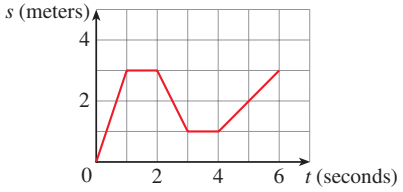
11. A cliff diver plunges from a height of 100 ft above the water surface. The distance the diver falls in t seconds is given by the function $d(t) = 16t^2$ ft.
(a) After how many seconds will the diver hit the water?
(b) With what velocity does the diver hit the water?

12. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height (in meters) after t seconds is given by $H = 10t - 1.86t^2$.
(a) Find the velocity of the rock after one second.
(b) Find the velocity of the rock when $t = a$.
(c) When will the rock hit the surface?
(d) With what velocity will the rock hit the surface?

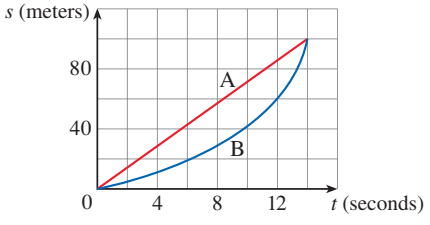
13. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $s = 1/t^2$, where t is measured in seconds. Find the velocity of the particle at times $t = a$, $t = 1$, $t = 2$, and $t = 3$.

14. The displacement (in feet) of a particle moving in a straight line is given by $s = \frac{1}{2}t^2 - 6t + 23$, where t is measured in seconds.
(a) Find the average velocity over each time interval:
(i) $[4, 8]$ (ii) $[6, 8]$
(iii) $[8, 10]$ (iv) $[8, 12]$
(b) Find the instantaneous velocity when $t = 8$.
(c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a). Then draw the tangent line whose slope is the instantaneous velocity in part (b).

15. (a) A particle starts by moving to the right along a horizontal line; the graph of its position function is shown in the figure. When is the particle moving to the right? Moving to the left? Standing still?
(b) Draw a graph of the velocity function.

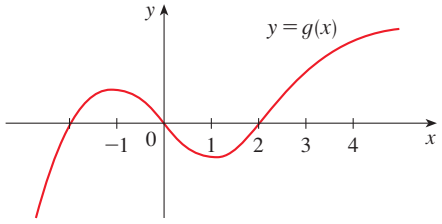


16. Shown are graphs of the position functions of two runners, A and B, who run a 100-meter race and finish in a tie.
(a) Describe and compare how the runners run the race.
(b) At what time is the distance between the runners the greatest?
(c) At what time do they have the same velocity?



17. For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

$$0 \qquad g'(-2) \qquad g'(0) \qquad g'(2) \qquad g'(4)$$



18. The graph of a function f is shown.

(a) Find the average rate of change of f on the interval $[20, 60]$.

(b) Identify an interval on which the average rate of change of f is 0.

(c) Compute

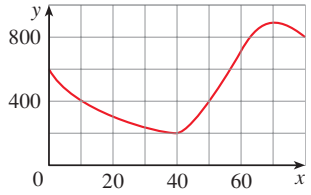
$$\frac{f(40) - f(10)}{40 - 10}$$

What does this value represent geometrically?

(d) Estimate the value of $f'(50)$.

(e) Is $f'(10) > f'(30)$?

(f) Is $f'(60) > \dfrac{f(80) - f(40)}{80 - 40}$? Explain.



19–20 Use Definition 4 to find $f'(a)$ at the given number a.

19. $f(x) = \sqrt{4x + 1}, \quad a = 6$

20. $f(x) = 5x^4, \quad a = -1$

21–22 Use Equation 5 to find $f'(a)$ at the given number a.

21. $f(x) = \dfrac{x^2}{x + 6}, \quad a = 3$

22. $f(x) = \dfrac{1}{\sqrt{2x + 2}}, \quad a = 1$

23–26 Find $f'(a)$.

23. $f(x) = 2x^2 - 5x + 3$

24. $f(t) = t^3 - 3t$

25. $f(t) = \dfrac{1}{t^2 + 1}$

26. $f(x) = \dfrac{x}{1 - 4x}$

27. Find an equation of the tangent line to the graph of $y = B(x)$ at $x = 6$ if $B(6) = 0$ and $B'(6) = -\frac{1}{2}$.

28. Find an equation of the tangent line to the graph of $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$.

29. If $f(x) = 3x^2 - x^3$, find $f'(1)$ and use it to find an equation of the tangent line to the curve $y = 3x^2 - x^3$ at the point $(1, 2)$.

30. If $g(x) = x^4 - 2$, find $g'(1)$ and use it to find an equation of the tangent line to the curve $y = x^4 - 2$ at the point $(1, -1)$.

31. (a) If $F(x) = 5x/(1 + x^2)$, find $F'(2)$ and use it to find an equation of the tangent line to the curve $y = 5x/(1 + x^2)$ at the point $(2, 2)$.

(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

32. (a) If $G(x) = 4x^2 - x^3$, find $G'(a)$ and use it to find equations of the tangent lines to the curve $y = 4x^2 - x^3$ at the points $(2, 8)$ and $(3, 9)$.

(b) Illustrate part (a) by graphing the curve and the tangent lines on the same screen.

33. If an equation of the tangent line to the curve $y = f(x)$ at the point where $a = 2$ is $y = 4x - 5$, find $f(2)$ and $f'(2)$.

34. If the tangent line to $y = f(x)$ at $(4, 3)$ passes through the point $(0, 2)$, find $f(4)$ and $f'(4)$.

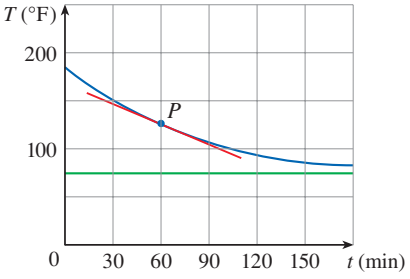
35–36 A particle moves along a straight line with equation of motion $s = f(t)$, where s is measured in meters and t in seconds. Find the velocity and the speed when $t = 4$.

35. $f(t) = 80t - 6t^2$

36. $f(t) = 10 + \dfrac{45}{t + 1}$

37. A warm can of soda is placed in a cold refrigerator. Sketch the graph of the temperature of the soda as a function of time. Is the initial rate of change of temperature greater or less than the rate of change after an hour?

38. A roast turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F. The graph shows how the temperature of the turkey decreases and eventually approaches room temperature. By measuring the slope of the tangent, estimate the rate of change of the temperature after an hour.



39. Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = 3$, $f'(1) = 0$, and $f'(2) = -1$.

40. Sketch the graph of a function g for which $g(0) = g(2) = g(4) = 0$, $g'(1) = g'(3) = 0$, $g'(0) = g'(4) = 1$, $g'(2) = -1$, $\lim_{x\to\infty} g(x) = \infty$, and $\lim_{x\to-\infty} g(x) = -\infty$.

41. Sketch the graph of a function g that is continuous on its domain $(-5, 5)$ and where $g(0) = 1$, $g'(0) = 1$, $g'(-2) = 0$, $\lim_{x\to-5^+} g(x) = \infty$, and $\lim_{x\to5^-} g(x) = 3$.

42. Sketch the graph of a function f where the domain is $(-2, 2)$, $f'(0) = -2$, $\lim_{x\to2^-} f(x) = \infty$, f is continuous at all numbers in its domain except ±1, and f is odd.

43–48 Each limit represents the derivative of some function f at some number a. State such an f and a in each case.

43. $\displaystyle\lim_{h\to0} \frac{\sqrt{9+h} - 3}{h}$

44. $\displaystyle\lim_{h\to0} \frac{e^{-2+h} - e^{-2}}{h}$

45. $\displaystyle\lim_{x\to2} \frac{x^6 - 64}{x - 2}$

46. $\displaystyle\lim_{x\to1/4} \frac{\frac{1}{x} - 4}{x - \frac{1}{4}}$

47. $\displaystyle\lim_{h\to0} \frac{\tan\left(\frac{\pi}{4} + h\right) - 1}{h}$

48. $\displaystyle\lim_{\theta\to\pi/6} \frac{\sin\theta - \frac{1}{2}}{\theta - \frac{\pi}{6}}$

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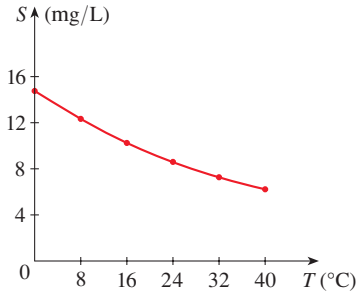
49. The cost (in dollars) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.
(a) Find the average rate of change of C with respect to x when the production level is changed
(i) from $x = 100$ to $x = 105$
(ii) from $x = 100$ to $x = 101$
(b) Find the instantaneous rate of change of C with respect to x when $x = 100$. (This is called the *marginal cost*. Its significance will be explained in Section 3.7.)

50. Let $H(t)$ be the daily cost (in dollars) to heat an office building when the outside temperature is t degrees Fahrenheit.
(a) What is the meaning of $H'(58)$? What are its units?
(b) Would you expect $H'(58)$ to be positive or negative? Explain.

51. The cost of producing x ounces of gold from a new gold mine is $C = f(x)$ dollars.
(a) What is the meaning of the derivative $f'(x)$? What are its units?
(b) What does the statement $f'(800) = 17$ mean?
(c) Do you think the values of $f'(x)$ will increase or decrease in the short term? What about the long term? Explain.

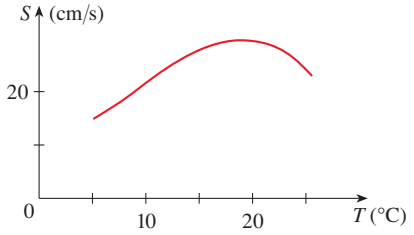
52. The quantity (in pounds) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per pound is $Q = f(p)$.
(a) What is the meaning of the derivative $f'(8)$? What are its units?
(b) Is $f'(8)$ positive or negative? Explain.

53. The quantity of oxygen that can dissolve in water depends on the temperature of the water. (So thermal pollution influences the oxygen content of water.) The graph shows how oxygen solubility S varies as a function of the water temperature T.
(a) What is the meaning of the derivative $S'(T)$? What are its units?
(b) Estimate the value of $S'(16)$ and interpret it.



Source: C. Kupchella et al., *Environmental Science: Living Within the System of Nature*, 2d ed. (Boston: Allyn and Bacon, 1989).

54. The graph shows the influence of the temperature T on the maximum sustainable swimming speed S of Coho salmon.
(a) What is the meaning of the derivative $S'(T)$? What are its units?
(b) Estimate the values of $S'(15)$ and $S'(25)$ and interpret them.



55. Researchers measured the average blood alcohol concentration $C(t)$ of eight men starting one hour after consumption of 30 mL of ethanol (corresponding to two alcoholic drinks).

t (hours)	1.0	1.5	2.0	2.5	3.0
$C(t)$ (g/dL)	0.033	0.024	0.018	0.012	0.007

(a) Find the average rate of change of C with respect to t over each time interval:
(i) $[1.0, 2.0]$ (ii) $[1.5, 2.0]$
(iii) $[2.0, 2.5]$ (iv) $[2.0, 3.0]$
In each case, include the units.
(b) Estimate the instantaneous rate of change at $t = 2$ and interpret your result. What are the units?

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

56. The number N of locations of a popular coffeehouse chain is given in the table. (The numbers of locations as of October 1 are given.)

Year	N
2008	16,680
2010	16,858
2012	18,066
2014	21,366
2016	25,085

(a) Find the average rate of growth
(i) from 2008 to 2010
(ii) from 2010 to 2012
In each case, include the units. What can you conclude?
(b) Estimate the instantaneous rate of growth in 2010 by taking the average of two average rates of change. What are its units?
(c) Estimate the instantaneous rate of growth in 2010 by measuring the slope of a tangent.

57–58 Determine whether $f'(0)$ exists.

57. $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

58. $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

59. (a) Graph the function $f(x) = \sin x - \frac{1}{1000} \sin(1000x)$ in the viewing rectangle $[-2\pi, 2\pi]$ by $[-4, 4]$. What slope does the graph appear to have at the origin?
(b) Zoom in to the viewing window $[-0.4, 0.4]$ by $[-0.25, 0.25]$ and estimate the value of $f'(0)$. Does this agree with your answer from part (a)?
(c) Now zoom in to the viewing window $[-0.008, 0.008]$ by $[-0.005, 0.005]$. Do you wish to revise your estimate for $f'(0)$?

60. Symmetric Difference Quotients In Example 8 we approximated an instantaneous rate of change by averaging two average rates of change. An alternative method is to use a single average rate of change over an interval *centered* at the desired value. We define the *symmetric difference quotient* of a function f at $x = a$ on the interval $[a - d, a + d]$ as

$$\frac{f(a + d) - f(a - d)}{(a + d) - (a - d)} = \frac{f(a + d) - f(a - d)}{2d}$$

(a) Compute the symmetric difference quotient for the function D in Example 8 on the interval $[2004, 2012]$ and verify that your result agrees with the estimate for $D'(2008)$ computed in the example.
(b) Show that the symmetric difference quotient of a function f at $x = a$ is equivalent to averaging the average rates of change of f over the intervals $[a - d, a]$ and $[a, a + d]$.
(c) Use a symmetric difference quotient to estimate $f'(1)$ for $f(x) = x^3 - 2x^2 + 2$ with $d = 0.4$. Draw a graph of f along with secant lines corresponding to average rates of change over the intervals $[1 - d, 1]$, $[1, 1 + d]$, and $[1 - d, 1 + d]$. Which of these secant lines appears to have slope closest to that of the tangent line at $x = 1$?

WRITING PROJECT | EARLY METHODS FOR FINDING TANGENTS

The first person to explicitly formulate the ideas of limits and derivatives was Sir Isaac Newton in the 1660s. But Newton acknowledged that "If I have seen further than other men, it is because I have stood on the shoulders of giants." Two of those giants were Pierre Fermat (1601–1665) and Newton's mentor at Cambridge, Isaac Barrow (1630–1677). Newton was familiar with the methods that these men used to find tangent lines, and their methods played a role in Newton's eventual formulation of calculus.

Learn about these methods by researching on the Internet or reading one of the references listed here. Write an essay comparing the methods of either Fermat or Barrow to modern methods. In particular, use the method of Section 2.7 to find an equation of the tangent line to the curve $y = x^3 + 2x$ at the point $(1, 3)$ and show how either Fermat or Barrow would have solved the same problem. Although you used derivatives and they did not, point out similarities between the methods.

1. C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), pp. 124, 132.