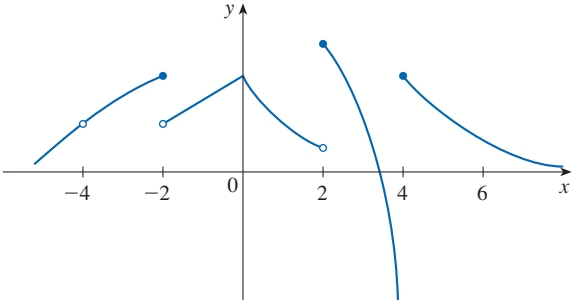
2.5 Exercises

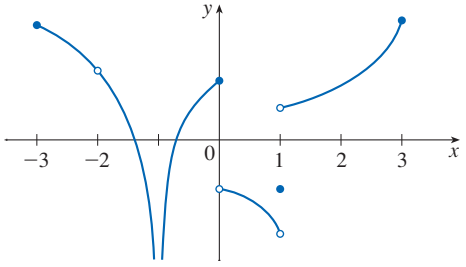
1. Write an equation that expresses the fact that a function f is continuous at the number 4.

2. If f is continuous on $(-\infty, \infty)$, what can you say about its graph?

3. (a) From the given graph of f, state the numbers at which f is discontinuous and explain why.
(b) For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left, or neither.

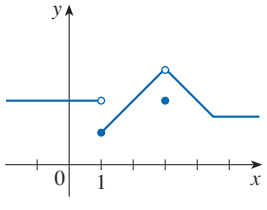


4. From the given graph of g, state the numbers at which g is discontinuous and explain why.

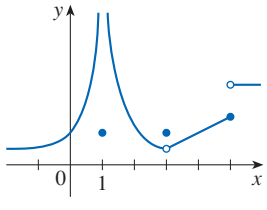


5–6 The graph of a function f is given.
(a) At what numbers a does $\lim_{x \to a} f(x)$ not exist?
(b) At what numbers a is f not continuous?
(c) At what numbers a does $\lim_{x \to a} f(x)$ exist but f is not continuous at a?

5.



6.



7–10 Sketch the graph of a function f that is defined on $\mathbb{R}$ and continuous except for the stated discontinuities.

7. Removable discontinuity at -2, infinite discontinuity at 2

8. Jump discontinuity at -3, removable discontinuity at 4

9. Discontinuities at 0 and 3, but continuous from the right at 0 and from the left at 3

10. Continuous only from the left at -1, not continuous from the left or right at 3

11. The toll T charged for driving on a certain stretch of a toll road is \$5 except during rush hours (between 7 AM and 10 AM and between 4 PM and 7 PM) when the toll is \$7.
(a) Sketch a graph of T as a function of the time t, measured in hours past midnight.
(b) Discuss the discontinuities of this function and their significance to someone who uses the road.

12. Explain why each function is continuous or discontinuous.
(a) The temperature at a specific location as a function of time
(b) The temperature at a specific time as a function of the distance due west from New York City
(c) The altitude above sea level as a function of the distance due west from New York City
(d) The cost of a taxi ride as a function of the distance traveled
(e) The current in the circuit for the lights in a room as a function of time

13–16 Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

13. $f(x) = 3x^2 + (x + 2)^5, \quad a = -1$

14. $g(t) = \dfrac{t^2 + 5t}{2t + 1}, \quad a = 2$

15. $p(v) = 2\sqrt{3v^2 + 1}, \quad a = 1$

16. $f(r) = \sqrt[3]{4r^2 - 2r + 7}, \quad a = -2$

17–18 Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

17. $f(x) = x + \sqrt{x - 4}\,, \quad [4, \infty)$

18. $g(x) = \dfrac{x - 1}{3x + 6}, \quad (-\infty, -2)$

19–24 Explain why the function is discontinuous at the given number a. Sketch the graph of the function.

19. $f(x) = \dfrac{1}{x + 2} \qquad a = -2$

20. $f(x) = \begin{cases} \dfrac{1}{x + 2} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases} \qquad a = -2$

21. $f(x) = \begin{cases} x + 3 & \text{if } x \leq -1 \\ 2^x & \text{if } x > -1 \end{cases}$ $a = -1$

22. $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$ $a = 1$

23. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}$ $a = 0$

24. $f(x) = \begin{cases} \dfrac{2x^2 - 5x - 3}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$ $a = 3$

25–26

(a) Show that f has a removable discontinuity at $x = 3$.
(b) Redefine $f(3)$ so that f is continuous at $x = 3$ (and thus the discontinuity is "removed").

25. $f(x) = \dfrac{x - 3}{x^2 - 9}$

26. $f(x) = \dfrac{x^2 - 7x + 12}{x - 3}$

27–34 Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain. State the domain.

27. $f(x) = \dfrac{x^2}{\sqrt{x^4 + 2}}$

28. $g(v) = \dfrac{3v - 1}{v^2 + 2v - 15}$

29. $h(t) = \dfrac{\cos(t^2)}{1 - e^t}$

30. $B(u) = \sqrt{3u - 2} + \sqrt[3]{2u - 3}$

31. $L(v) = v \ln(1 - v^2)$

32. $f(t) = e^{-t^2} \ln(1 + t^2)$

33. $M(x) = \sqrt{1 + \dfrac{1}{x}}$

34. $g(t) = \cos^{-1}(e^t - 1)$

35–38 Use continuity to evaluate the limit.

35. $\lim_{x \to 2} x\sqrt{20 - x^2}$

36. $\lim_{\theta \to \pi/2} \sin(\tan(\cos\theta))$

37. $\lim_{x \to 1} \ln\left(\dfrac{5 - x^2}{1 + x}\right)$

38. $\lim_{x \to 4} 3^{\sqrt{x^2 - 2x - 4}}$

39–40 Locate the discontinuities of the function and illustrate by graphing.

39. $f(x) = \dfrac{1}{\sqrt{1 - \sin x}}$

40. $y = \arctan\dfrac{1}{x}$

41–42 Show that f is continuous on $(-\infty, \infty)$.

41. $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leq 1 \\ \ln x & \text{if } x > 1 \end{cases}$

42. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$

43–45 Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

43. $f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ x & \text{if } -1 \leq x < 1 \\ 1/x & \text{if } x \geq 1 \end{cases}$

44. $f(x) = \begin{cases} 2^x & \text{if } x \leq 1 \\ 3 - x & \text{if } 1 < x \leq 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$

45. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } x > 1 \end{cases}$

46. The gravitational force exerted by the planet Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{if } r < R \\ \dfrac{GM}{r^2} & \text{if } r \geq R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

47. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

48. Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leq x < 3 \\ 2x - a + b & \text{if } x \geq 3 \end{cases}$$

49. Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim_{x \to 2} [3f(x) + f(x)g(x)] = 36$. Find $f(2)$.

50. Let $f(x) = 1/x$ and $g(x) = 1/x^2$.
(a) Find $(f \circ g)(x)$.
(b) Is $f \circ g$ continuous everywhere? Explain.

51. Which of the following functions f has a removable discontinuity at a? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous at a.

(a) $f(x) = \dfrac{x^4 - 1}{x - 1}, \quad a = 1$

(b) $f(x) = \dfrac{x^3 - x^2 - 2x}{x - 2}, \quad a = 2$

(c) $f(x) = [\![\sin x]\!], \quad a = \pi$

52. Suppose that a function f is continuous on $[0, 1]$ except at 0.25 and that $f(0) = 1$ and $f(1) = 3$. Let $N = 2$. Sketch two possible graphs of f, one showing that f might not satisfy the conclusion of the Intermediate Value Theorem and one showing that f might still satisfy the conclusion of the Intermediate Value Theorem (even though it doesn't satisfy the hypothesis).

53. If $f(x) = x^2 + 10 \sin x$, show that there is a number c such that $f(c) = 1000$.

54. Suppose f is continuous on $[1, 5]$ and the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) = 8$, explain why $f(3) > 6$.

55–58 Use the Intermediate Value Theorem to show that there is a solution of the given equation in the specified interval.

55. $-x^3 + 4x + 1 = 0, \quad (-1, 0)$

56. $\ln x = x - \sqrt{x}, \quad (2, 3)$

57. $e^x = 3 - 2x, \quad (0, 1)$

58. $\sin x = x^2 - x, \quad (1, 2)$

59–60

(a) Prove that the equation has at least one real solution.
(b) Use a calculator to find an interval of length 0.01 that contains a solution.

59. $\cos x = x^3$

60. $\ln x = 3 - 2x$

61–62

(a) Prove that the equation has at least one real solution.
(b) Find the solution correct to three decimal places, by graphing.

61. $100e^{-x/100} = 0.01x^2$

62. $\arctan x = 1 - x$

63–64 Prove, without graphing, that the graph of the function has at least two x-intercepts in the specified interval.

63. $y = \sin x^3, \quad (1, 2)$

64. $y = x^2 - 3 + 1/x, \quad (0, 2)$

65. Prove that f is continuous at a if and only if

$$\lim_{h \to 0} f(a + h) = f(a)$$

66. To prove that sine is continuous, we need to show that $\lim_{x \to a} \sin x = \sin a$ for every real number a. By Exercise 65 an equivalent statement is that

$$\lim_{h \to 0} \sin(a + h) = \sin a$$

Use (6) to show that this is true.

67. Prove that cosine is a continuous function.

68. (a) Prove Theorem 4, part 3.
(b) Prove Theorem 4, part 5.

69. Use Theorem 8 to prove Limit Laws 6 and 7 from Section 2.3.

70. Is there a number that is exactly 1 more than its cube?

71. For what values of x is f continuous?

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

72. For what values of x is g continuous?

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$$

73. Show that the function

$$f(x) = \begin{cases} x^4 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous on $(-\infty, \infty)$.

74. If a and b are positive numbers, prove that the equation

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0$$

has at least one solution in the interval $(-1, 1)$.

75. A Tibetan monk leaves the monastery at 7:00 AM and takes his usual path to the top of the mountain, arriving at 7:00 PM. The following morning, he starts at 7:00 AM at the top and takes the same path back, arriving at the monastery at 7:00 PM. Use the Intermediate Value Theorem to show that there is a point on the path that the monk will cross at exactly the same time of day on both days.

76. Absolute Value and Continuity

(a) Show that the absolute value function $F(x) = |x|$ is continuous everywhere.
(b) Prove that if f is a continuous function on an interval, then so is $|f|$.
(c) Is the converse of the statement in part (b) also true? In other words, if $|f|$ is continuous, does it follow that f is continuous? If so, prove it. If not, find a counterexample.