EXAMPLE 11 Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

SOLUTION First note that we **cannot** rewrite the limit as the product of the limits $\lim_{x \to 0} x^2$ and $\lim_{x \to 0} \sin(1/x)$ because $\lim_{x \to 0} \sin(1/x)$ does not exist (see Example 2.2.5).

We *can* find the limit by using the Squeeze Theorem. To apply the Squeeze Theorem we need to find a function f smaller than $g(x) = x^2 \sin(1/x)$ and a function h bigger than g such that both $f(x)$ and $h(x)$ approach 0 as $x \to 0$. To do this we use our knowledge of the sine function. Because the sine of any number lies between -1 and 1, we can write

$$-1 \leqslant \sin \frac{1}{x} \leqslant 1 \tag{4}$$

Any inequality remains true when multiplied by a positive number. We know that $x^2 \geqslant 0$ for all x and so, multiplying each side of the inequalities in (4) by x^2, we get

$$-x^2 \leqslant x^2 \sin \frac{1}{x} \leqslant x^2$$

as illustrated by Figure 8. We know that

$$\lim_{x \to 0} x^2 = 0 \qquad \text{and} \qquad \lim_{x \to 0} (-x^2) = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$$

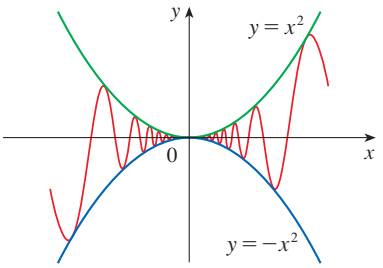


FIGURE 8
$y = x^2 \sin(1/x)$

2.3 Exercises

1. Given that

$$\lim_{x \to 2} f(x) = 4 \qquad \lim_{x \to 2} g(x) = -2 \qquad \lim_{x \to 2} h(x) = 0$$

find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x \to 2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 2} [g(x)]^3$

(c) $\lim_{x \to 2} \sqrt{f(x)}$ (d) $\lim_{x \to 2} \dfrac{3f(x)}{g(x)}$

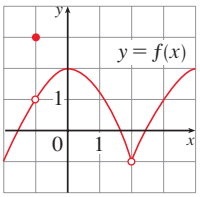
(e) $\lim_{x \to 2} \dfrac{g(x)}{h(x)}$ (f) $\lim_{x \to 2} \dfrac{g(x)h(x)}{f(x)}$

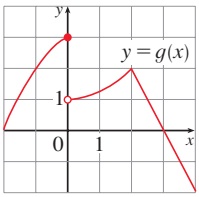
2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.

(a) $\lim_{x \to 2} [f(x) + g(x)]$ (b) $\lim_{x \to 0} [f(x) - g(x)]$

(c) $\lim_{x \to -1} [f(x)g(x)]$ (d) $\lim_{x \to 3} \dfrac{f(x)}{g(x)}$

(e) $\lim_{x \to 2} [x^2 f(x)]$ (f) $f(-1) + \lim_{x \to -1} g(x)$





3–9 Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x \to 5} (4x^2 - 5x)$

4. $\lim_{x \to -3} (2x^3 + 6x^2 - 9)$

5. $\lim_{v \to 2} (v^2 + 2v)(2v^3 - 5)$

6. $\lim_{t \to 7} \dfrac{3t^2 + 1}{t^2 - 5t + 2}$

7. $\lim_{u \to -2} \sqrt{9 - u^3 + 2u^2}$

8. $\lim_{x \to 3} \sqrt[3]{x + 5}\,(2x^2 - 3x)$

9. $\lim_{t \to -1} \left(\dfrac{2t^5 - t^4}{5t^2 + 4} \right)^3$

10. (a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2} (x + 3)$$

is correct.

11–34 Evaluate the limit, if it exists.

11. $\lim_{x \to -2} (3x - 7)$

12. $\lim_{x \to 6} \left(8 - \frac{1}{2}x\right)$

13. $\lim_{t \to 4} \frac{t^2 - 2t - 8}{t - 4}$

14. $\lim_{x \to -3} \frac{x^2 + 3x}{x^2 - x - 12}$

15. $\lim_{x \to 2} \frac{x^2 + 5x + 4}{x - 2}$

16. $\lim_{x \to 4} \frac{x^2 + 3x}{x^2 - x - 12}$

17. $\lim_{x \to -2} \frac{x^2 - x - 6}{3x^2 + 5x - 2}$

18. $\lim_{x \to -5} \frac{2x^2 + 9x - 5}{x^2 - 25}$

19. $\lim_{t \to 3} \frac{t^3 - 27}{t^2 - 9}$

20. $\lim_{u \to -1} \frac{u + 1}{u^3 + 1}$

21. $\lim_{h \to 0} \frac{(h - 3)^2 - 9}{h}$

22. $\lim_{x \to 9} \frac{9 - x}{3 - \sqrt{x}}$

23. $\lim_{h \to 0} \frac{\sqrt{9 + h} - 3}{h}$

24. $\lim_{x \to 2} \frac{2 - x}{\sqrt{x + 2} - 2}$

25. $\lim_{x \to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3}$

26. $\lim_{h \to 0} \frac{(-2 + h)^{-1} + 2^{-1}}{h}$

27. $\lim_{t \to 0} \frac{\sqrt{1 + t} - \sqrt{1 - t}}{t}$

28. $\lim_{t \to 0} \left(\frac{1}{t} - \frac{1}{t^2 + t}\right)$

29. $\lim_{x \to 16} \frac{4 - \sqrt{x}}{16x - x^2}$

30. $\lim_{x \to 2} \frac{x^2 - 4x + 4}{x^4 - 3x^2 - 4}$

31. $\lim_{t \to 0} \left(\frac{1}{t\sqrt{1 + t}} - \frac{1}{t}\right)$

32. $\lim_{x \to -4} \frac{\sqrt{x^2 + 9} - 5}{x + 4}$

33. $\lim_{h \to 0} \frac{(x + h)^3 - x^3}{h}$

34. $\lim_{h \to 0} \frac{\frac{1}{(x + h)^2} - \frac{1}{x^2}}{h}$

35. (a) Estimate the value of

$$\lim_{x \to 0} \frac{x}{\sqrt{1 + 3x} - 1}$$

by graphing the function $f(x) = x/\left(\sqrt{1 + 3x} - 1\right)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

36. (a) Use a graph of

$$f(x) = \frac{\sqrt{3 + x} - \sqrt{3}}{x}$$

to estimate the value of $\lim_{x \to 0} f(x)$ to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Use the Limit Laws to find the exact value of the limit.

37. Use the Squeeze Theorem to show that

$$\lim_{x \to 0} x^2 \cos 20\pi x = 0$$

Illustrate by graphing the functions $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$, and $h(x) = x^2$ on the same screen.

38. Use the Squeeze Theorem to show that

$$\lim_{x \to 0} \sqrt{x^3 + x^2} \sin \frac{\pi}{x} = 0$$

Illustrate by graphing the functions f, g, and h (in the notation of the Squeeze Theorem) on the same screen.

39. If $4x - 9 \leq f(x) \leq x^2 - 4x + 7$ for $x \geq 0$, find $\lim_{x \to 4} f(x)$.

40. If $2x \leq g(x) \leq x^4 - x^2 + 2$ for all x, evaluate $\lim_{x \to 1} g(x)$.

41. Prove that $\lim_{x \to 0} x^4 \cos \frac{2}{x} = 0$.

42. Prove that $\lim_{x \to 0^+} \sqrt{x}\, e^{\sin(\pi/x)} = 0$.

43–48 Find the limit, if it exists. If the limit does not exist, explain why.

43. $\lim_{x \to -4} \left(|x + 4| - 2x\right)$

44. $\lim_{x \to -4} \frac{|x + 4|}{2x + 8}$

45. $\lim_{x \to 0.5^-} \frac{2x - 1}{|2x^3 - x^2|}$

46. $\lim_{x \to -2} \frac{2 - |x|}{2 + x}$

47. $\lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{|x|}\right)$

48. $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right)$

49. The Signum Function The *signum* (or sign) *function*, denoted by sgn, is defined by

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

(a) Sketch the graph of this function.

(b) Find each of the following limits or explain why it does not exist.

(i) $\lim_{x \to 0^+} \operatorname{sgn} x$

(ii) $\lim_{x \to 0^-} \operatorname{sgn} x$

(iii) $\lim_{x \to 0} \operatorname{sgn} x$

(iv) $\lim_{x \to 0} |\operatorname{sgn} x|$

50. Let $g(x) = \operatorname{sgn}(\sin x)$.

(a) Find each of the following limits or explain why it does not exist.

(i) $\lim_{x \to 0^+} g(x)$ (ii) $\lim_{x \to 0^-} g(x)$ (iii) $\lim_{x \to 0} g(x)$

(iv) $\lim_{x \to \pi^+} g(x)$ (v) $\lim_{x \to \pi^-} g(x)$ (vi) $\lim_{x \to \pi} g(x)$

(b) For which values of a does $\lim_{x \to a} g(x)$ not exist?

(c) Sketch a graph of g.

51. Let $g(x) = \dfrac{x^2 + x - 6}{|x - 2|}$.

(a) Find

(i) $\lim_{x \to 2^+} g(x)$ (ii) $\lim_{x \to 2^-} g(x)$

(b) Does $\lim_{x \to 2} g(x)$ exist?

(c) Sketch the graph of g.

52. Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x - 2)^2 & \text{if } x \geqslant 1 \end{cases}$$

(a) Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$.

(b) Does $\lim_{x \to 1} f(x)$ exist?

(c) Sketch the graph of f.

53. Let

$$B(t) = \begin{cases} 4 - \frac{1}{2}t & \text{if } t < 2 \\ \sqrt{t + c} & \text{if } t \geqslant 2 \end{cases}$$

Find the value of c so that $\lim_{t \to 2} B(t)$ exists.

54. Let

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leqslant 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

(a) Evaluate each of the following, if it exists.

(i) $\lim_{x \to 1^-} g(x)$ (ii) $\lim_{x \to 1} g(x)$ (iii) $g(1)$

(iv) $\lim_{x \to 2^-} g(x)$ (v) $\lim_{x \to 2^+} g(x)$ (vi) $\lim_{x \to 2} g(x)$

(b) Sketch the graph of g.

55. (a) If the symbol $[\![\ ]\!]$ denotes the greatest integer function defined in Example 10, evaluate

(i) $\lim_{x \to -2^+} [\![x]\!]$ (ii) $\lim_{x \to -2} [\![x]\!]$ (iii) $\lim_{x \to -2.4} [\![x]\!]$

(b) If n is an integer, evaluate

(i) $\lim_{x \to n^-} [\![x]\!]$ (ii) $\lim_{x \to n^+} [\![x]\!]$

(c) For what values of a does $\lim_{x \to a} [\![x]\!]$ exist?

56. Let $f(x) = [\![\cos x]\!]$, $-\pi \leqslant x \leqslant \pi$.

(a) Sketch the graph of f.

(b) Evaluate each limit, if it exists.

(i) $\lim_{x \to 0} f(x)$ (ii) $\lim_{x \to (\pi/2)^-} f(x)$

(iii) $\lim_{x \to (\pi/2)^+} f(x)$ (iv) $\lim_{x \to \pi/2} f(x)$

(c) For what values of a does $\lim_{x \to a} f(x)$ exist?

57. If $f(x) = [\![x]\!] + [\![-x]\!]$, show that $\lim_{x \to 2} f(x)$ exists but is not equal to $f(2)$.

58. In the theory of relativity, the Lorentz contraction formula

$$L = L_0 \sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v \to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

59. If p is a polynomial, show that $\lim_{x \to a} p(x) = p(a)$.

60. If r is a rational function, use Exercise 59 to show that $\lim_{x \to a} r(x) = r(a)$ for every number a in the domain of r.

61. If $\lim_{x \to 1} \dfrac{f(x) - 8}{x - 1} = 10$, find $\lim_{x \to 1} f(x)$.

62. If $\lim_{x \to 0} \dfrac{f(x)}{x^2} = 5$, find the following limits.

(a) $\lim_{x \to 0} f(x)$ (b) $\lim_{x \to 0} \dfrac{f(x)}{x}$

63. If

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x \to 0} f(x) = 0$.

64. Show by means of an example that $\lim_{x \to a} [f(x) + g(x)]$ may exist even though neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists.

65. Show by means of an example that $\lim_{x \to a} [f(x)\, g(x)]$ may exist even though neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists.

66. Evaluate $\lim_{x \to 2} \dfrac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1}$.

67. Is there a number a such that

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

68. The figure shows a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?

