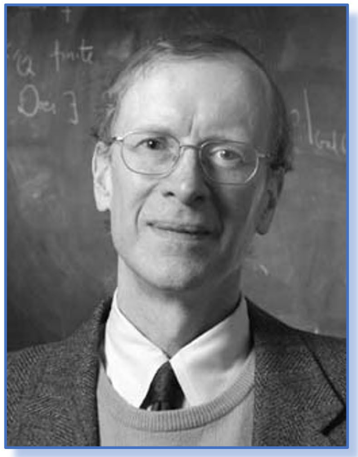
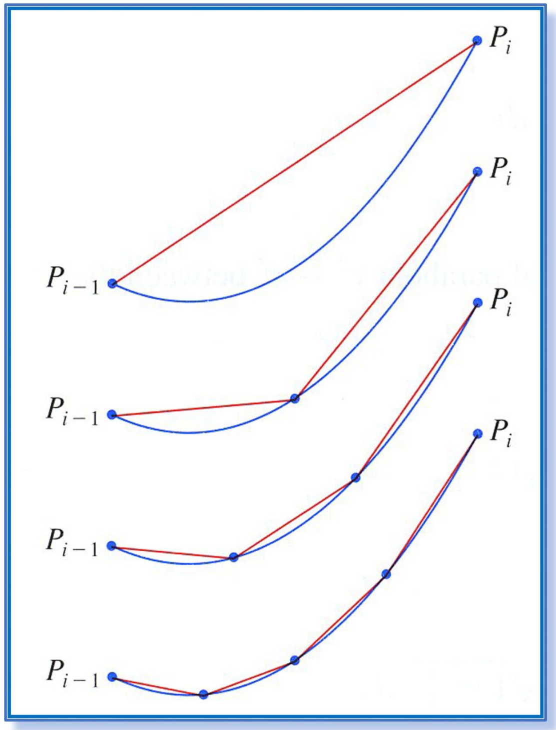
8.1 Arc Length

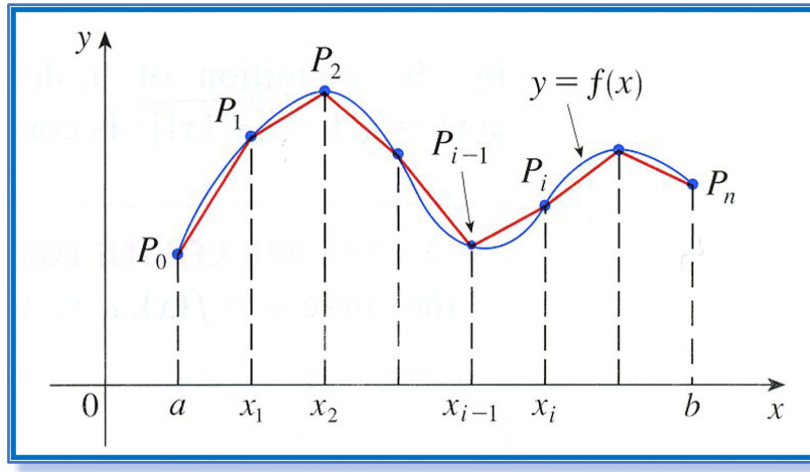


Andrew John Wiles
1953 – Present

Sir Andrew John Wiles, is a British mathematician and a Royal Society Research Professor at Oxford University, specializing in number theory. He is most notable for proving Fermat's Last Theorem.

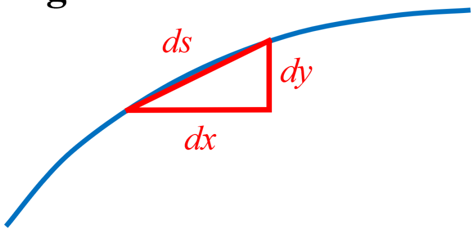
8.1 ARC LENGTH





$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

Lengths of Curves:



If we want to approximate the length of a curve, over a short distance we could measure a straight line.

By the pythagorean theorem:

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2}$$

$$ds = \sqrt{\left(1 + \frac{dy^2}{dx^2}\right) dx^2}$$

$$ds = \sqrt{1 + \frac{dy^2}{dx^2}}\ dx$$

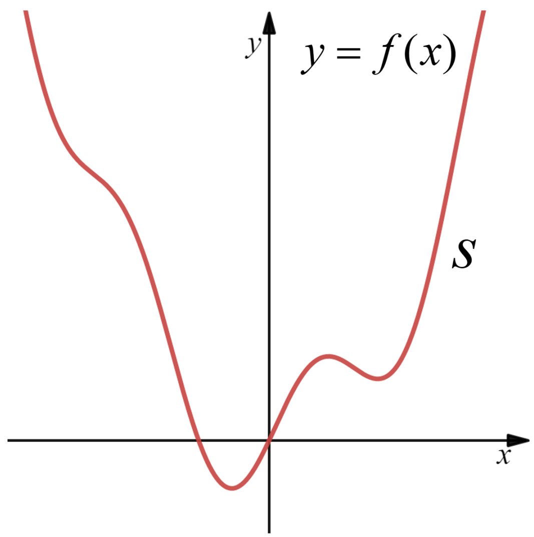
$$\int ds = \int \sqrt{1 + \frac{dy^2}{dx^2}}\ dx$$

$$s = \int \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}\ dx$$

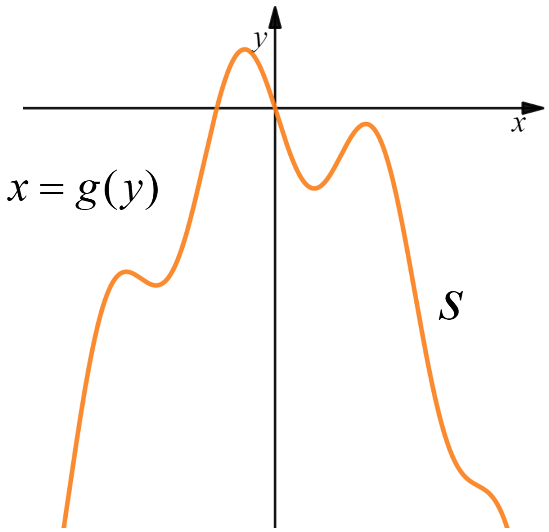
Length of Curve (Cartesian)

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Arc Length Formula

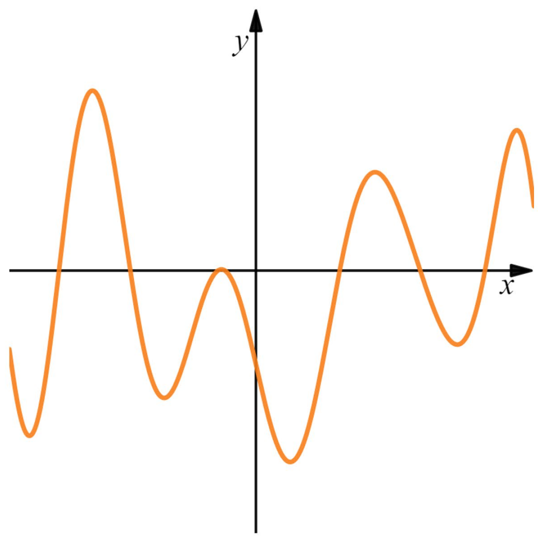


$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

General arc length formula $s = \int ds$.



$$s = \int ds$$

$$y = f(x) \qquad\qquad x = g(y)$$

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad\qquad s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Example: Find the arc length for the function over the specified interval.

$y = x^{3/2}$, over $x \in [0, 5]$.

SOLUTION

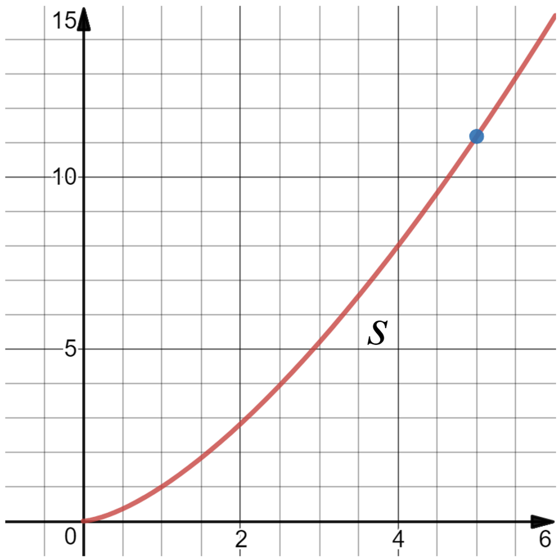
$$s = \int ds$$

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2}$$

$$s = \int_0^5 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\ dx$$

$$s = \int_0^5 \sqrt{1 + \frac{9}{4}x}\ dx$$



$$s=\int_0^5 \sqrt{1+\frac{9}{4}x}\ dx$$

$$s=\frac{4}{9}\cdot\int_1^{49/4} u^{1/2}\ du$$

$$s=\frac{4}{9}\cdot\frac{2}{3}\left[u^{3/2}\right]_1^{49/4}$$

$$s=\frac{8}{27}\left[\left(\frac{49}{4}\right)^{3/2}-(1)^{3/2}\right]$$

$$s=\frac{8}{27}\left[\left(\frac{7}{2}\right)^{3}-1\right]$$

$$s=\frac{8}{27}\left[\frac{343}{8}-\frac{8}{8}\right]=\frac{335}{27}\text{ units}$$

$u-$Substitution

$$u=1+\frac{9}{4}x$$

$$du=\frac{9}{4}dx$$

$$\frac{4}{9}du=dx$$

New Bounds

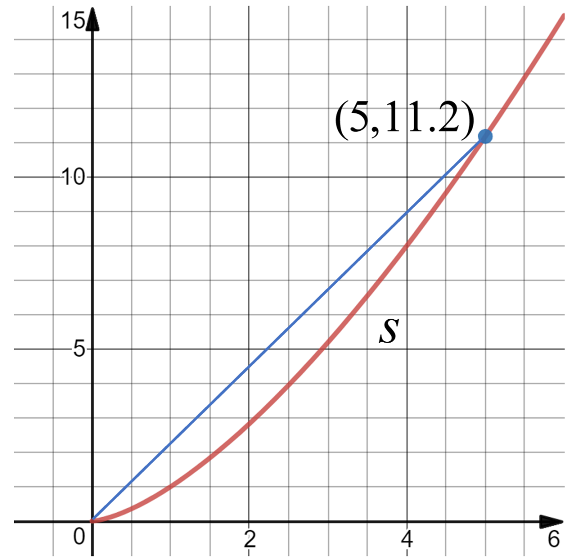
Lower Bound

$$u(0)=1+\frac{9}{4}(0)=1$$

Upper Bound

$$u(5)=1+\frac{9}{4}(5)=\frac{49}{4}$$

$$\frac{335}{27}\text{ units}=12.407\text{ units}$$



$$\sqrt{5^2+11.2^2}=12.265\text{ units}$$

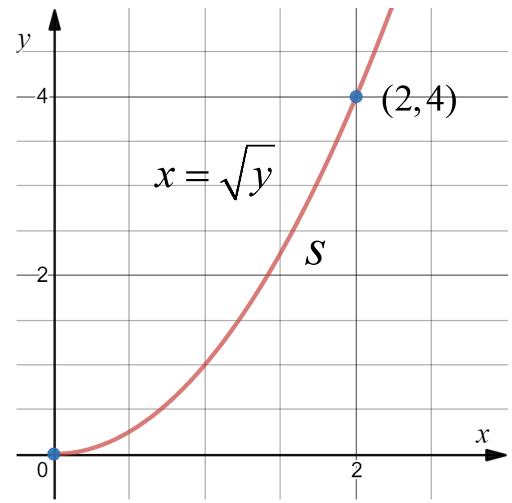
Example: Find the arc length for the function over the specified interval.

$x = \sqrt{y}$, over $y \in [0, 4]$.

SOLUTION

$$s = \int ds$$

$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$



$$\frac{dx}{dy} = \frac{1}{2} y^{-1/2}$$

$$s = \int_0^4 \sqrt{1 + \left(\frac{1}{2} y^{-1/2}\right)^2}\, dy$$

$$s = \int_0^4 \sqrt{1 + \frac{1}{4} y^{-1}}\, dy$$

$$s=\int_0^4\sqrt{1+\frac{1}{4}y^{-1}}\;dy$$

$$s=\int_0^4\sqrt{1+\frac{1}{4y}}\;dy$$

$$s=\int_0^4\sqrt{\frac{y+\frac{1}{4}}{y}}\;dy$$

$$s=\int_0^4\frac{\sqrt{y+\frac{1}{4}}}{\sqrt{y}}\;dy$$

$$s=\int_0^2\frac{\sqrt{u^2+\frac{1}{4}}}{u}\cdot 2u\;du$$

$$s=2\cdot\int_0^2\sqrt{u^2+\frac{1}{4}}\;du$$

Now what?

Math 1B!!

u – Substitution

$u=\sqrt{y}$ $\qquad u^2=y$

$2u\;du=dy$

New Bounds

Lower Bound

$u(0)=\sqrt{0}=0$

Upper Bound

$u(4)=\sqrt{4}=2$

Trig – Substitution

Let $u=\frac{1}{2}\tan\theta$

So $du=\frac{1}{2}\sec^2\theta\;d\theta$

Substituting we get $u^2+\frac{1}{4}=\frac{1}{4}\tan^2\theta+\frac{1}{4}$

$$=\tfrac{1}{4}(\tan^2\theta+1)$$

$$=\tfrac{1}{4}\cdot\sec^2\theta$$

$$s=2\cdot\int\sqrt{\tfrac{1}{4}\sec^2\theta}\cdot\tfrac{1}{2}\sec^2\theta\;d\theta$$

$$s=2\cdot\int\tfrac{1}{2}\sec\theta\cdot\tfrac{1}{2}\sec^2\theta\;d\theta$$

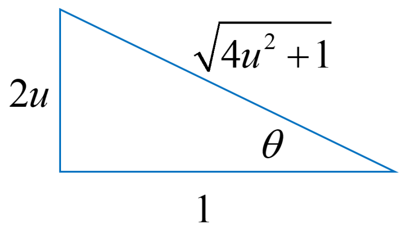
$$s=\frac{1}{2}\cdot\int\sec^3\theta\;d\theta$$

71. $\int\sec^3u\;du=\frac{1}{2}\sec u\tan u+\frac{1}{2}\ln|\sec u+\tan u|+C$

$$s=\frac{1}{4}\cdot\left[\sec\theta\cdot\tan\theta+\ln\left|\sec\theta+\tan\theta\right|\right]$$

$$s = \frac{1}{4} \cdot \left[\sec\theta \cdot \tan\theta + \ln\left| \sec\theta + \tan\theta \right| \right]$$

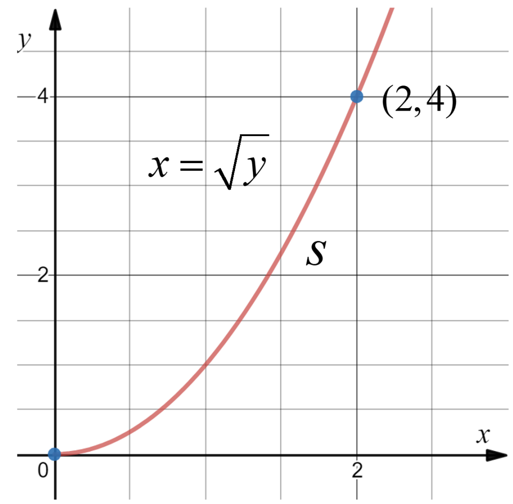
Recall $u = \frac{1}{2}\tan\theta$ So $\tan\theta = 2u$



$$s = \frac{1}{4} \cdot \left[\sqrt{4u^2+1} \cdot 2u + \ln\left| \sqrt{4u^2+1} + 2u \right| \right]_0^2$$

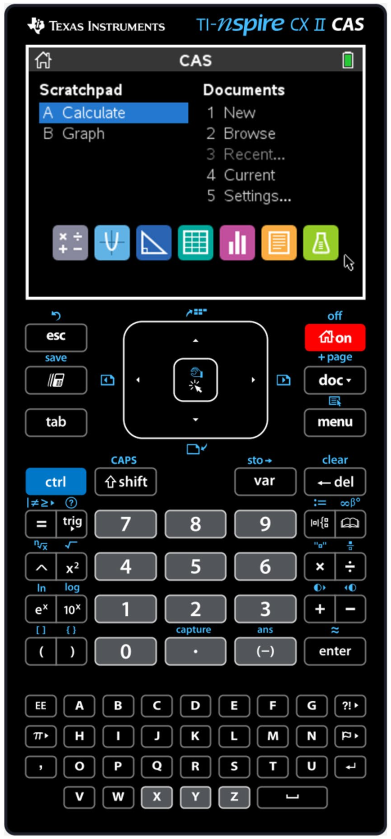
$$s = \frac{1}{4} \cdot \left[\left(\sqrt{17} \cdot 4 + \ln\left| \sqrt{17} + 4 \right| \right) - (0+0) \right]$$

$$s = \sqrt{17} + \tfrac{1}{4} \cdot \ln\left(\sqrt{17} + 4 \right) \text{ units}$$

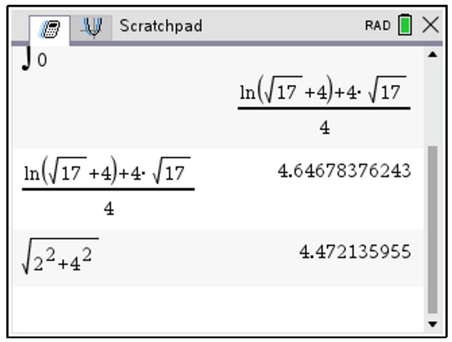


Wow!! This is called Trigonometric Substitution!!

Computer Algebra Systems

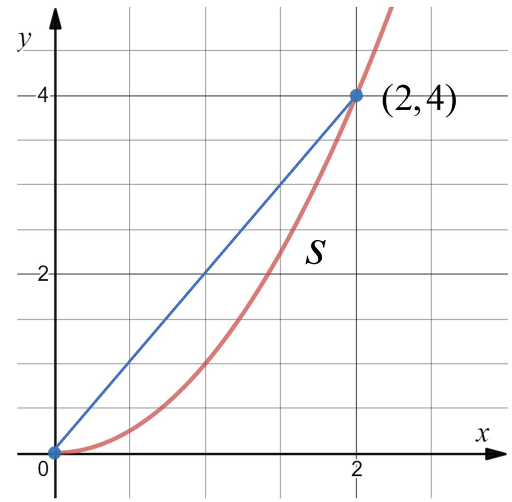


$$s=\int_0^4\sqrt{1+\left(\frac{1}{2}y^{-1/2}\right)^2}\,dy$$



$s=\sqrt{17}+\frac{1}{4}\cdot\ln\left(\sqrt{17}+4\right)$ units

$s\approx 4.647$ units



$\sqrt{2^2+4^2}\approx 4.472$ units

Example: Find the arc length for the function over the specified interval.

$y = \sin x, \text{ over } x \in [0, \pi].$

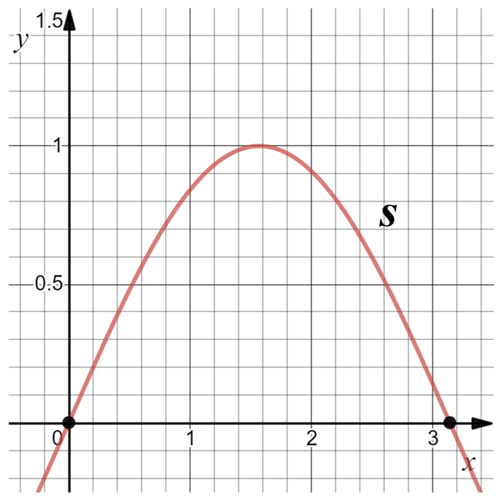
SOLUTION

$$s = \int ds$$

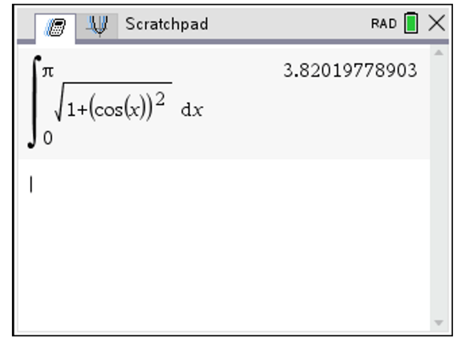
$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \cos x$$

$$s = \int_0^\pi \sqrt{1 + (\cos x)^2}\, dx$$



F, the antiderivative, is not an elementary function.



No exact answer, just an approximation.

$s \approx 3.820$ units

Example: Find the arc length for the function over the specified interval.

$$y = \sin x, \ \text{over } x \in [0, \pi].$$

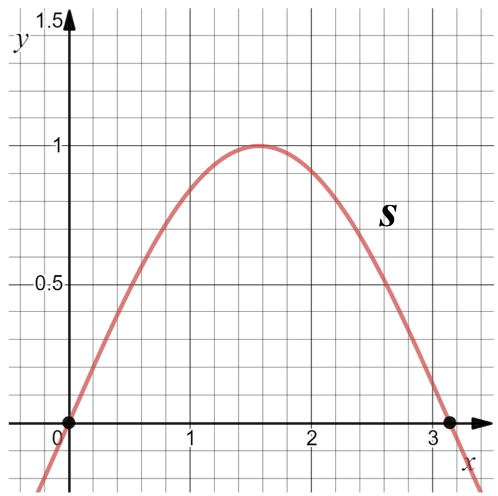
SOLUTION

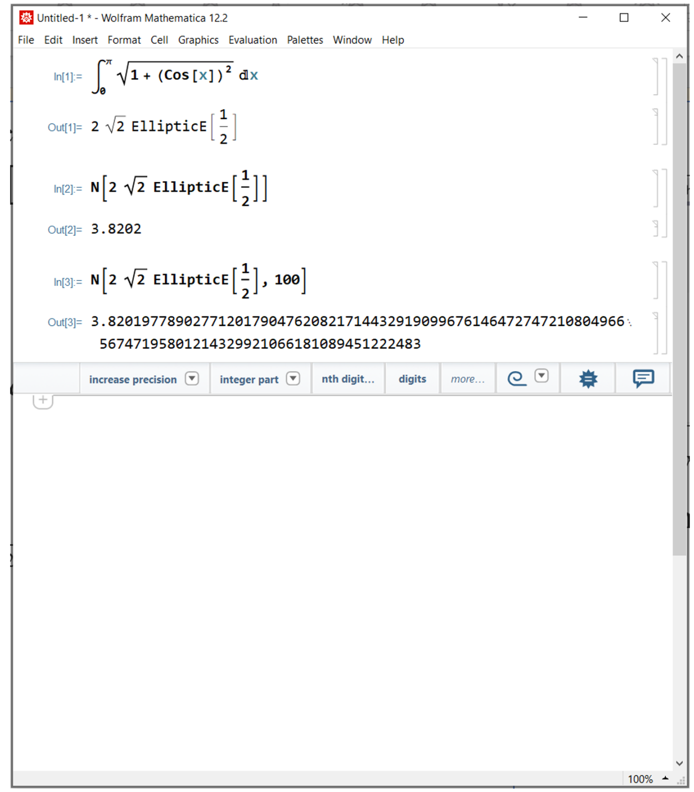
$$s = \int ds$$

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$\frac{dy}{dx} = \cos x$$

$$s = \int_0^{\pi} \sqrt{1 + (\cos x)^2} \, dx$$





F, the antiderivative, is not an elementary function.