7.7 Approximate Integration

Srinivasa Ramanujan
1887 - 1920

Ramanujan made substantial contributions to the analytical theory of numbers and worked on elliptic functions, continued fractions, and infinite series.

Using integrals to find area works extremely well as long as we can find the antiderivative of the function.

Sometimes it is impossible to find the antiderivative of a function. For example, it is impossible to evaluate the following integrals exactly:

$$\int_0^1 e^{x^2} dx \qquad\qquad \int_{-1}^1 \sqrt{1 + x^3}\, dx$$

At other times, we don't even have a function, but only measurements taken from real life.

What we need is an efficient method to estimate area when we can not find the antiderivative.

In this section, we will introduce three numerical techniques for calculation the definite integral.

Example Find the area under the curve $y = \frac{1}{8}x^2 + 1$ on $0 \leq x \leq 4$.

Solution

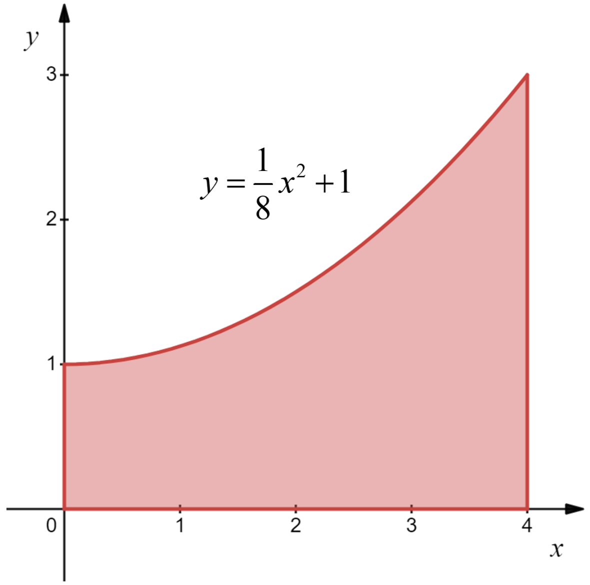
Actual area under curve:

$$A = \int_0^4 \frac{1}{8}x^2 + 1\, dx$$

$$A = \left[\frac{1}{24}x^3 + x\right]_0^4$$

$$A = \frac{20}{3} \text{ units}$$

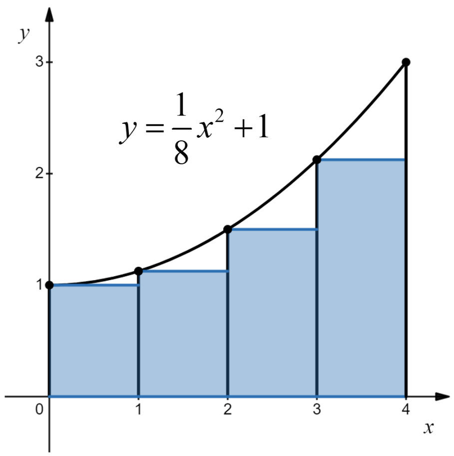
$$A = 6.\overline{6} \text{ units}$$



Example Find the area under the curve $y = \frac{1}{8}x^2 + 1$ on $0 \le x \le 4$.

Solution

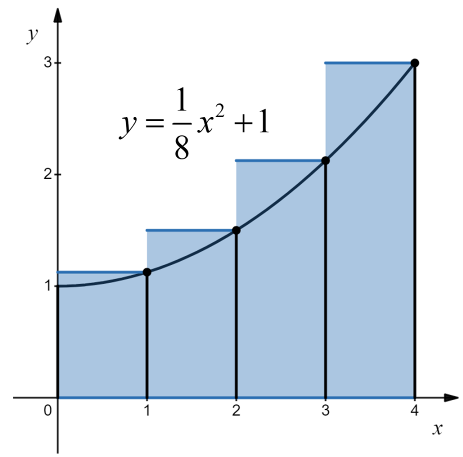
Left-hand rectangular approximation:



Approximate area:

$$L_4 = 1 + 1\frac{1}{8} + 1\frac{1}{2} + 2\frac{1}{8} = 5\frac{3}{4} = 5.75$$ (too low)

Right-hand rectangular approximation:



Approximate area:

$$R_4 = 1\frac{1}{8} + 1\frac{1}{2} + 2\frac{1}{8} + 3 = 7\frac{3}{4} = 7.75$$ (too high)

Averaging the two: $\frac{7.75+5.75}{2}=6.75$ What's the error?

Absolute and Relative Error

Definitions **Absolute and Relative Error**

Suppose c is a computed numerical solution to a problem having an exact solution x. There are two common measures of the error in c as an approximation to x:

$$\textbf{absolute error} = |c - x|$$

and

$$\textbf{relative error} = \frac{|c - x|}{|x|} \quad (\text{if } x \neq 0).$$

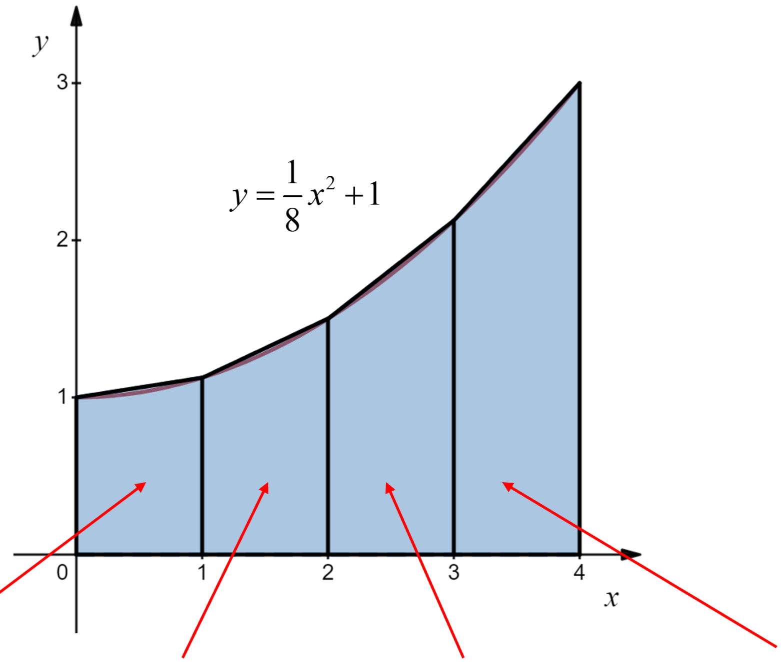
Relative Error $= \frac{|6.75-6.\overline{6}|}{6.\overline{6}} = .0125$ So, 1.25% error (too high)

What is the visual calculus of averaging right and left sums?

Averaging right and left sums gives us trapezoids!

$$T = \frac{1}{2}\left(b_1 + b_2\right)h$$

where $h = \Delta x$



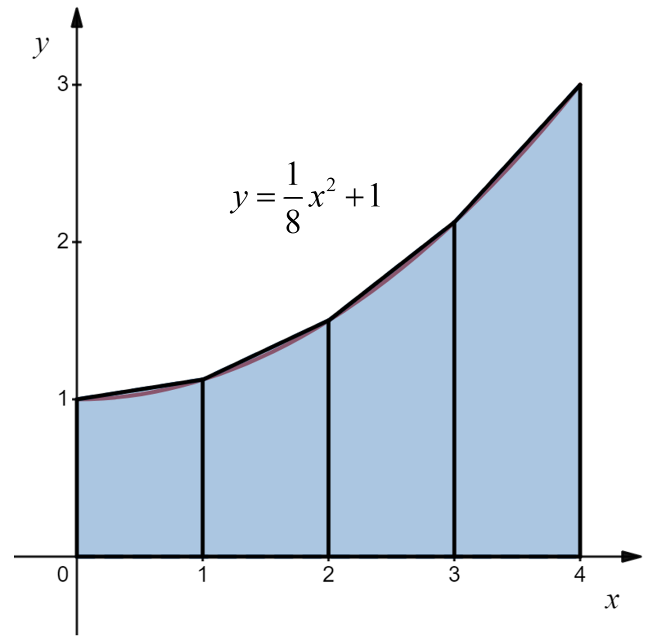
$$T_4 = \frac{1}{2}\left(1 + \frac{9}{8}\right)\Delta x + \frac{1}{2}\left(\frac{9}{8} + \frac{3}{2}\right)\Delta x + \frac{1}{2}\left(\frac{3}{2} + \frac{17}{8}\right)\Delta x + \frac{1}{2}\left(\frac{17}{8} + 3\right)\Delta x$$

$$T_4 = \frac{1}{2}\left(1+\frac{9}{8}\right)\Delta x + \frac{1}{2}\left(\frac{9}{8}+\frac{3}{2}\right)\Delta x + \frac{1}{2}\left(\frac{3}{2}+\frac{17}{8}\right)\Delta x + \frac{1}{2}\left(\frac{17}{8}+3\right)\Delta x$$

$$T_4 = \frac{\Delta x}{2}\left(1+\frac{9}{8}+\frac{9}{8}+\frac{3}{2}+\frac{3}{2}+\frac{17}{8}+\frac{17}{8}+3\right)$$

$$T_4 = \frac{\Delta x}{2}\left(1+2\cdot\frac{9}{8}+2\cdot\frac{3}{2}+2\cdot\frac{17}{8}+3\right)$$

$$T_4 = \frac{\Delta x}{2}\left(f(x_0)+2\cdot f(x_1)+2\cdot f(x_2)+2\cdot f(x_3)+f(x_4)\right) \qquad h = \Delta x$$



$$T_4 = \frac{1}{2}\left(\frac{27}{2}\right)$$

$$T_4 = \frac{27}{4}$$

$T_4 = 6.75$ As we calculated before.

Trapezoidal Rule

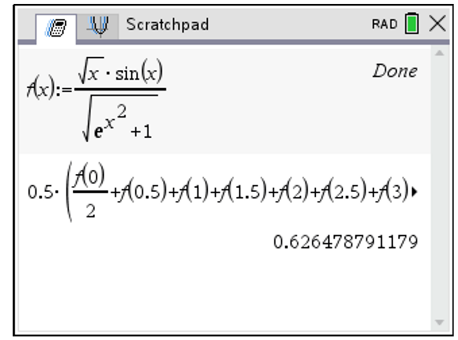
$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

where $\Delta x = (b - a)/n$ and $x_i = a + i\,\Delta x$.

$$T_n = \frac{\Delta x}{2}\left(y_0 + 2\cdot y_1 + 2\cdot y_2 + \cdots + 2\cdot y_{n-1} + y_n\right) = \Delta x\left(\frac{y_0}{2} + y_1 + y_2 + \cdots + y_{n-1} + \frac{y_n}{2}\right)$$

Example Use the Trapezodial Rule to approximate to six decimal places the integral

$$\int_0^4 \frac{\sqrt{x}\sin x}{\sqrt{e^{x^2}+1}}\,dx,\ n=8.$$



Solution

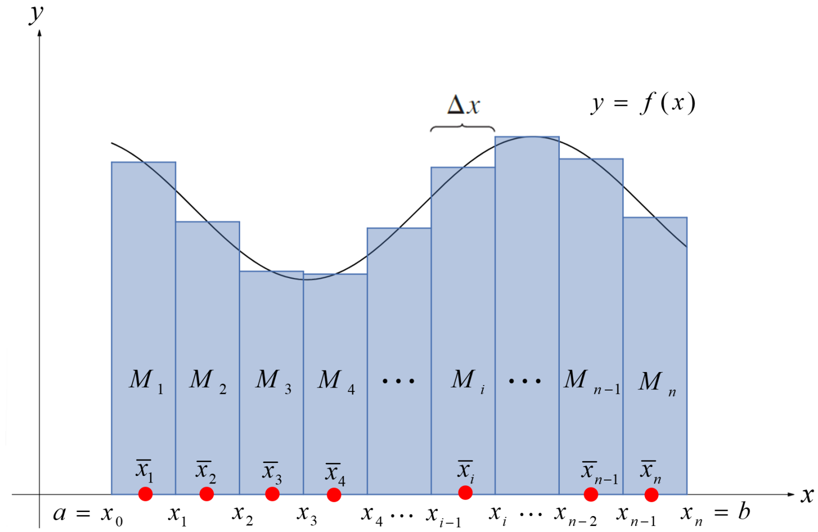
$$\Delta x=\frac{4-0}{8}=0.5 \qquad x_i=a+i\cdot\Delta x=0.5i$$

$$T_n=\frac{\Delta x}{2}\left(y_0+2\cdot y_1+2\cdot y_2+\cdots+2\cdot y_{n-1}+y_n\right)=\Delta x\left(\frac{y_0}{2}+y_1+y_2+\cdots+y_{n-1}+\frac{y_n}{2}\right)$$

$$T_8=0.5\cdot\left(\frac{f(0)}{2}+f(.5)+f(1)+f(1.5)+f(2)+f(2.5)+f(3)+f(3.5)+\frac{f(4)}{2}\right)$$

$$T_8\approx 0.626479$$

Recall Midpoint Rectangles



Using Midpoint Rectangles (midpoint grid or sample points) we get,

$M_n = (f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + \cdots + f(\bar{x}_i) + \cdots + f(\bar{x}_{n-1}) + f(\bar{x}_n)) \cdot \Delta x$

Using sigma notation we get,

$M_n = \sum_{i=1}^{n} f(\bar{x}_i) \cdot \Delta x$, sometimes called a midpoint sum.

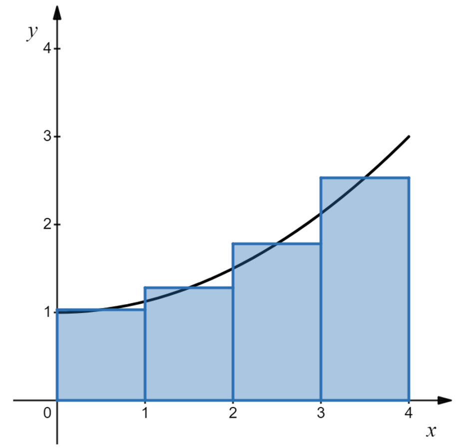
Where, $\Delta x = \dfrac{b-a}{n}$ and $\bar{x}_i = a + \left(i - \dfrac{1}{2}\right) \cdot \Delta x$, the midpoint grid or sample points. Remember i is counting each rectangle.

$M_n = \Delta x \, [f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$

Example Find the area under the curve $y = \frac{1}{8}x^2 + 1$ on $0 \le x \le 4$ with 4 rectangles.

Solution

$$\Delta x = \frac{4-0}{4} = 1 \qquad \bar{x}_i = a + \left(i - \frac{1}{2}\right) \cdot \Delta x = i - \frac{1}{2}$$

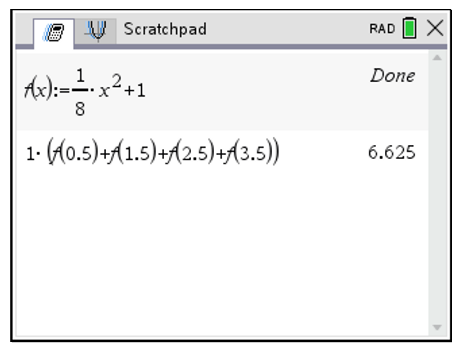


$$M_n = \Delta x\,[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$$

$$M_4 = 1 \cdot \left(f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + f(\bar{x}_4)\right)$$

$$M_4 = 1 \cdot \left(f(0.5) + f(1.5) + f(2.5) + f(3.5)\right)$$

$$M_4 = 6.625$$



0.625% error (too low)

The midpoint rule gives a closer approximation than the trapezoidal rule, but in the opposite direction.

Trapezoidal Rule:	6.750	1.25% error	(too high)
Midpoint Rule:	6.625	0.625% error	(too low)

Notice that the trapezoidal rule gives us an answer that has twice as much error as the midpoint rule, but in the opposite direction.

Ahhh!

If we use a weighted average:

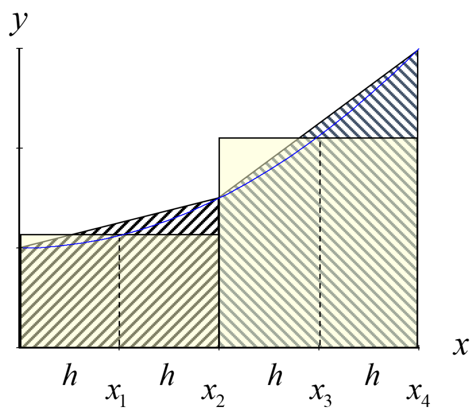
$$\frac{2(6.625)+6.750}{3}=6.\overline{6}$$

This is the exact answer!

Oooh!

Wow!

This weighted approximation gives us a closer approximation than the midpoint or trapezoidal rules.



Midpoint: $h = \Delta x$

$$M = 2\Delta x \cdot y_1 + 2\Delta x \cdot y_3 = 2\Delta x\left(y_1 + y_3\right)$$

Trapezoidal:

$$T = \frac{1}{2}\left(y_0 + y_2\right)2\Delta x + \frac{1}{2}\left(y_2 + y_4\right)2\Delta x$$

$$T = \Delta x\left(y_0 + y_2\right) + \Delta x\left(y_2 + y_4\right)$$

$$T = \Delta x\left(y_0 + 2y_2 + y_4\right)$$

$$\frac{2M+T}{3} = \frac{1}{3}\Big[\underbrace{4\Delta x\left(y_1 + y_3\right)}_{\text{twice midpoint}} + \underbrace{\Delta x\left(y_0 + 2y_2 + y_4\right)}_{\text{trapezoidal}}\Big]$$

$$= \frac{\Delta x}{3}\left[4y_1 + 4y_3 + y_0 + 2y_2 + y_4\right]$$

$$= \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$$

Simpson's Rule

$$S_n = \frac{\Delta x}{3}\left(y_0 + 4\cdot y_1 + 2\cdot y_2 + 4\cdot y_3 + \cdots + 2\cdot y_{n-2} + 4\cdot y_{n-1} + y_n\right)$$

Simpson's Rule

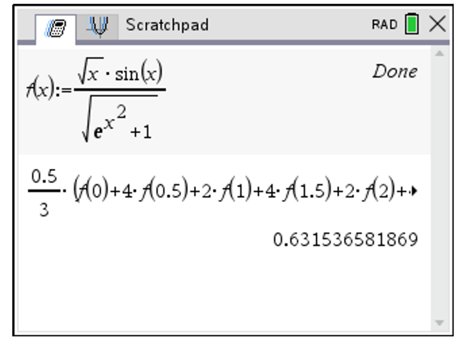
$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots$$

$$+ 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where n is even and $\Delta x = (b - a)/n$ and $x_i = a + i\,\Delta x$.

Example Use the Simpson's Rule to approximate to six decimal places the integral

$$\int_0^4 \frac{\sqrt{x}\sin x}{\sqrt{e^{x^2}+1}}\,dx,\ n=8.$$



Solution

$$\Delta x=\frac{4-0}{8}=0.5 \qquad x_i=a+i\cdot\Delta x=0.5i$$

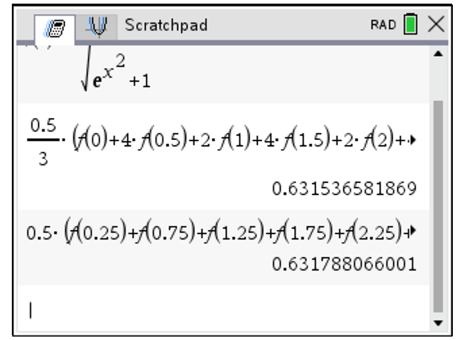
$$S_n=\frac{\Delta x}{3}\left(y_0+4\cdot y_1+2\cdot y_2+4\cdot y_3+\cdots+2\cdot y_{n-2}+4\cdot y_{n-1}+y_n\right)$$

$$S_8=\frac{0.5}{3}\cdot(f(0)+4f(.5)+2f(1)+4f(1.5)+2f(2)+4f(2.5)+2f(3)+4f(3.5)+f(4))$$

$$S_8\approx 0.631537$$

Example Use the Midpoint Sum to approximate to six decimal places the integral

$$\int_0^4 \frac{\sqrt{x}\sin x}{\sqrt{e^{x^2}+1}}\,dx,\;\; n = 8.$$



Solution

$$\Delta x = \frac{4-0}{8} = 0.5 \qquad \bar{x}_i = a + \left(i - \frac{1}{2}\right)\cdot \Delta x = \left(i - \frac{1}{2}\right)\cdot (0.5)$$

$$M_n = \Delta x\,[\,f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$$

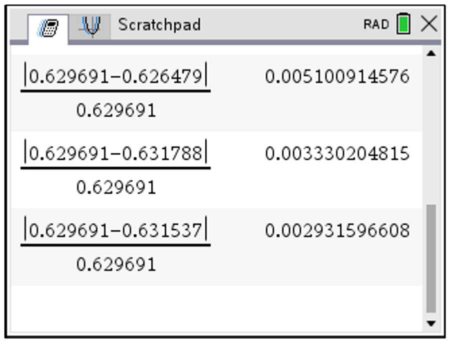
$$M_8 = 0.5\cdot(f(0.25)+f(0.75)+f(1.25)+f(1.75)+f(2.25)+f(2.75)+f(3.25)+f(3.75))$$

$$M_8 \approx 0.631788$$

Example Use a Computer Algebra System to approximate to six decimal places the integral

$$\int_0^4 \frac{\sqrt{x}\sin x}{\sqrt{e^{x^2}+1}}\,dx.$$

Solution



$CAS \approx 0.629691$

$T_8 \approx 0.626479$

$$\text{Relative Error} = \frac{|CAS - T_8|}{CAS} = 0.00510 = 0.510\%$$

$M_8 \approx 0.631788$

$$\text{Relative Error} = \frac{|CAS - M_8|}{CAS} = 0.00333 = 0.333\%$$

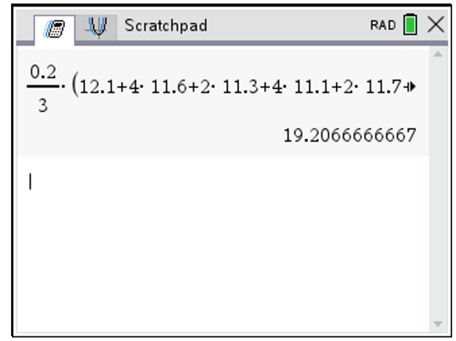
$S_8 \approx 0.631537$

$$\text{Relative Error} = \frac{|CAS - S_8|}{CAS} = 0.00293 = 0.293\%$$

Simpson's Rule Wins

Example A table of values of a function g is given. Use Simpson's Rule to estimate $\int_0^{1.6} g(x)dx$ to three decimal places of accuracy.

Solution



$$n = 8 \quad \Delta x = \frac{1.6-0}{8} = 0.2 \quad x_i = a + i \cdot \Delta x = 0.2i$$

$$\int_0^{1.6} g(x)\,dx \approx S_8$$

$$S_n = \frac{\Delta x}{3}\left(y_0 + 4 \cdot y_1 + 2 \cdot y_2 + 4 \cdot y_3 + \cdots + 2 \cdot y_{n-2} + 4 \cdot y_{n-1} + y_n\right)$$

$$S_8 = \frac{0.2}{3} \cdot [g(0) + 4g(0.2) + 2g(0.4) + 4g(0.6) + 2g(0.8) + 4g(1.0) + 2g(1.2) + 4g(1.4) + g(1.6)]$$

$$S_8 = \frac{0.2}{3} \cdot [12.1 + 4(11.6) + 2(11.3) + 4(11.1) + 2(11.7) + 4(12.2) + 2(12.6) + 4(13.0) + 13.2]$$

$$S_8 \approx 19.207$$