

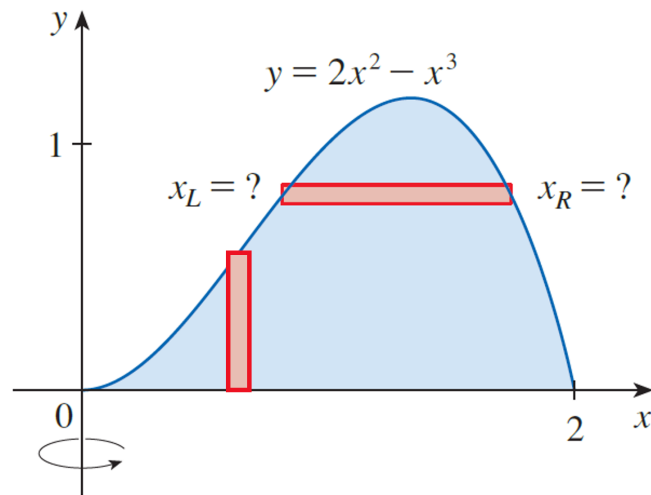
6.3 Volumes by Cylindrical Shells



Johann Peter Gustav Lejeune Dirichlet
1805 – 1859

Dirichlet proved the convergence of Fourier series, as well as developed the modern definition of a function, also he contributed to analytic number theory.

■ The Method of Cylindrical Shells



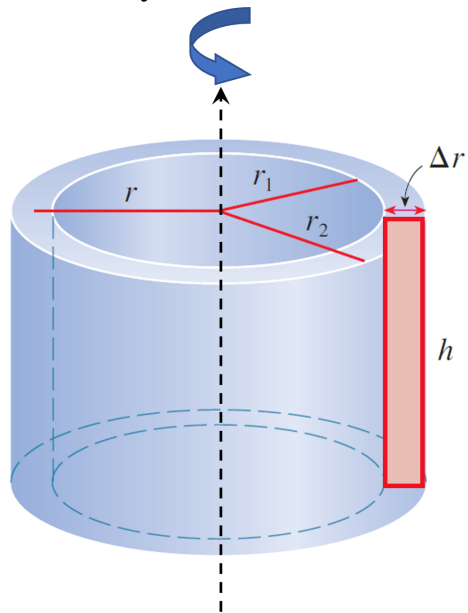
If we slice perpendicular to the y -axis, we get a washer. But to compute the inner radius and the outer radius of the washer, we have to solve the cubic equation $y = 2x^2 - x^3$ for x in terms of y ; that's not easy to do!

What if we take parallel cross-sections to the y -axis? What's the cross-section?

■ The Method of Cylindrical Shells

There is a method, called the *method of cylindrical shells*, that is easier to use in a case like the one shown in the previous slide.

Its volume V is calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder:



$$\begin{aligned}V &= V_2 - V_1 \\&= \pi r_2^2 h - \pi r_1^2 h = \pi(r_2^2 - r_1^2)h \\&= \pi(r_2 + r_1)(r_2 - r_1)h \\&= 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1)\end{aligned}$$

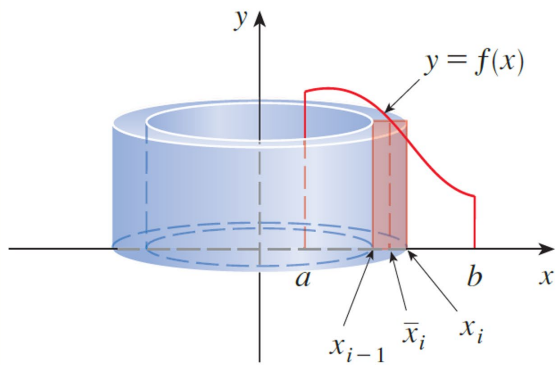
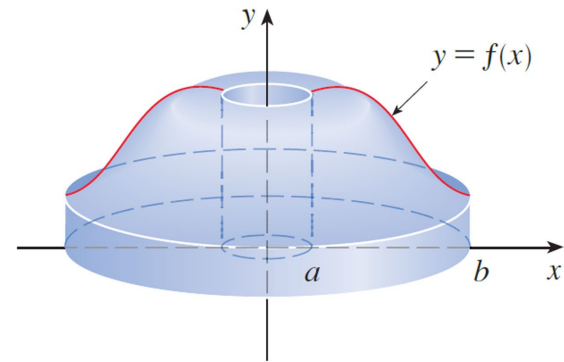
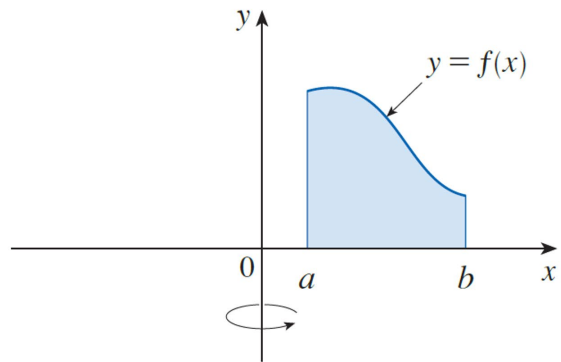
If we let $\Delta r = r_2 - r_1$ (the thickness of the shell)
and $r = \frac{1}{2}(r_2 + r_1)$ (the average radius of the shell)

then this formula for the volume of a cylindrical shell becomes

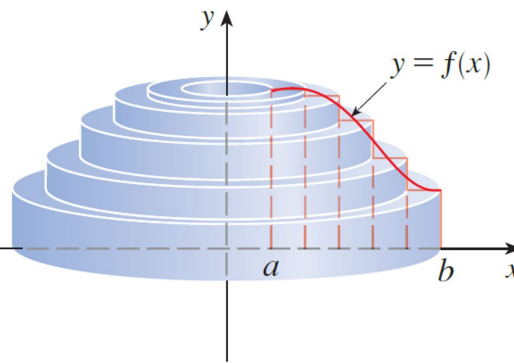
$$V = 2\pi r h \Delta r$$

$V =$ [circumference][height][thickness]

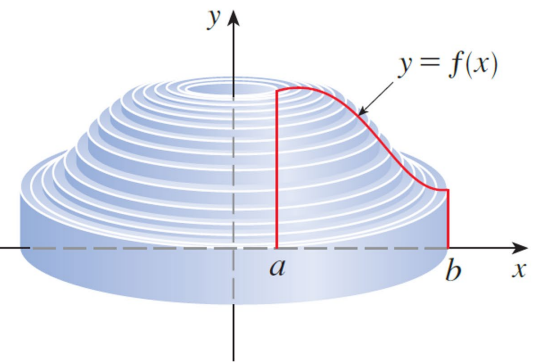
■ The Method of Cylindrical Shells



$$V = [\text{circumference}][\text{height}][\text{thickness}]$$



$$V_i = (2\pi\bar{x}_i)[f(\bar{x}_i)] \Delta x$$

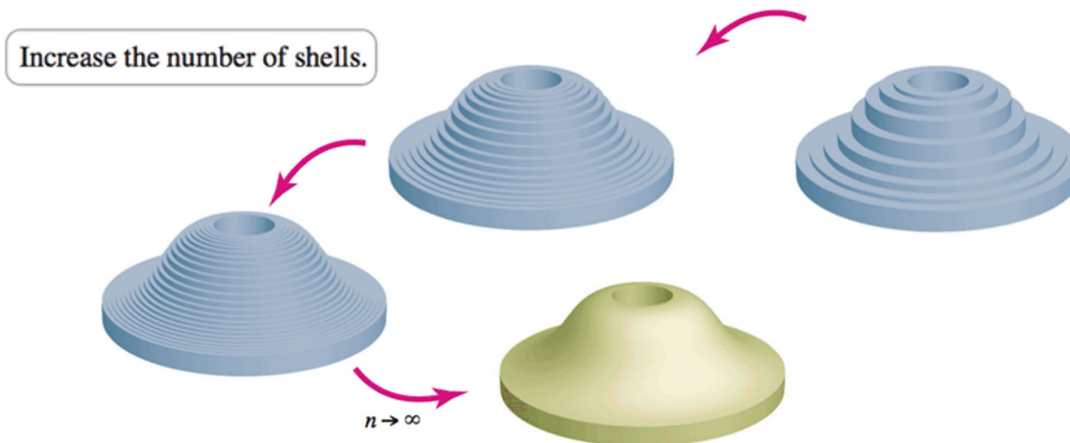


■ The Method of Cylindrical Shells

Therefore an approximation to the volume V of S is given by the sum of the volumes of these shells:

$$V \approx \sum_{i=1}^n V_i = \sum_{i=1}^n 2\pi \bar{x}_i f(\bar{x}_i) \Delta x$$

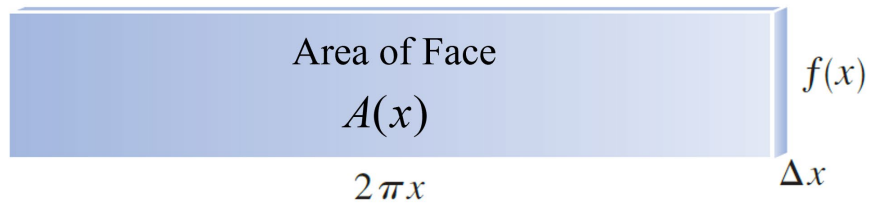
This approximation appears to become better as $n \rightarrow \infty$. But, from the definition of an integral, we know that



$$\lim_{n \rightarrow \infty} \sum_{i=1}^n 2\pi \bar{x}_i f(\bar{x}_i) \Delta x = \int_a^b 2\pi x f(x) dx$$

■ The Method of Cylindrical Shells

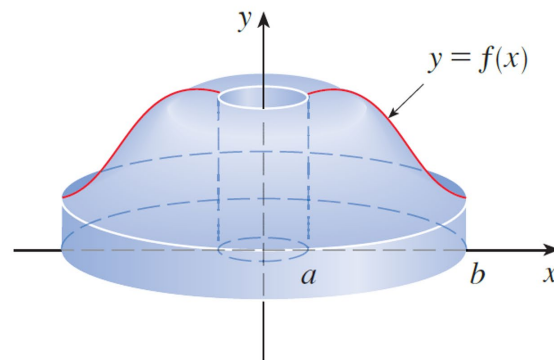
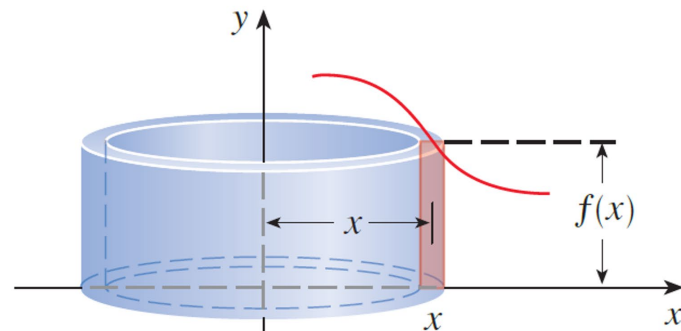
$$V = \int_a^b A(x) dx$$



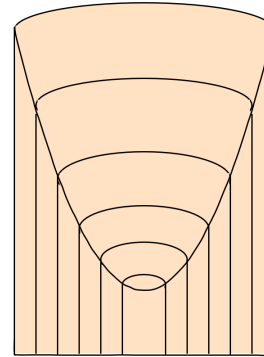
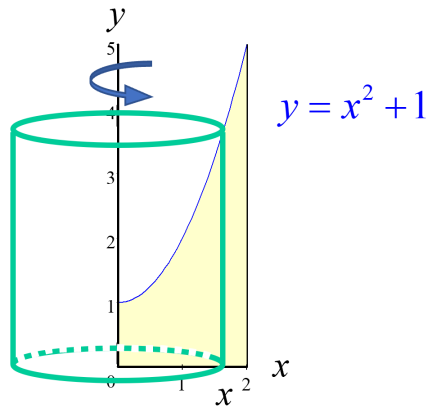
$$A(x) = 2\pi x \cdot f(x)$$

$$\text{Volume of Shell} = 2\pi x \cdot f(x) \cdot \Delta x$$

$$V = \int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \underbrace{[f(x)]}_{\text{height}} \underbrace{dx}_{\text{thickness}}$$



Example



cross section

This is called the **Shell method** because we use cylindrical shells.

Solution

The volume of the shell is given by:

$$\begin{aligned} V &= \text{Lateral surface area of cylinder} \cdot \text{thickness} \\ &= \text{circumference} \cdot \text{height} \cdot \text{thickness} \\ &= 2\pi r \cdot h \cdot dx \end{aligned}$$

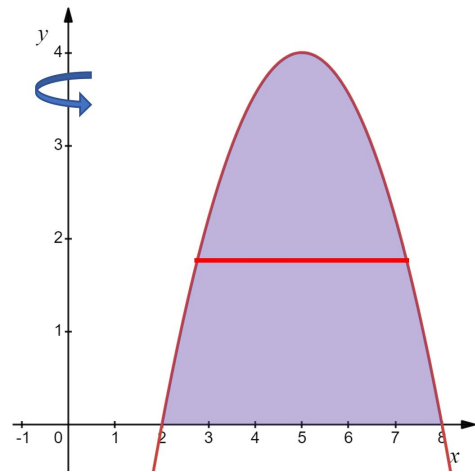
circumference

$$= \underbrace{2\pi x}_{\substack{\uparrow \\ r}} \underbrace{(x^2 + 1)}_{\substack{\uparrow \\ h}} \underbrace{dx}_{\substack{\uparrow \\ \text{thickness}}}$$

Now we bring out the Super Sum, the Great Accumulator

$$\begin{aligned} &\int_0^2 2\pi x(x^2 + 1) dx \\ &= 2\pi \int_0^2 (x^3 + x) dx \\ &= 2\pi \left[\frac{1}{4}x^4 + \frac{1}{2}x^2 \right]_0^2 \\ &= 2\pi [4 + 2] = 12\pi \text{ units}^3 \end{aligned}$$

Example Find the volume generated when this shape is revolved about the y -axis.



$$f(x) = -\frac{4}{9}(x^2 - 10x + 16)$$

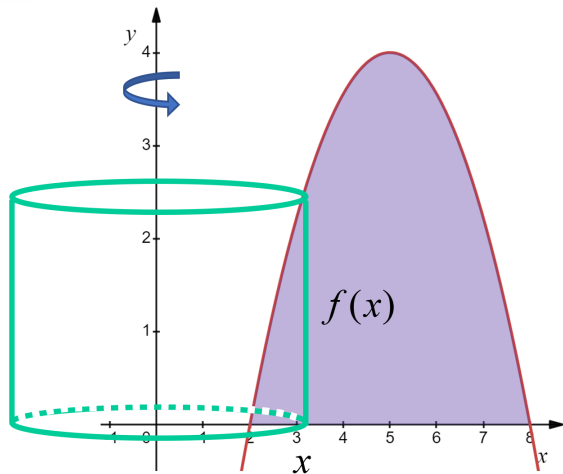


Solution

Washers? $V = \pi \int_c^d (r_o^2 - r_i^2) dy$

It's not easy to solve for x , so we do not want to use washers to find the volume.

Example Find the volume generated when this shape is revolved about the y -axis.



$$f(x) = -\frac{4}{9}(x^2 - 10x + 16)$$

Solution

If we take a vertical rectangle and revolve it about the y -axis we get a cylindrical shell.

$$V = \int_a^b A(x) dx$$

Shell method:

Lateral surface area of cylinder

$$A(x) = \text{circumference} \cdot \text{height}$$

$$A(x) = 2\pi r \cdot h$$

$$A(x) = 2\pi x \cdot f(x)$$

$$\text{Volume of the shell} = 2\pi x \cdot f(x) \cdot dx$$

$$V = \int_a^b A(x) dx$$

$$V = \int_2^8 \underbrace{2\pi x}_{\substack{\text{circumference} \\ r}} \left[\underbrace{-\frac{4}{9}(x^2 - 10x + 16)}_{h} \right] \underbrace{dx}_{\text{thickness}} = 160\pi \text{ units}^3$$

$A(x)$

Example

Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis.

$$x = 2y^2, y \geq 0, x = 2; \text{ about } y = 2$$

Solution

$$A(y) = 2\pi r \cdot h$$

$$A(y) = 2\pi(2 - y) \cdot (2 - 2y^2)$$

$$V = \int_c^d A(y) dy$$

$$V = \int_0^1 2\pi(2 - y)(2 - 2y^2) dy$$

$$= 4\pi \int_0^1 (2 - y)(1 - y^2) dy$$

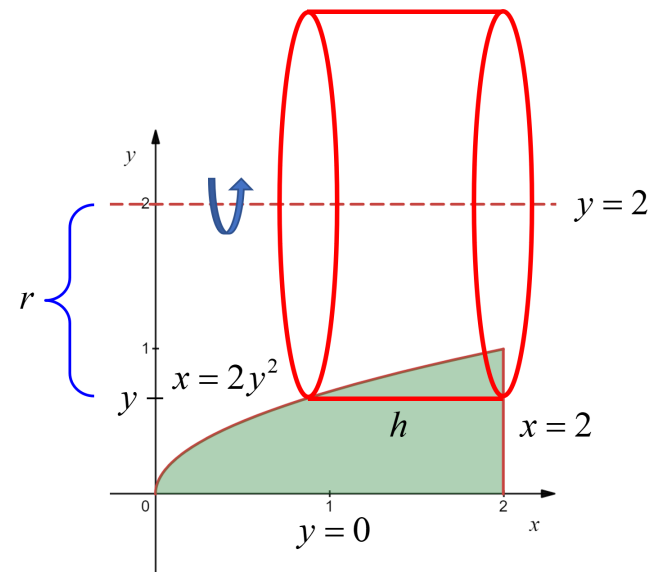
$$= 4\pi \int_0^1 (y^3 - 2y^2 - y + 2) dy$$

$$= 4\pi \left[\frac{1}{4}y^4 - \frac{2}{3}y^3 - \frac{1}{2}y^2 + 2y \right]_0^1$$

$$= 4\pi \left(\frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2 \right)$$

$$= 4\pi \left(\frac{13}{12} \right)$$

$$= \frac{13\pi}{3} \text{ units}^3$$



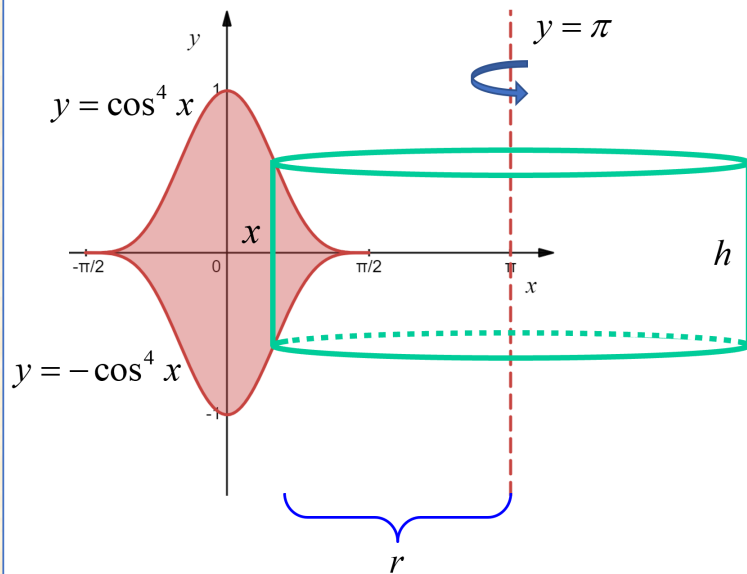
Example

(a) Set up an integral for the volume of the solid obtained by rotating the region bounded by the given curve about the specified axis.

T

(b) Use a calculator or computer to evaluate the integral correct to three decimal places.

$$y = \cos^4 x, y = -\cos^4 x, -\pi/2 \leq x \leq \pi/2; \text{ about } x = \pi$$

Solution

$$(a) A(x) = 2\pi r \cdot h$$

$$A(x) = 2\pi(\pi - x) \cdot (\cos^4 x - (-\cos^4 x))$$

$$V = \int_a^b A(x) dx$$

$$V = 2\pi \int_{-\pi/2}^{\pi/2} (\pi - x) [\cos^4 x - (-\cos^4 x)] dx$$

$$= 4\pi \int_{-\pi/2}^{\pi/2} (\pi - x) \cos^4 x dx$$

Example

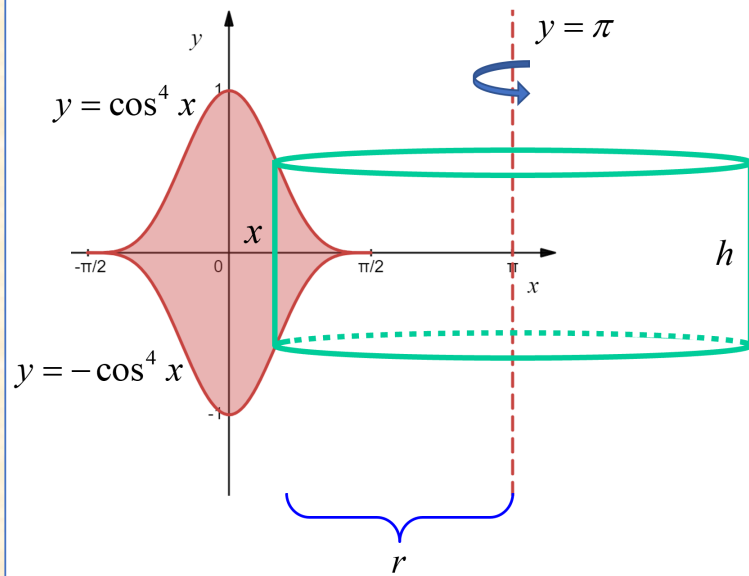
(a) Set up an integral for the volume of the solid obtained by rotating the region bounded by the given curve about the specified axis.

T

(b) Use a calculator to evaluate the integral correct to three decimal places.

$$y = \cos^4 x, y = -\cos^4 x, -\pi/2 \leq x \leq \pi/2; \text{ about } x = \pi$$

Solution



(b)
$$V = 4\pi \int_{-\pi/2}^{\pi/2} (\pi - x) \cos^4 x \, dx$$

$$V = \frac{3\pi^3}{2} \text{ units}^3 \approx 46.509 \text{ units}^3$$

