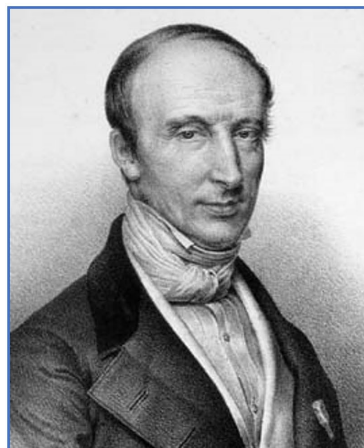




6.2

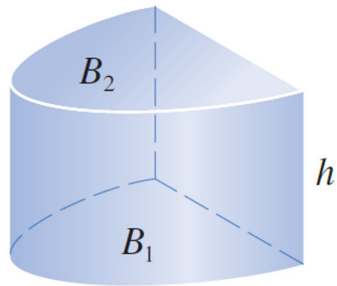
Volumes



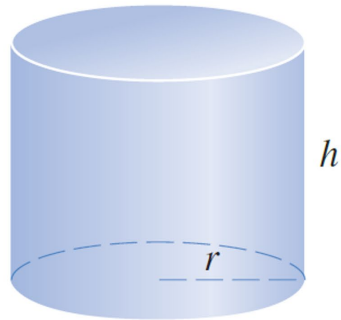
Augustin Louis Cauchy
1789 – 1857

Cauchy pioneered the study of analysis, both real and complex, and the theory of permutation groups. He also researched in convergence and divergence of infinite series, differential equations, determinants, probability and mathematical physics.

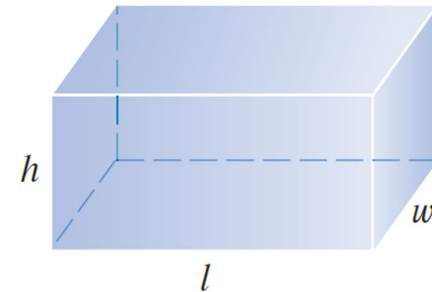
■ Definition of Volume



(a) Cylinder $V = Ah$



(b) Circular cylinder $V = \pi r^2 h$



(c) Rectangular box $V = lwh$

■ Definition of Volume

We start by intersecting S with a plane and obtaining a plane region that is called a **cross-section** of S . Let $A(x)$ be the area of the cross-section of S in a plane P_x perpendicular to the x -axis and passing through the point x , where $a \leq x \leq b$. (See Figure 2. Think of slicing S with a knife through x and computing the area of this slice.) The cross-sectional area $A(x)$ will vary as x increases from a to b .

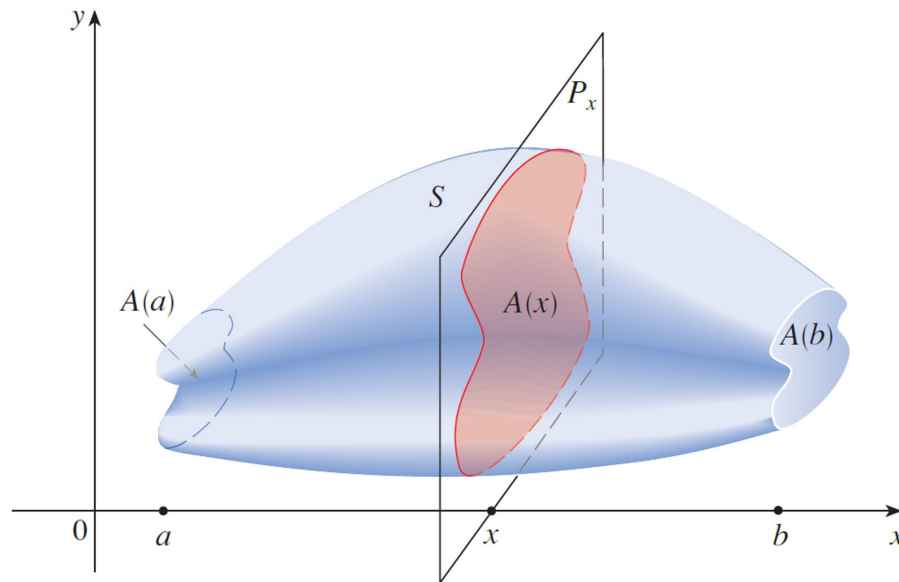
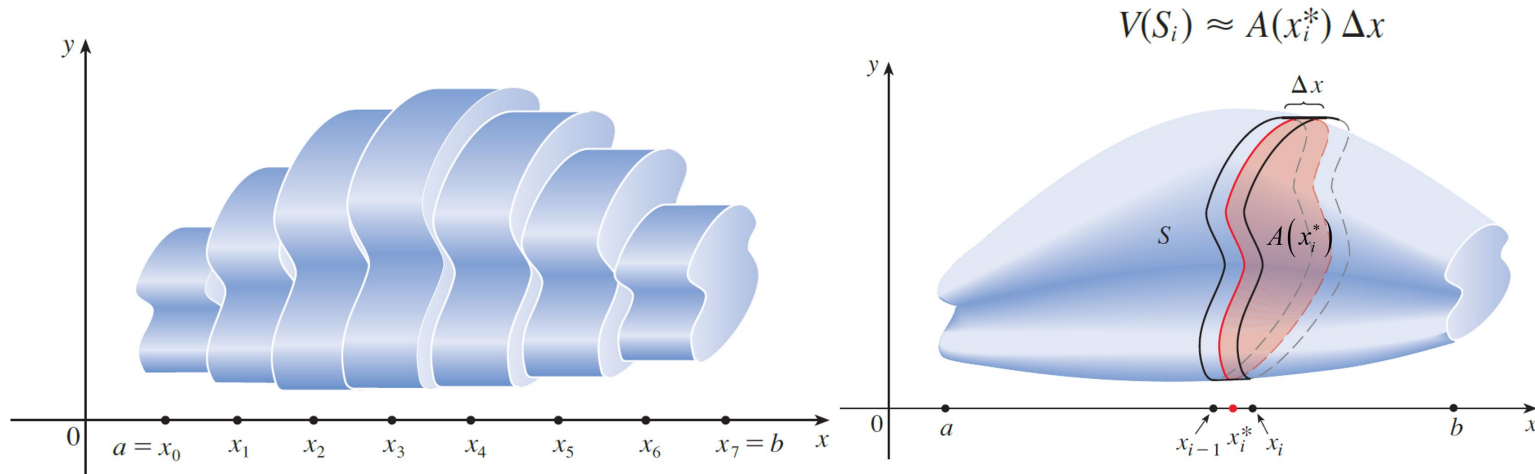


FIGURE 2

■ Definition of Volume

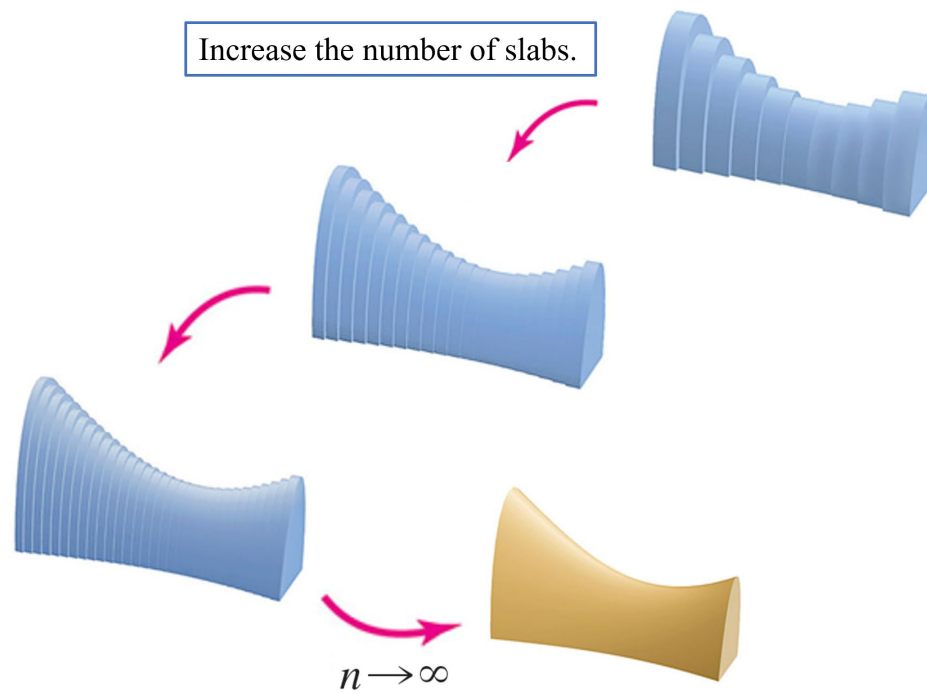
Let's divide S into n "slabs" of equal width Δx by using the planes P_{x_1}, P_{x_2}, \dots to slice the solid. (Think of slicing a loaf of bread.) If we choose sample points x_i^* in $[x_{i-1}, x_i]$, we can approximate the i th slab S_i (the part of S that lies between the planes $P_{x_{i-1}}$ and P_{x_i}) by a cylinder with base area $A(x_i^*)$ and "height" Δx .



Adding the volumes of these slabs, we get an approximation to the total volume (that is, what we think of intuitively as the volume):

$$V \approx \sum_{i=1}^n A(x_i^*) \Delta x$$

■ Definition of Volume

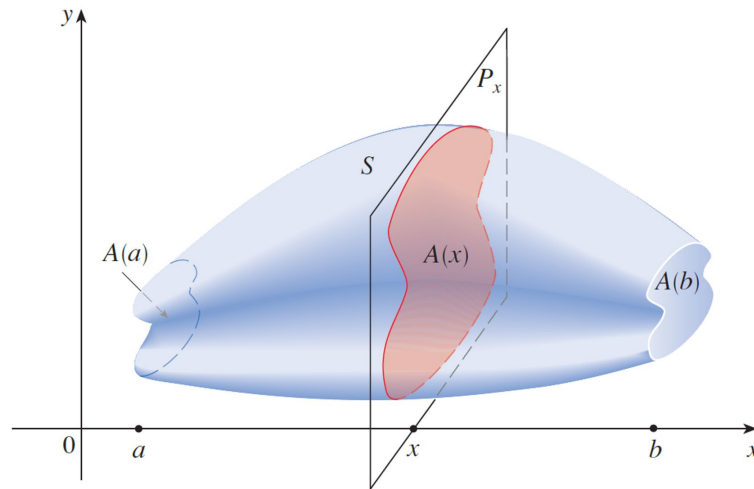


We recognize the limit of a Riemann sum $V = \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i^*) \Delta x$

■ Definition of Volume

Definition of Volume Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x , through x and perpendicular to the x -axis, is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i^*) \Delta x = \int_a^b A(x) dx$$



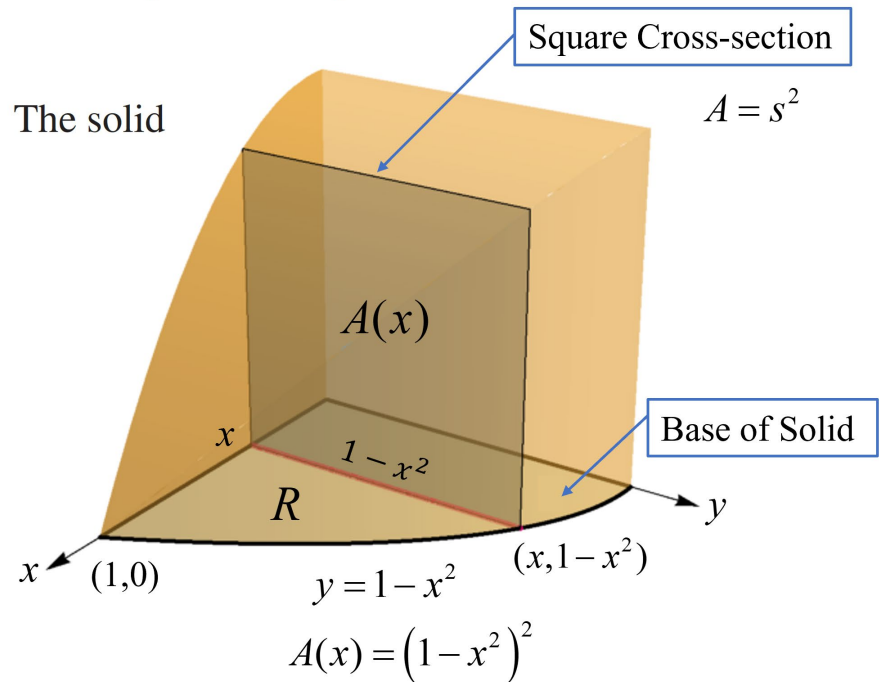
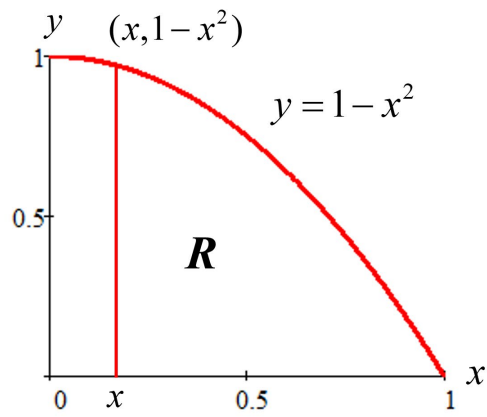
Example

Volume of a Parabolic Cube

Let R be the region in the first quadrant bounded by the coordinate axes and the curve $y = 1 - x^2$. A solid has a base formed by R , and cross sections through the solid perpendicular to the base and parallel to the y -axis are squares. Find the volume of the solid.

Solution

The base

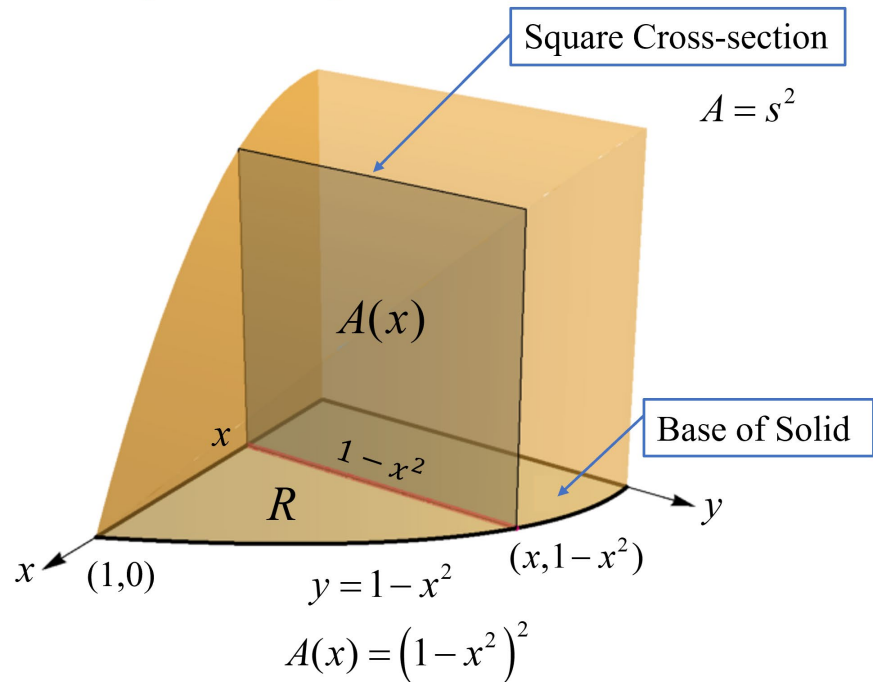


Example**Volume of a Parabolic Cube**

Let R be the region in the first quadrant bounded by the coordinate axes and the curve $y = 1 - x^2$. A solid has a base formed by R , and cross sections through the solid perpendicular to the base and parallel to the y -axis are squares. Find the volume of the solid.

Solution

$$\begin{aligned} V &= \int_0^1 A(x) dx \\ &= \int_0^1 (1 - x^2)^2 dx \\ &= \int_0^1 (1 - 2x^2 + x^4) dx \\ &= \frac{8}{15} \text{ units}^3 \end{aligned}$$

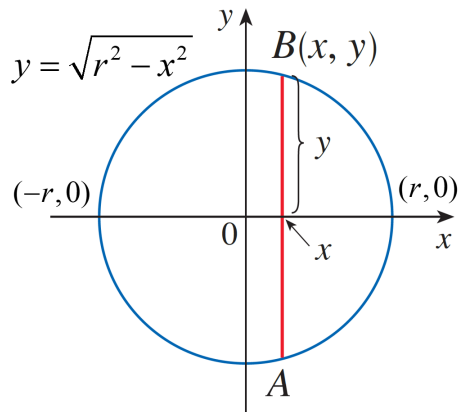


Example

Find the volume of the solid below with a circular base of radius r . For this solid, each cross section made with a plane P_x perpendicular to the x -axis is an equilateral triangle.

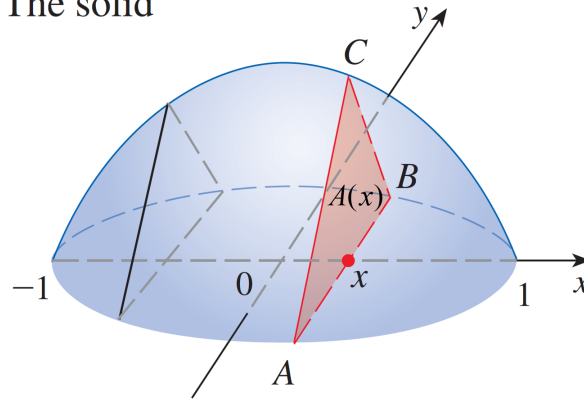
Solution

The base

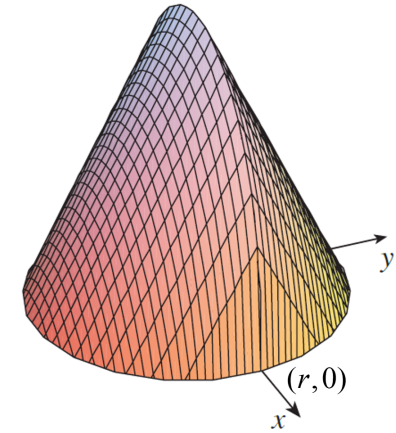
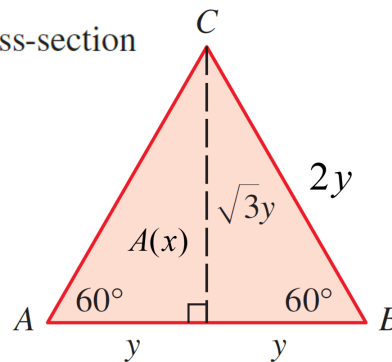


$$\begin{aligned}x^2 + y^2 &= r^2 \\y^2 &= r^2 - x^2 \\y &= \pm\sqrt{r^2 - x^2}\end{aligned}$$

The solid



A cross-section



$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2} \cdot 2y \cdot \sqrt{3}y$$

$$A(x) = \frac{1}{2} \cdot 2\sqrt{r^2 - x^2} \cdot \sqrt{3} \cdot \sqrt{r^2 - x^2}$$

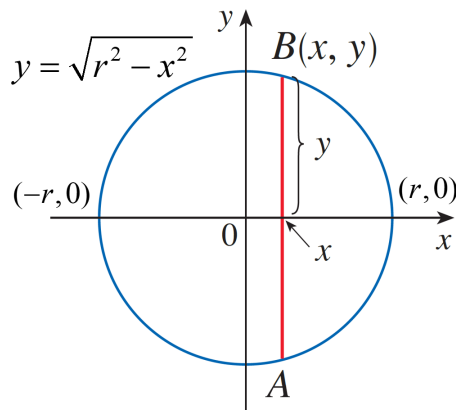
$$A(x) = \sqrt{3} \cdot (r^2 - x^2)$$

Example

Find the volume of the solid below with a circular base or radius r . For this solid, each cross section made with a plane P_x perpendicular to the x -axis is an equilateral triangle.

Solution

Its base

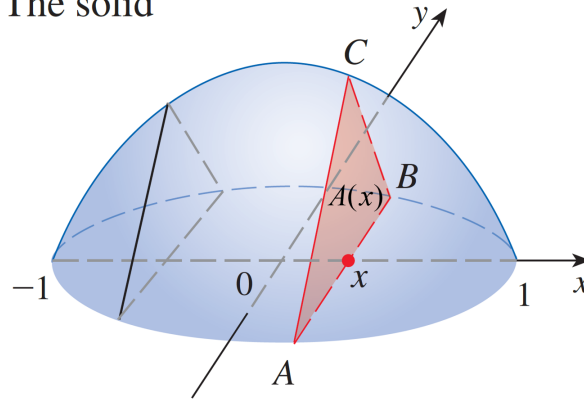


$$x^2 + y^2 = r^2$$

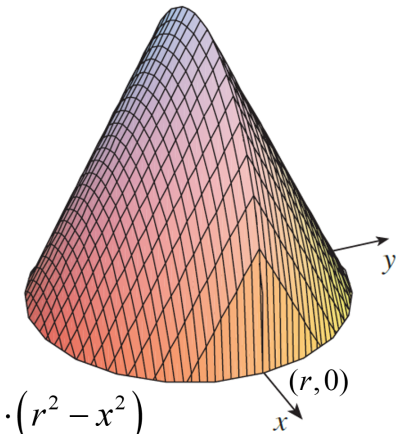
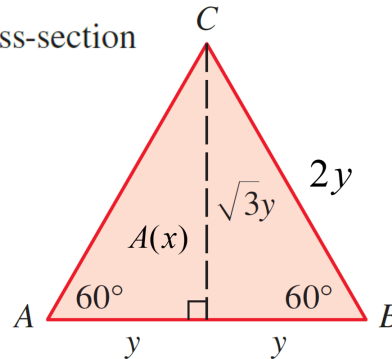
$$y^2 = r^2 - x^2$$

$$y = \pm\sqrt{r^2 - x^2}$$

The solid



A cross-section



$$A(x) = \sqrt{3} \cdot (r^2 - x^2)$$

$$V = \int_{-r}^r A(x) dx$$

(Symmetry) $= 2 \cdot \int_0^r \sqrt{3} \cdot (r^2 - x^2) dx$

$$= 2\sqrt{3} \cdot \left[r^2 x - \frac{1}{3} x^3 \right]_0^r$$

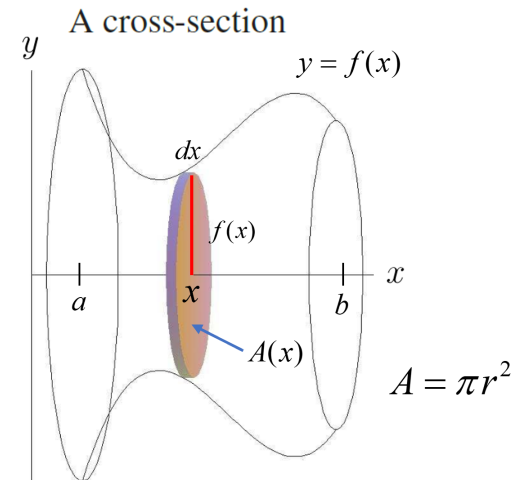
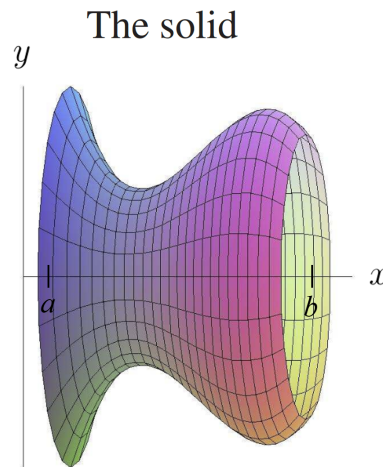
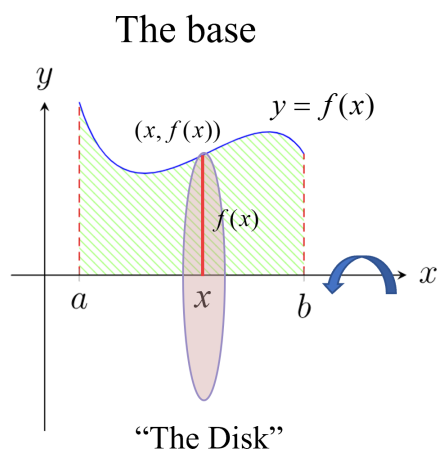
$$= 2\sqrt{3} \cdot \left(r^3 - \frac{1}{3} r^3 \right)$$

$$= 2\sqrt{3} \cdot \left(\frac{2}{3} r^3 \right) = \frac{4\sqrt{3}}{3} r^3 \text{ units}^3$$

■ Volumes of Solids of Revolution

If we revolve a region about a line, we obtain a **solid of revolution**. In the following examples we see that for such a solid, cross-sections perpendicular to the axis of rotation are circular.

■ Disk Method



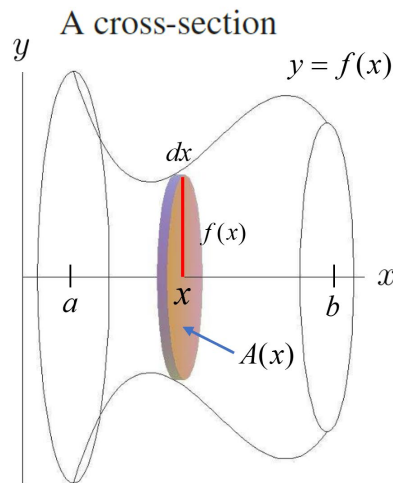
Area of the Disk $A(x) = \pi [f(x)]^2$

$$V = \int_a^b A(x) dx = \int_a^b \pi [f(x)]^2 dx = \pi \int_a^b [f(x)]^2 dx$$

Disk Method about the x -Axis

Let f be continuous with $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the graph of f , the x -axis, and the lines $x = a$ and $x = b$ is revolved about the x -axis, the volume of the resulting solid of revolution is

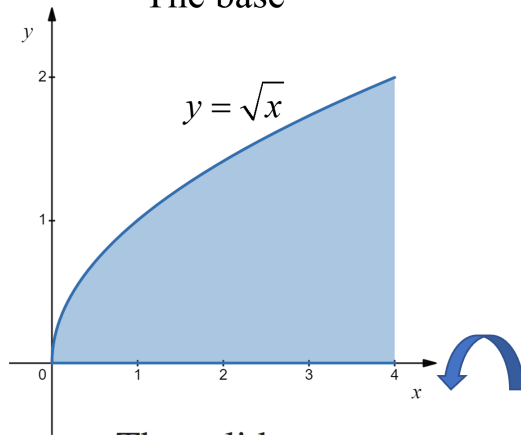
$$V = \int_a^b A(x) \, dx = \int_a^b \pi [f(x)]^2 \, dx = \pi \int_a^b [f(x)]^2 \, dx$$



Area of the Disk $A(x) = \pi [f(x)]^2$

Example

The base

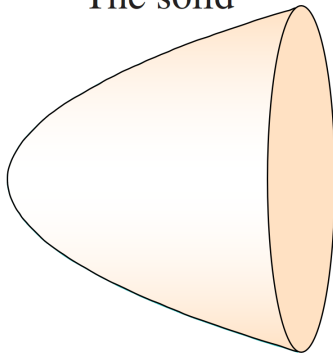


Suppose I start with this curve.

My boss at the ACME Rocket Company has assigned me to build a nose cone in this shape.

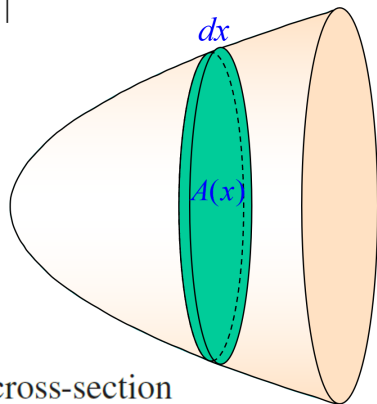
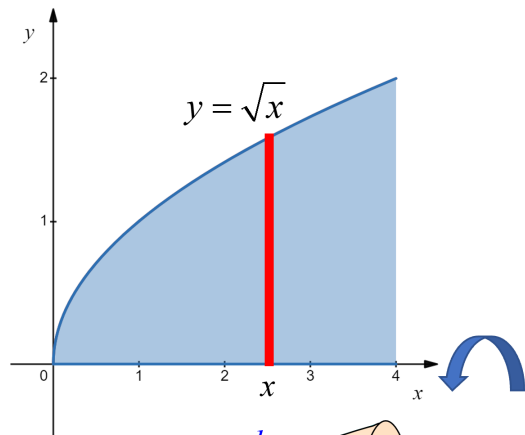
I revolve the curve about x -axis.

The solid



So, what is the volume of the nose cone?

Solution



A cross-section

How could we find the volume of the cone? Disk Method!

Draw the radius of the disk, a very thin rectangle.

Rotate about the x -axis.

A cross-section is a disk.

The volume of this disk is the area the face times thickness:

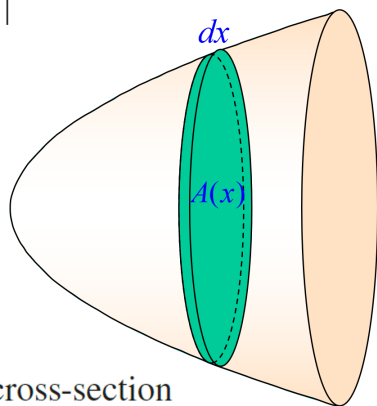
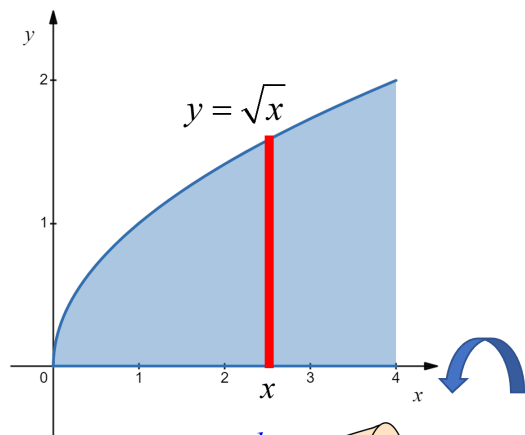
$$\pi r^2 \cdot \text{the thickness}$$
$$\pi (\sqrt{x})^2 dx \quad A(x) dx$$

In this case:

$r =$ the y value of the function

thickness = a small change in x
= dx

Solution



A cross-section

If we add the volumes, we get:

$$\begin{aligned} V &= \int_a^b A(x) \, dx \\ &= \int_0^4 \pi (\sqrt{x})^2 \, dx \\ &= \pi \int_0^4 x \, dx \\ &= \pi \left[\frac{1}{2} x^2 \right]_0^4 \\ &= 8\pi \text{ units}^3 \end{aligned}$$

In summary, if the bounded region is rotated about the x -axis, and the plane perpendicular to the x -axis intersects the region with circular disks, then the formula is:

$$V = \int_a^b \underbrace{\pi [f(x)]^2}_{\text{Area of Face}} \underbrace{dx}_{\text{Thickness}}$$

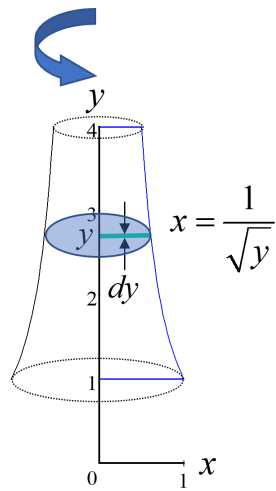
$$V = \int_a^b \underbrace{\pi [\text{Radius}]^2}_{\text{Volume of the Disk}} \cdot \text{Thickness}$$

A shape rotated about the y -axis would be:

$$V = \int_c^d \pi \cdot [f(y)]^2 dy$$

Example The region between the curve $x = \frac{1}{\sqrt{y}}$, $1 \leq y \leq 4$ and the y -axis is revolved about the y -axis. Find the volume.

Solution



We use a horizontal disk.

The thickness is dy .

The radius is the x value of the function $x = \frac{1}{\sqrt{y}}$.

$$V = \int_a^b A(x) dx$$

$$V = \int_1^4 \underbrace{\pi \left(\frac{1}{\sqrt{y}} \right)^2}_{\text{volume of disk}} dy$$

$$= \int_1^4 \pi \frac{1}{y} dy$$

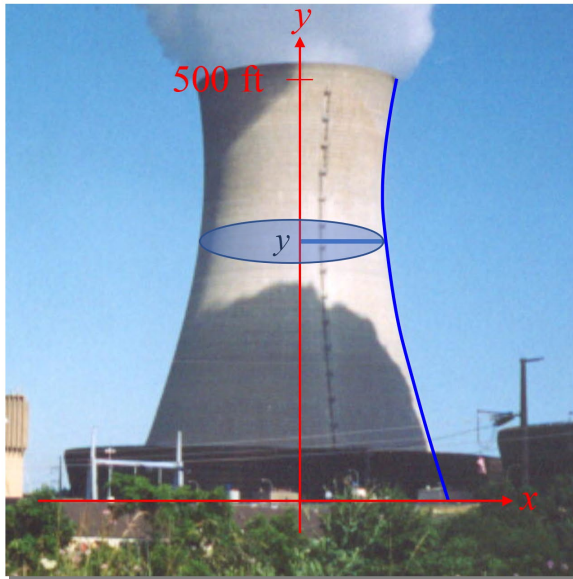
$$= \pi [\ln y]_1^4$$

$$= \pi (\ln 4 - \ln 1)$$

$$= \pi \ln 2^2$$

$$= 2\pi \ln 2 \text{ units}^3$$

Example



The natural draft cooling tower shown at left is about 500 feet high and its shape can be approximated by the graph of this equation revolved about the y -axis:

$$x = .000574y^2 - .439y + 185$$

Find the total of the cooling tower.

Solution

The volume can be calculated using the disk method with a horizontal disk.

$$V = \int_c^d A(y) dy$$

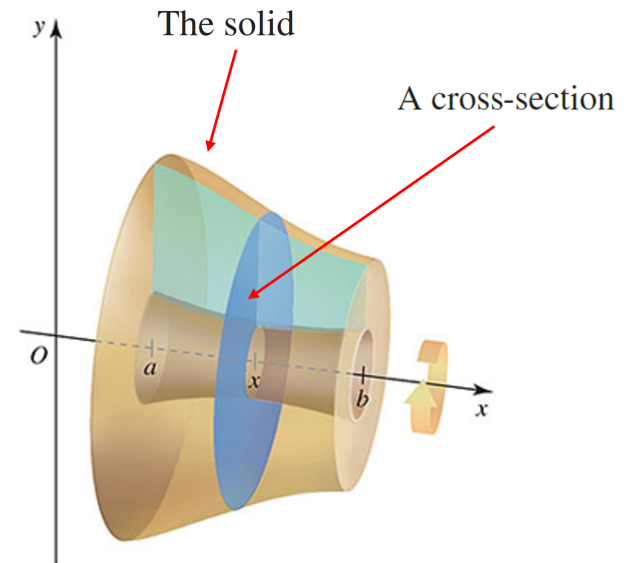
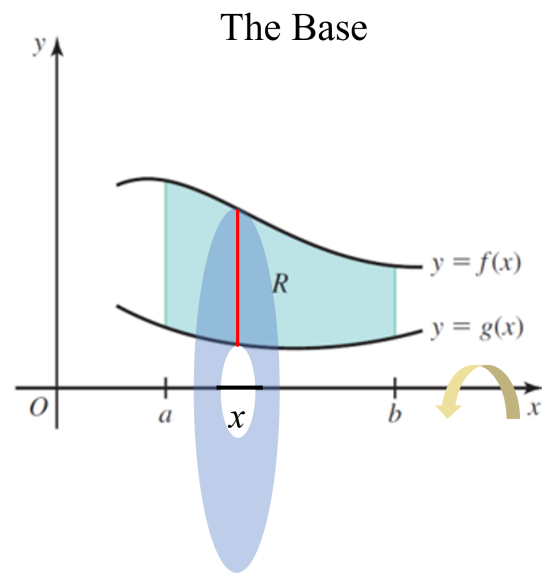
$$V = \int_c^d \pi \cdot [f(y)]^2 dy$$

$$V = \pi \int_0^{500} (.000574y^2 - .439y + 185)^2 dy$$

$$\stackrel{CAS}{\approx} 24,700,000 \text{ ft}^3$$

Washer Method

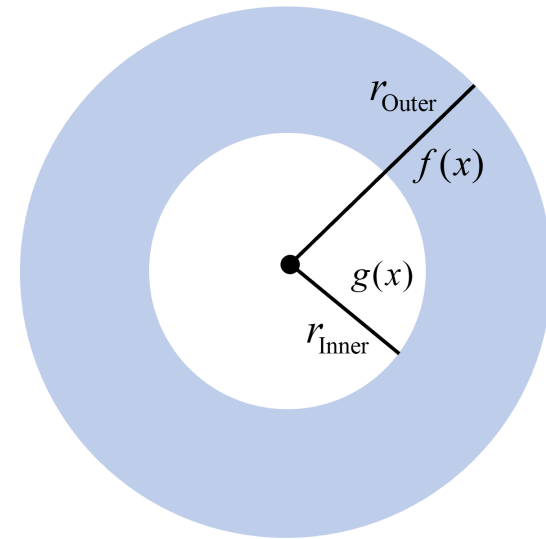
In the following examples we see that some solids of revolution have a hollow core surrounding the axis of revolution.



A disk with a hole in it, we will call the cross-section a washer.

Washer Method

$$A(x) = \pi r_{\text{Outer}}^2 - \pi r_{\text{Inner}}^2 = \pi (r_{\text{Outer}}^2 - r_{\text{Inner}}^2) = \pi (f(x)^2 - g(x)^2)$$



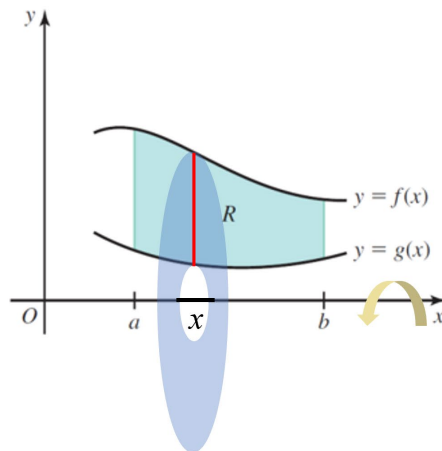
$$\text{Volume of Washer} = \underbrace{\pi \left[(\text{OUTER RADIUS})^2 - (\text{INNER RADIUS})^2 \right]}_{\text{Area of Face } A(x)} \cdot \underbrace{\text{Thickness}}_{dx}$$

Washer Method about the x -Axis

Let f and g be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let R be the region bounded by the curves $y = f(x)$ and $y = g(x)$, and the lines $x = a$ and $x = b$. When R is revolved about the x -axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi (f(x)^2 - g(x)^2) dx.$$

$$V = \pi \int_a^b [(\text{OUTER RADIUS})^2 - (\text{INNER RADIUS})^2] \cdot \text{thickness}$$



Example

Write an and evaluate an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

$$y = x^3, y = \sqrt{x}; \text{ about the } x\text{-axis}$$

Solution

A cross-section is a washer with inner radius x^3 and outer radius \sqrt{x} , so its area is

$$A(x) = \pi (r_{\text{Outer}}^2 - r_{\text{Inner}}^2) = \pi (f(x)^2 - g(x)^2)$$

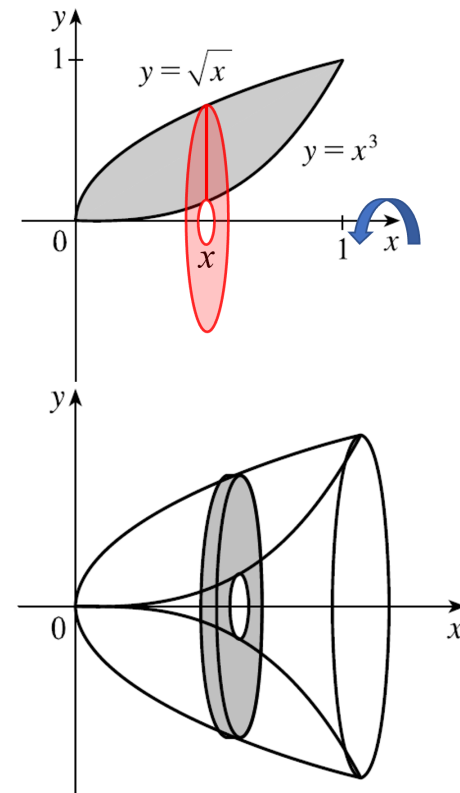
$$A(x) = \pi \left[(\sqrt{x})^2 - (x^3)^2 \right] = \pi (x - x^6)$$

$$V = \int_a^b A(x) dx$$

$$V = \int_0^1 \pi (x - x^6) dx$$

$$= \pi \left[\frac{1}{2}x^2 - \frac{1}{7}x^7 \right]_0^1$$

$$= \pi \left[\left(\frac{1}{2} - \frac{1}{7} \right) - 0 \right] = \frac{5}{14} \pi \text{ units}^3$$

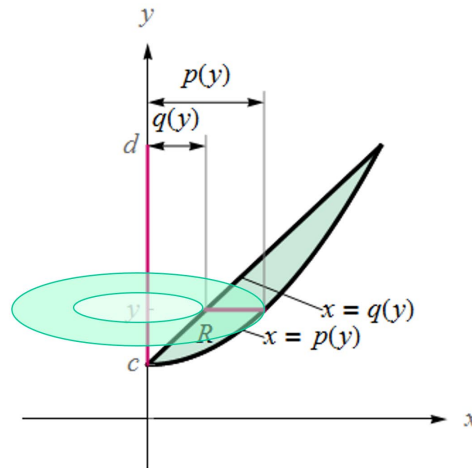


Revolving About the y -Axis

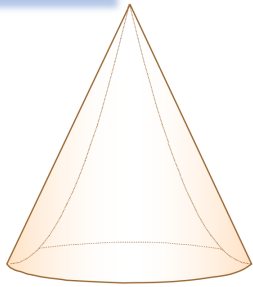
Disk and Washer Methods About the y -Axis

Let p and q be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let R be the region bounded by the curves $x = p(y)$ and $x = q(y)$, and the lines $y = c$ and $y = d$. When R is revolved about the y -axis, the volume of the resulting solid of revolution is given by

$$V = \int_c^d \pi (p(y)^2 - q(y)^2) dy.$$



Example



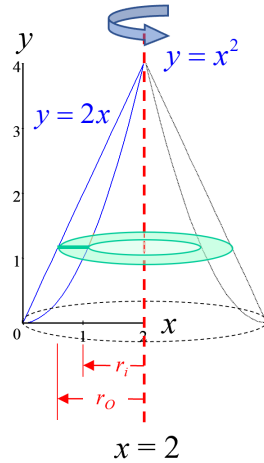
$$y = x^2 \quad y = 2x$$

$$x = \sqrt{y} \quad x = \frac{y}{2}$$

$$V = \int_c^d A(y) dy$$

$$V = \pi \int_0^4 (r_o^2 - r_i^2) dy$$

$$= \pi \int_0^4 \left[\left(2 - \frac{y}{2} \right)^2 - (2 - \sqrt{y})^2 \right] dy$$



The region bounded by $y = x^2$ and $y = 2x$ is revolved about the line $x = 2$. Find the volume.

The outer radius is:

$$r_o = 2 - \frac{y}{2}$$

The inner radius is:

$$r_i = 2 - \sqrt{y}$$

$$\begin{aligned} &= \pi \int_0^4 \left[\left(4 - 2y + \frac{y^2}{4} \right) - (4 - 4\sqrt{y} + y) \right] dy \\ &= \pi \int_0^4 \left[\cancel{4} - 2y + \frac{y^2}{4} - \cancel{4} + 4\sqrt{y} - y \right] dy \\ &= \pi \int_0^4 \left[-3y + \frac{1}{4}y^2 + 4y^{\frac{1}{2}} \right] dy \\ &= \pi \cdot \left[-\frac{3}{2}y^2 + \frac{1}{12}y^3 + \frac{8}{3}y^{\frac{3}{2}} \right]_0^4 \\ &= \pi \cdot \left[-24 + \frac{16}{3} + \frac{64}{3} \right] = \frac{8\pi}{3} \text{ units}^3 \end{aligned}$$

Example

Write an and evaluate an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

$$y = \sin x, y = \cos x, 0 \leq x \leq \pi/4; \text{ about } y = -1$$

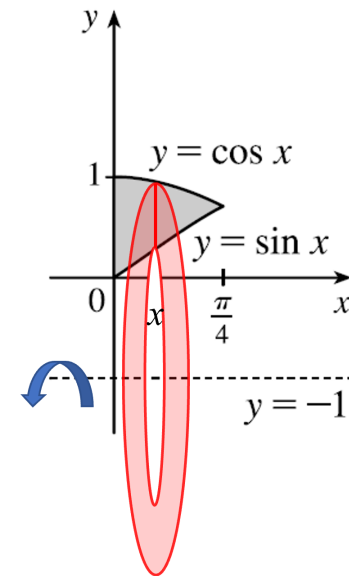
Solution

$$A(x) = \pi (r_{\text{Outer}}^2 - r_{\text{Inner}}^2) = \pi (f(x)^2 - g(x)^2)$$

$$r_{\text{Outer}} = \cos x + 1 \quad r_{\text{Inner}} = \sin x + 1$$

$$\begin{aligned} A(x) &= \pi [(\cos x + 1)^2 - (\sin x + 1)^2] \\ &= \pi (\cos^2 x + 2 \cos x - \sin^2 x - 2 \sin x) \\ &= \pi (\cos 2x + 2 \cos x - 2 \sin x) \end{aligned}$$

$$\begin{aligned} V &= \int_0^{\pi/4} A(x) dx \\ &= \int_0^{\pi/4} \pi (\cos 2x + 2 \cos x - 2 \sin x) dx \\ &= \pi \left[\frac{1}{2} \sin 2x + 2 \sin x + 2 \cos x \right]_0^{\pi/4} \end{aligned}$$



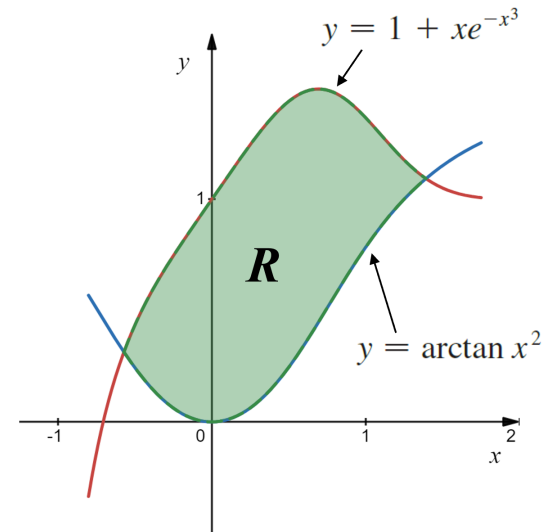
$$\begin{aligned} &= \pi \left[\left(\frac{1}{2} + \sqrt{2} + \sqrt{2} \right) - (0 + 0 + 2) \right] \\ &= \left(2\sqrt{2} - \frac{3}{2} \right) \pi \text{ units}^3 \end{aligned}$$

Example Let R be the region bounded by the curves,

$$y = 1 + xe^{-x^3}, \quad y = \arctan x^2$$

use a calculator to approximate the following. **T**

- Find approximate x -coordinates of the points of intersection of the given curves.
- Find the area of the region R .
- Find the volume of the solid obtained by rotating the region R about the x -axis.
- Find the volume of the solid obtained by rotating the region R about the line $y = -2$.
- Find the volume of the solid obtained by using the region R as a base of a solid where each cross section perpendicular to the x -axis is an equilateral triangle.



Solution

- (a) Find approximate x -coordinates of the points of intersection of the given curves.

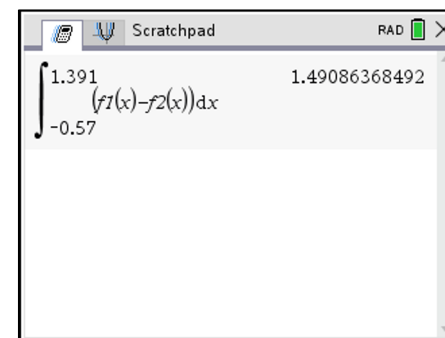
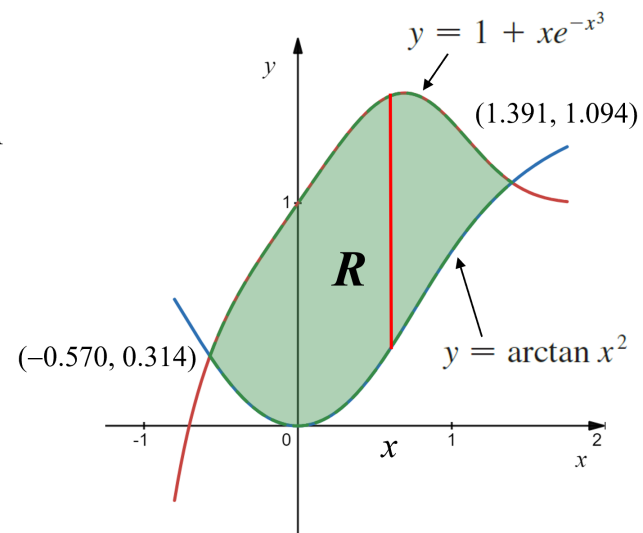
The points of intersection are

$$(-0.570, 0.314) \text{ and } (1.391, 1.094).$$

- (b) Find the area of the region R .

$$\text{Area} = \int_a^b (y_T - y_B) dx$$

$$A = \int_{-0.570}^{1.391} (f_1(x) - f_2(x)) dx \stackrel{CAS}{\approx} 1.491 \text{ unit}^2$$



Solution

(c) Find the volume of the solid obtained by rotating the region R about the x -axis.

$$A(x) = \pi (r_{\text{Outer}}^2 - r_{\text{Inner}}^2)$$

$$r_{\text{Outer}} = 1 + xe^{-x^3} = f1(x) \quad r_{\text{Inner}} = \arctan x^2 = f2(x)$$

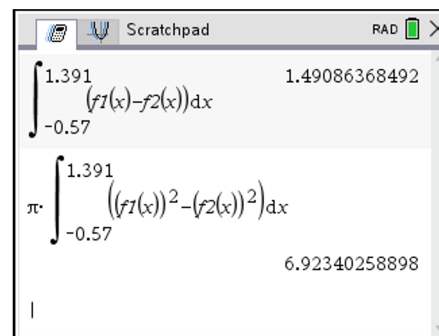
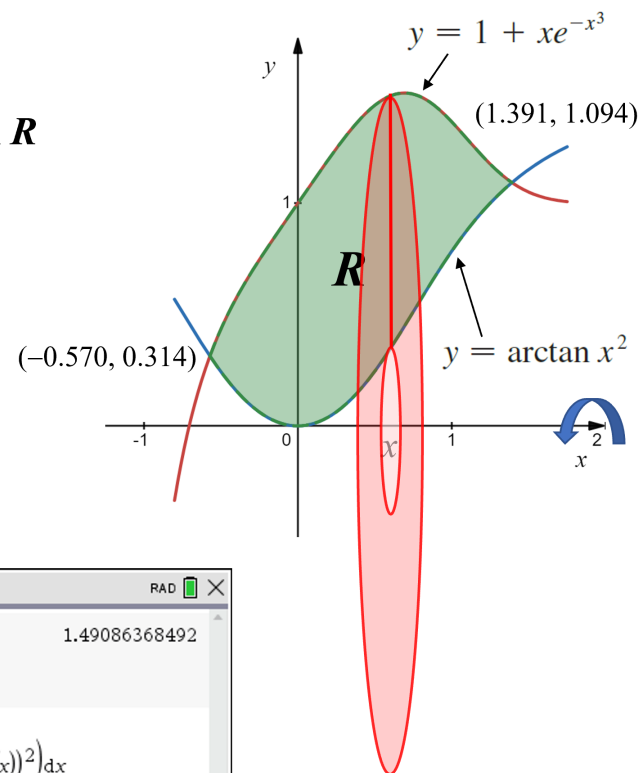
$$A(x) = \pi \left[\left(1 + xe^{-x^3}\right)^2 - (\arctan x^2)^2 \right]$$

$$A(x) = \pi \left[f1(x)^2 - f2(x)^2 \right]$$

$$V = \int_{-0.570}^{1.391} A(x) dx$$

$$V = \pi \int_{-0.570}^{1.391} (f1(x)^2 - f2(x)^2) dx$$

$$\stackrel{\text{CAS}}{\approx} 6.923 \text{ unit}^3$$



Solution

(d) Find the volume of the solid obtained by rotating the region R about the line $y = -2$.

$$A(x) = \pi (r_{\text{Outer}}^2 - r_{\text{Inner}}^2)$$

$$r_{\text{Outer}} = (1 + xe^{-x^3}) + 2 = f1(x) + 2$$

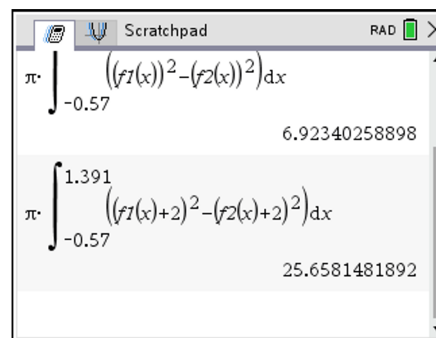
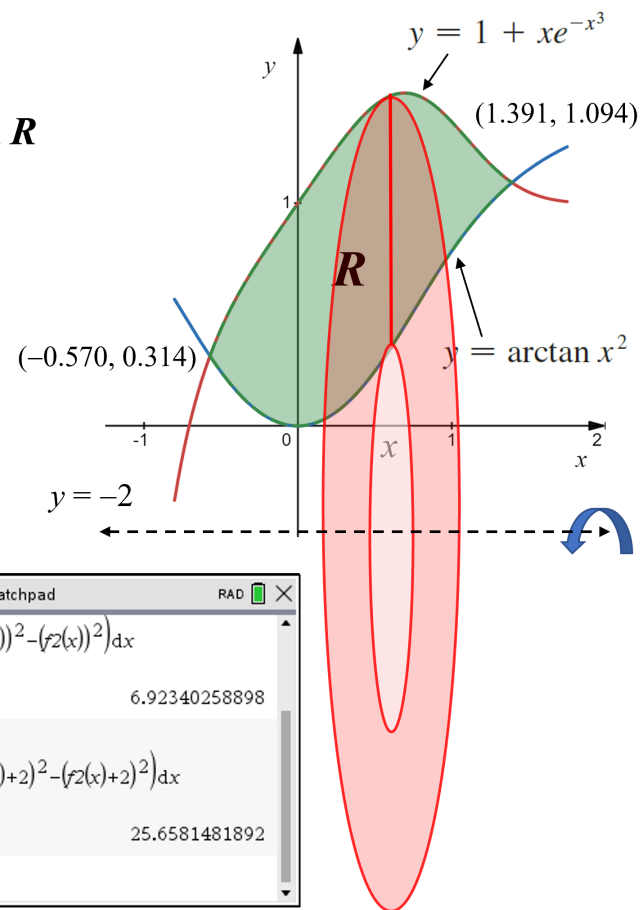
$$r_{\text{Inner}} = \arctan x^2 + 2 = f2(x) + 2$$

$$A(x) = \pi [(f1(x) + 2)^2 - (f2(x) + 2)^2]$$

$$V = \int_{-0.570}^{1.391} A(x) dx$$

$$V = \pi \int_{-0.570}^{1.391} [(f1(x) + 2)^2 - (f2(x) + 2)^2] dx$$

$$\stackrel{\text{CAS}}{\approx} 25.658 \text{ unit}^3$$



Solution

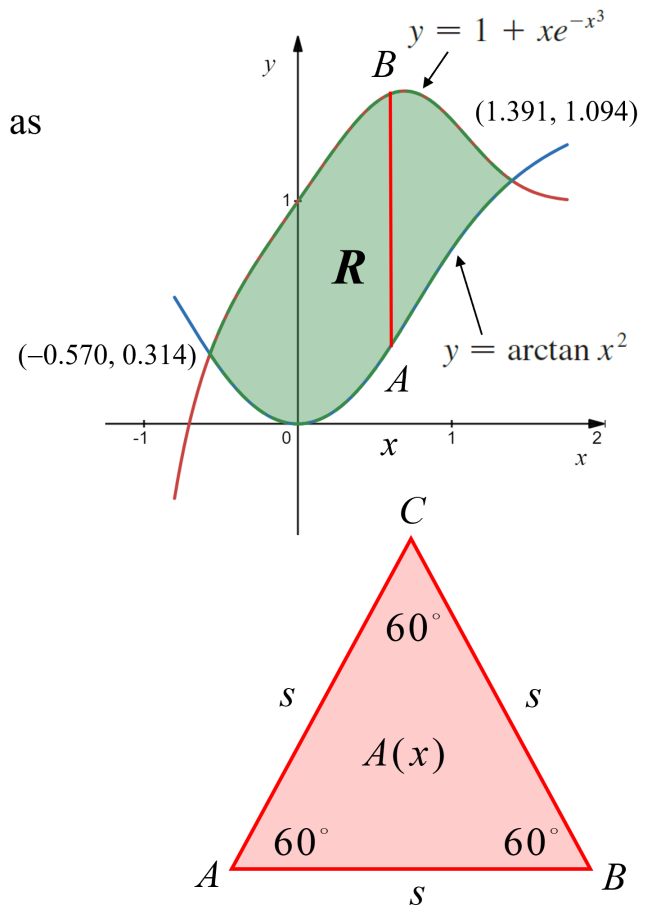
(e) Find the volume of the solid obtained by using the region R as a base of a solid where each cross section perpendicular to the x -axis is an equilateral triangle.

$$A(x) = \frac{1}{2}(AC)(AB) \sin(A)$$

$$A(x) = \frac{1}{2}(s)(s) \sin(60^\circ)$$

$$A(x) = \frac{1}{2}s^2 \cdot \frac{\sqrt{3}}{2}$$

$$A(x) = \frac{\sqrt{3}}{4}s^2$$



Solution

(e) Find the volume of the solid obtained by using the region **R** as a base of a solid where each cross section perpendicular to the x -axis is an equilateral triangle.

$$A(x) = \frac{\sqrt{3}}{4} s^2$$

$$s = y_T - y_B$$

$$s = (1 + xe^{-x^3}) - (\arctan x^2)$$

$$s = f_1(x) - f_2(x)$$

$$A(x) = \frac{\sqrt{3}}{4} (f_1(x) - f_2(x))^2$$

$$V = \int_{-0.570}^{1.391} A(x) dx$$

$$V = \frac{\sqrt{3}}{4} \cdot \int_{-0.570}^{1.391} (f_1(x) - f_2(x))^2 dx$$

$$\stackrel{CAS}{\approx} 0.614 \text{ unit}^3$$

