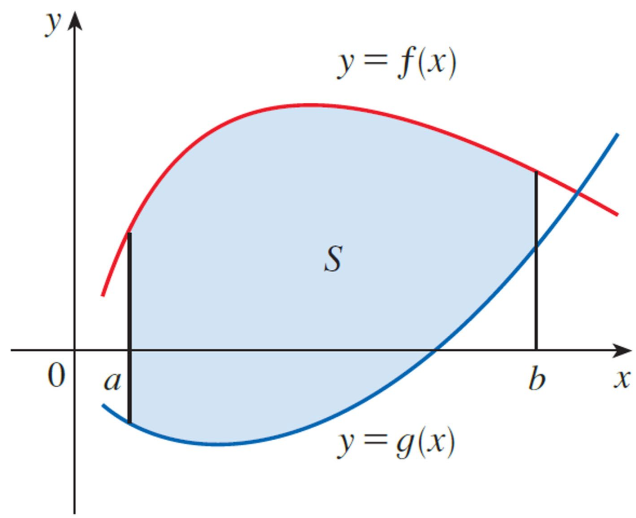
6.1 Areas Between Curves

Johann Carl Friedrich Gauss
1777 – 1855

Gauss worked in a wide variety of fields in both mathematics and physics including number theory, analysis, differential geometry, geodesy, magnetism, astronomy and optics. His work has had an immense influence in many areas.

Area Between Curves: Integrating With Respect to x

Consider the region S shown in Figure 1 that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geqslant g(x)$ for all x in $[a, b]$.



Area Between Curves: Integrating With Respect to x

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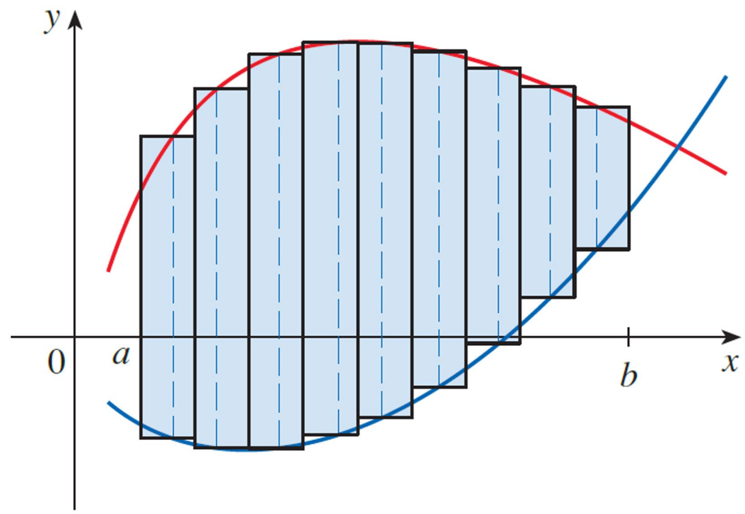
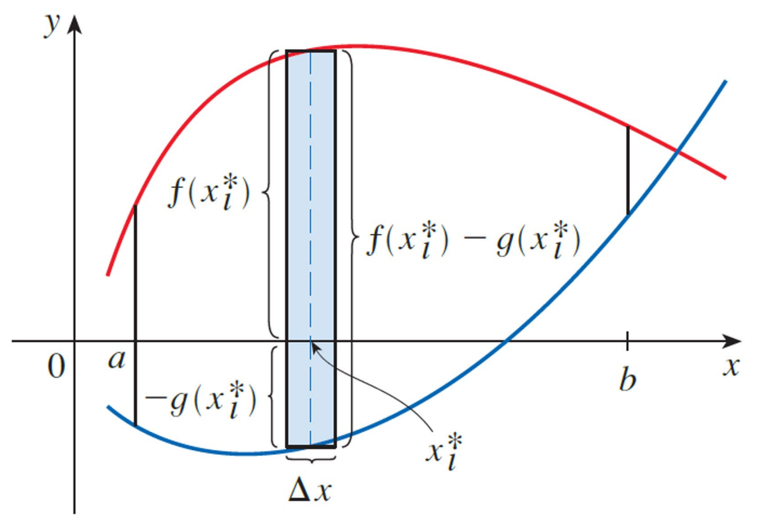


Approximating rectangles

Area Between Curves: Integrating With Respect to x

Consider the region S shown in Figure 1 that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geqslant g(x)$ for all x in $[a, b]$.



Typical rectangle

The Riemann sum $S \approx \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x$

Area Between Curves: Integrating With Respect to x

This approximation appears to become better and better as $n \to \infty$. Therefore, we define the **area** A of the region S as the limiting value of the sum of the areas of these approximating rectangles.

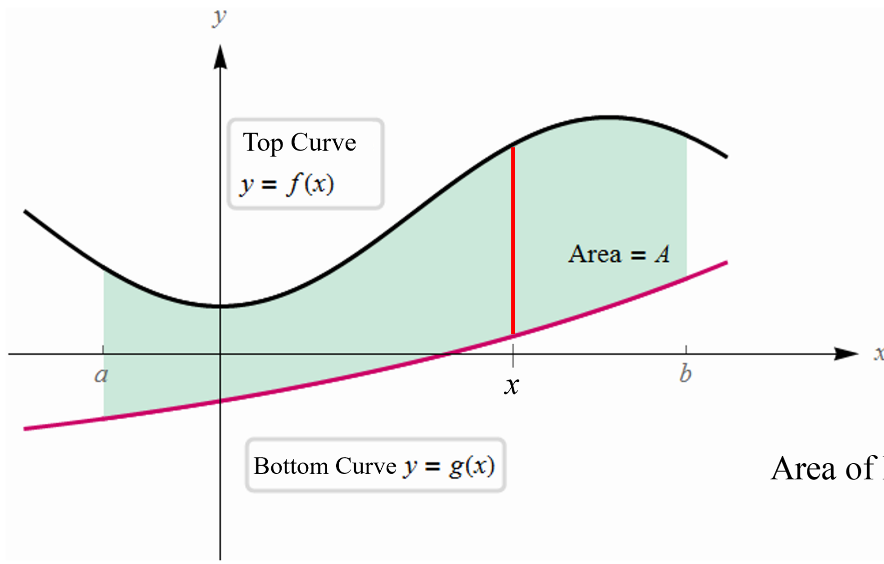
$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x$$

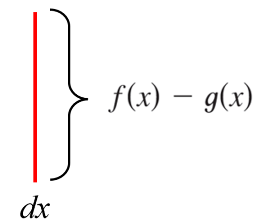
Therefore, we have the following formula for area.

2 The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geqslant g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\, dx$$

Area Between Curves: Integrating With Respect to x





Area of Rectangle = Height · Width

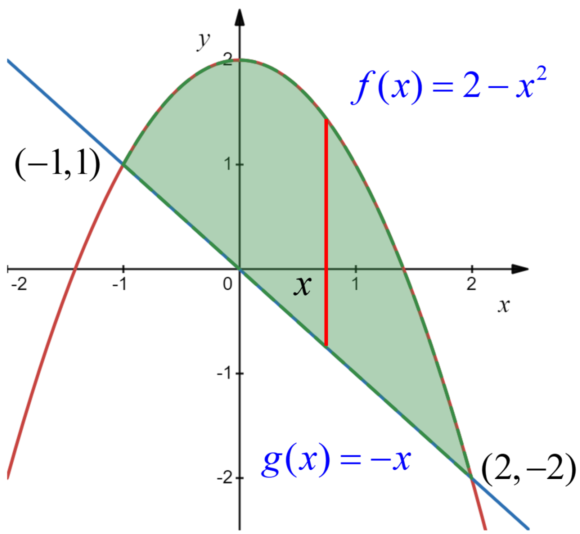
$$A = [f(x) - g(x)]\,dx$$

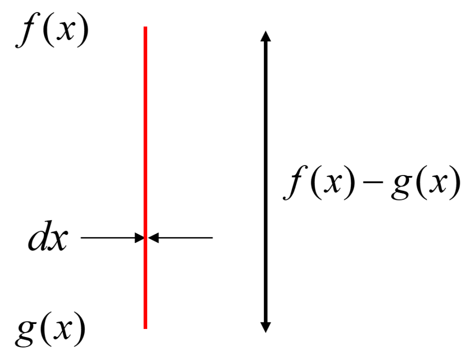
$$\text{Area between the curves} = \int_a^b [f(x) - g(x)]\,dx$$

$$\text{Area} = \int_a^b (\text{Top Curve} - \text{Bottom Curve}) \cdot \text{Width}$$

$$\text{Area} = \int_a^b (y_T - y_B)\,dx$$

Example Find the area of the region bounded between the two curves.





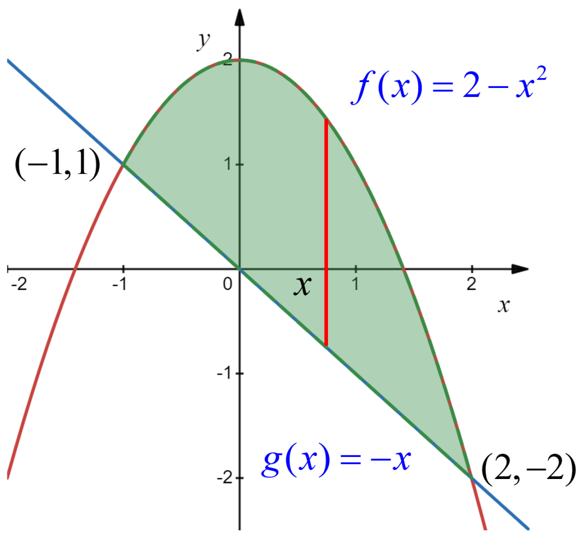
Since this strip is a long thin rectangle, the area is:

$$A = \text{length} \cdot \text{width} = \big(f(x) - g(x)\big)\,dx$$

$$A = \text{length} \cdot \text{width} = \big(2 - x^2 + x\big)\,dx$$

If we add all the rectangles, we get: $\int_{-1}^{2} (2 - x^2 + x)\; dx$

Example Find the area of the region bounded between the two curves.



Solution

$$A=\int_{-1}^{2}(2-x^2+x)\,dx=\left[2x-\frac{1}{3}x^3+\frac{1}{2}x^2\right]_{-1}^{2}$$

$$=\left(4-\frac{8}{3}+2\right)-\left(-2+\frac{1}{3}+\frac{1}{2}\right)$$

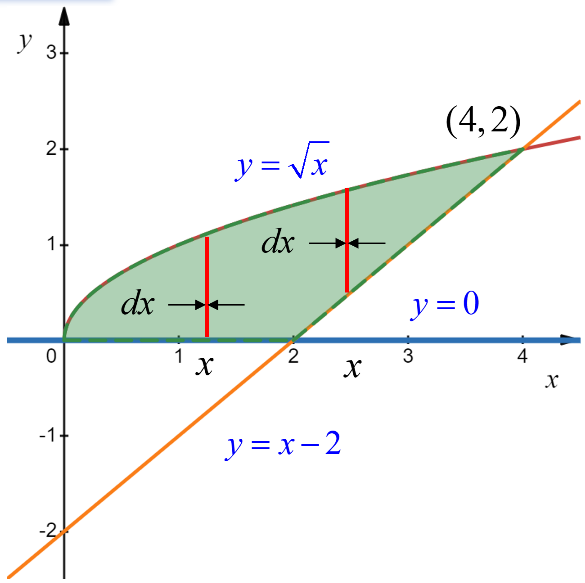
$$=6-\frac{8}{3}+2-\frac{1}{3}-\frac{1}{2}$$

$$=\frac{36-16+12-2-3}{6}$$

$$=\frac{27}{6}$$

$$=\frac{9}{2}\ \text{units}^2$$

Example Find the area bounded by the curves $y=\sqrt{x},\ y=x-2,$ and $y=0$.



Solution

We must integrate in two separate regions:

$$A=\int_0^2 \sqrt{x}\ dx+\int_2^4\left[\sqrt{x}-(x-2)\right]dx$$

$$=\int_0^2 x^{1/2}\ dx+\int_2^4\left[x^{1/2}-x+2\right]dx$$

$$=\left[\frac{2}{3}x^{3/2}\right]_0^2+\left[\frac{2}{3}x^{3/2}-\frac{1}{2}x^2+2x\right]_2^4$$

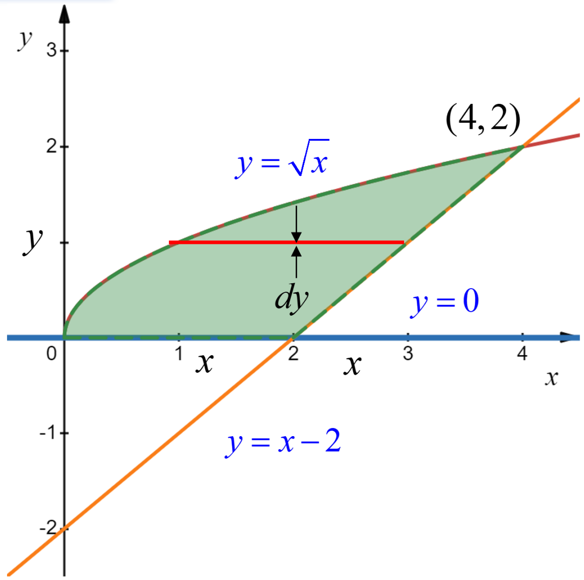
$$=\frac{2}{3}\cdot 2^{3/2}+\left(\frac{2}{3}\cdot 4^{3/2}-\frac{1}{2}\cdot 4^2+2\cdot 4\right)-\left(\frac{2}{3}\cdot 2^{3/2}-\frac{1}{2}\cdot 2^2+2\cdot 2\right)$$

$$=\frac{4\sqrt{2}}{3}+\frac{16}{3}-8+8-\frac{4\sqrt{2}}{3}+2-4$$

$$=\frac{16}{3}-2=\frac{10}{3}\ \text{units}^2$$

Is there an easier way?

Example Find the area bounded by the curves $y=\sqrt{x},\ y=x-2,$ and $y=0$.



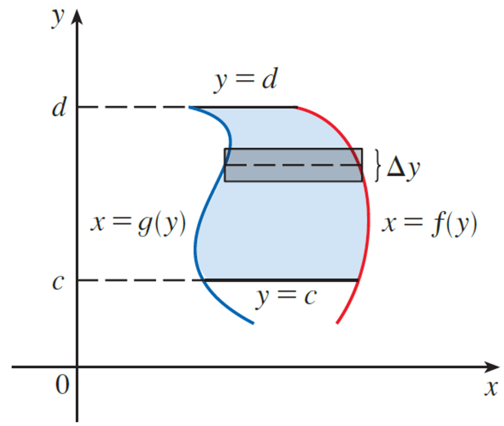
$$y=\sqrt{x} \qquad y=x-2$$

$$x=y^2 \qquad x=y+2$$

$$g(y)=y^2 \qquad f(y)=y+2$$

We can find the same area using a horizontal rectangle.

Since the width of the rectangle is dy, we find the length of the rectangle by solving for x in terms of y.

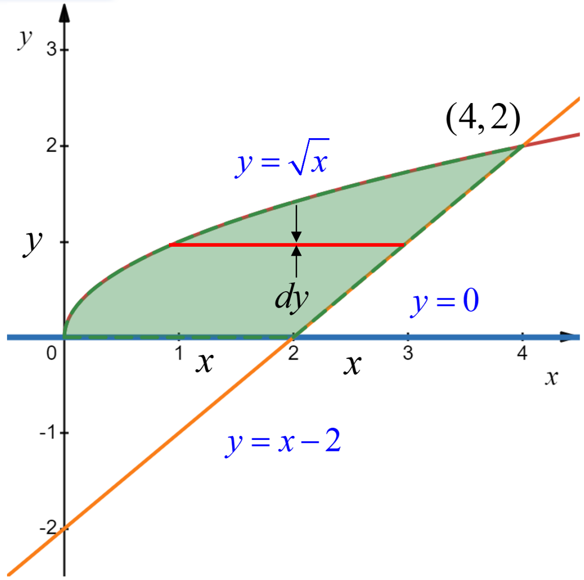


$$A=\int_c^d [f(y)-g(y)]\,dy$$

$$\text{Area}=\int_c^d (\text{Right Curve}-\text{Left Curve})\cdot \text{Width}$$

$$\text{Area}=\int_c^d (x_R-x_L)\,dy$$

Example Find the area bounded by the curves $y=\sqrt{x},\ y=x-2,$ and $y=0.$



$y=\sqrt{x}$ | $y=x-2$

$x=y^2$ | $x=y+2$

$g(y)=y^2$ | $f(y)=y+2$

Solution

$$A=\int_c^d \left[f(y)-g(y)\right]\,dy$$

$$A=\int_0^2 \left[(y+2)-y^2\right]\,dy$$

What are the limits?

length of rectangle

width of strip

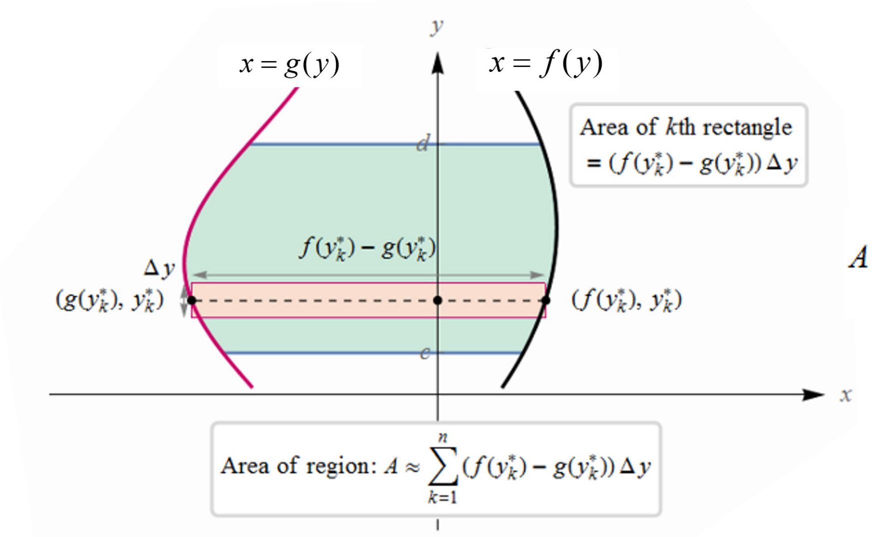
$$=\left[\frac{1}{2}y^2+2y-\frac{1}{3}y^3\right]_0^2$$

$$=2+4-\frac{8}{3}$$

$$=\frac{10}{3}\ \text{units}^2$$

That was a lot easier!

Area Between Curves: Integrating With Respect to *y*



$$A = \lim_{n\to\infty} \sum_{k=1}^{n} (f(y_k^*) - g(y_k^*))\,\Delta y = \int_c^d (f(y) - g(y))\,dy.$$

$$\text{Area} = \int_c^d (\text{Right Curve} - \text{Left Curve}) \cdot \text{Width}$$

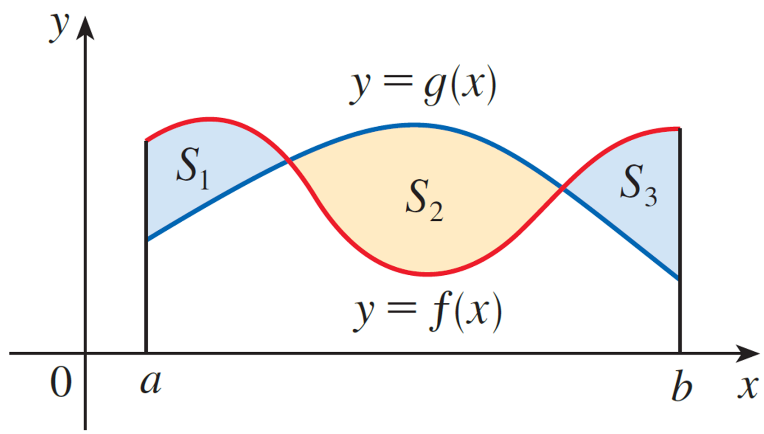
$$\text{Area} = \int_c^d (x_R - x_L)\,dy$$

Definition **Area of a Region Between Two Curves with Respect to *y***

Suppose that f and g are continuous functions with $f(y) \geq g(y)$ on the interval $[c, d]$. The area of the region bounded by the graphs $x = f(y)$ and $x = g(y)$ on $[c, d]$ is

$$A = \int_c^d (f(y) - g(y))\,dy.$$

If we are asked to find the area between the curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$ for some values of x but $g(x) \geq f(x)$ for other values of x, then we split the given region S into several regions $S_1, S_2, \ldots$ with areas $A_1, A_2, \ldots$ as shown in the figure.



3 The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)|\, dx$$

Example Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

Solution

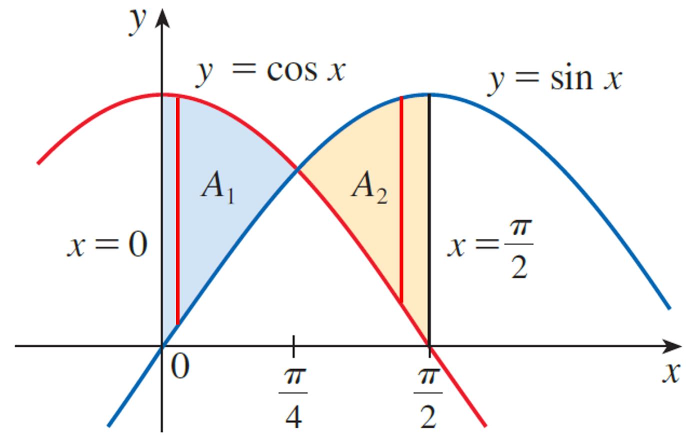
$$A = \int_0^{\pi/2} |\cos x - \sin x|\, dx = A_1 + A_2$$

$$= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$$

$$= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$= 2\sqrt{2} - 2 \text{ units}^2$$



What's the point of intersection?

$$\sin x = \cos x$$

$$x = \frac{\pi}{4}$$

General Strategy for Area Between Curves:

1. Sketch the curves.

2. Decide on vertical or horizontal rectangle connecting the boundary curves. (Pick whichever is easier to write formulas for the length of the rectangle, and/or whichever will let you integrate fewer times.)

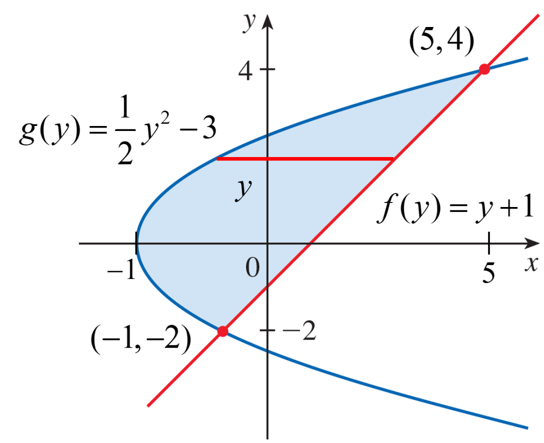
3. Write an expression for the area of the rectangle.
(If the width is dx, the length must be in terms of x.
If the width is dy, the length must be in terms of y.

4. Find the limits of integration. (If using dx, the limits are x values; if using dy, the limits are y values.)

5. Integrate to find area.

Example Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution



By solving the two equations simultaneously we find the points of intersection are

$$\begin{cases} x = y+1 \\ y^2 = 2x+6 \end{cases}$$

$$y^2 = 2(y+1)+6$$

$$y^2 - 2y - 8 = 0$$

$$(y-4)(y+2) = 0$$

$$y = 4 \quad \text{or} \quad y = -2$$

$$x = y+1$$

$$x = 5 \quad \text{or} \quad x = -1$$

Points of intersection are $(5,4)$ and $(-1,-2)$

Vertical or horizontal rectangles?

We solve the equations for x

$$x = y+1 \qquad x = \frac{y^2-6}{2} = \frac{1}{2}y^2 - 3$$

$$f(y) = y+1 \qquad g(y) = \frac{1}{2}y^2 - 3$$

$$A = \int_c^d [f(y) - g(y)]\, dy$$

$$A = \int_c^d (x_R - x_L)\, dy$$

$$\int_{-2}^{4} \left[(y+1) - \left(\tfrac{1}{2}y^2 - 3\right)\right] dy$$

$$= \int_{-2}^{4} \left(-\tfrac{1}{2}y^2 + y + 4\right) dy$$

$$= \left[-\frac{1}{2}\left(\frac{y^3}{3}\right) + \frac{y^2}{2} + 4y\right]_{-2}^{4} = -\tfrac{1}{6}(64) + 8 + 16 - \left(\tfrac{4}{3} + 2 - 8\right) = 18\,\text{units}^2$$

Example Sketch the region enclosed by the given curves and find its area.

$y = \cos x, \quad y = \sin 2x, \quad 0 \leqslant x \leqslant \pi/2$

Solution

By solving the two equations simultaneously we find the point of intersection is

$\cos x = \sin 2x$

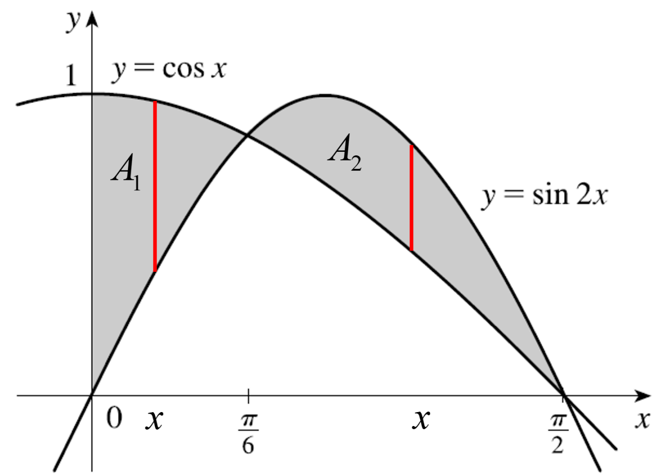
$\cos x = 2\sin x \cos x$

$2\sin x \cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0 \text{ or } \sin x = \frac{1}{2}$

$x = \frac{\pi}{2} \text{ or } x = \frac{\pi}{6} \text{ on } \left[0, \frac{\pi}{2}\right]$



$A = A_1 + A_2$

Horizontal or vertical rectangles?

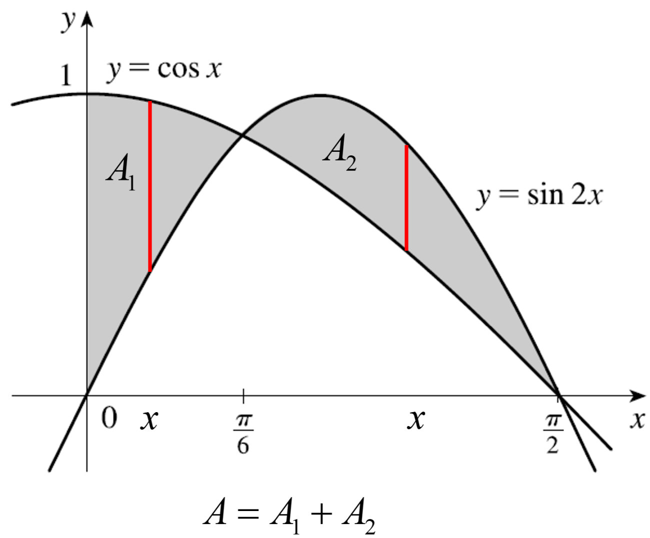
$A = \int_0^{\pi/6} (\cos x - \sin 2x)\, dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x)\, dx$

$= \left[\sin x + \frac{1}{2}\cos 2x\right]_0^{\pi/6} + \left[-\frac{1}{2}\cos 2x - \sin x\right]_{\pi/6}^{\pi/2}$

Example Sketch the region enclosed by the given curves and find its area.

$$y = \cos x, \quad y = \sin 2x, \quad 0 \leqslant x \leqslant \pi/2$$

Solution



$$= \left[\sin x + \tfrac{1}{2}\cos 2x\right]_0^{\pi/6} + \left[-\tfrac{1}{2}\cos 2x - \sin x\right]_{\pi/6}^{\pi/2}$$

$$= \left[\left(\sin\tfrac{\pi}{6} + \tfrac{1}{2}\cos\tfrac{\pi}{3}\right) - \left(\sin 0 + \tfrac{1}{2}\cos 0\right)\right] + \left[\left(-\tfrac{1}{2}\cos\pi - \sin\tfrac{\pi}{2}\right) - \left(-\tfrac{1}{2}\cos\tfrac{\pi}{3} - \sin\tfrac{\pi}{6}\right)\right]$$

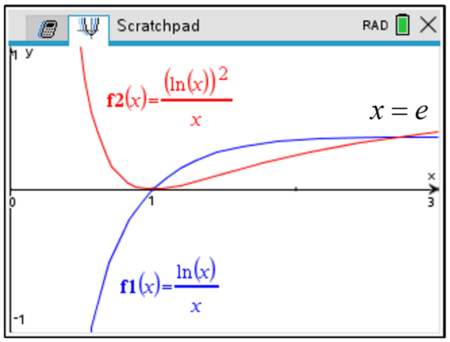
$$= \left[\left(\tfrac{1}{2} + \tfrac{1}{4}\right) - \left(0 + \tfrac{1}{2}\right)\right] + \left[\left(\tfrac{1}{2} - 1\right) - \left(-\tfrac{1}{4} - \tfrac{1}{2}\right)\right]$$

$$= \frac{1}{2} \text{ units}^2$$

Example Sketch the region enclosed by the given curves and find its area.

$$y = \frac{\ln x}{x}, \quad y = \frac{(\ln x)^2}{x}$$

Solution



By solving the two equations simultaneously with CAS we get

$x = 1$ or $x = e$

We will use vertical rectangles.

$$\text{Area} = \int_a^b (\text{Top Curve} - \text{Bottom Curve}) \cdot \text{Width}$$

$$A = \int_1^e (f1(x) - f2(x))\,dx$$

$$= \frac{1}{6} \text{ units}^2$$