5.5 The Substitution Rule

Sofia Kovalevskaya
1850 – 1891

Kovalevskaya was a Russian mathematician who made noteworthy contributions to analysis, partial differential equations and mechanics. She was a pioneer for women in mathematics around the world -- the first woman to obtain a doctorate (in the modern sense) in mathematics.

Substitution: Indefinite Integrals

Because of the Fundamental Theorem, it's important to be able to find antiderivatives. But our antidifferentiation formulas don't tell us how to evaluate integrals such as

$$\int 2x\sqrt{1 + x^2}\, dx$$

To find this integral we use the problem-solving strategy of *introducing something extra.* Here the "something extra" is a new variable; we change from the variable x to a new variable u. Making the integrand an obvious derivative.

Let's see how it works!

Example Evaluate the indefinite integral.

$$\int (x+2)^5 \, dx$$

Solution

$$\int (x+2)^5 \, dx$$

Let $u = x+2$

$$\frac{du}{dx} = 1$$

$$du = dx$$

$$= \int u^5 du$$

$$= \frac{1}{6}u^6 + C$$

Don't forget to substitute the value for u back into the problem!

$$= \frac{(x+2)^6}{6} + C$$

This technique is called the "**Substitute Rule**." It transforms the integrand into an obvious derivative.

Example Evaluate the indefinite integral.

$$\int \sqrt{1+x^2} \cdot 2x\, dx$$

Solution

$$\int \sqrt{1+x^2} \cdot \underbrace{2x\, dx}$$

$$= \int u^{\frac{1}{2}}\, du$$

$$= \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= \frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}} + C$$

One of the clues that we look for is if we can find a function and its derivative in the integrand.

The derivative of $1+x^2$ is $\underline{2x}$.

Let $u = 1 + x^2$ Differentiate

$$\frac{du}{dx} = 2x$$

$du = 2x\, dx$ Separation of variables

Note that this only worked because of the factor $2x$ in the original integrand. Many integrals can not be done by substitution.

Example Evaluate the indefinite integral.

$$\int \sqrt{4x-1}\ dx$$

Solution

$$\int \sqrt{4x-1}\ dx = \int u^{\frac{1}{2}} \cdot \frac{1}{4} du$$

$$= \frac{1}{4}\int u^{\frac{1}{2}}\ du$$

$$= \frac{1}{4}\cdot\frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{1}{6}u^{\frac{3}{2}} + C$$

$$= \frac{1}{6}(4x-1)^{\frac{3}{2}} + C$$

Let $u = 4x - 1$

$$\left.\begin{aligned} du &= 4\ dx \\ \frac{1}{4}du &= dx \end{aligned}\right\}\ \text{Solve for } dx.$$

Example Evaluate the indefinite integral.

$$\int \cos(7x+5)\ dx$$

Solution

$$\int \cos(7x+5)\ dx = \int \cos u \cdot \frac{1}{7} du$$

$$= \frac{1}{7}\int \cos u\ du$$

$$= \frac{1}{7}\sin u + C$$

$$= \frac{1}{7}\sin(7x+5) + C$$

Let $u = 7x+5$

$$du = 7\ dx$$

$$\frac{1}{7} du = dx$$

The Fundamental Theorem of Calculus, Part 1

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

$f(x) = \cos(7x+5)$

By FTC part 1

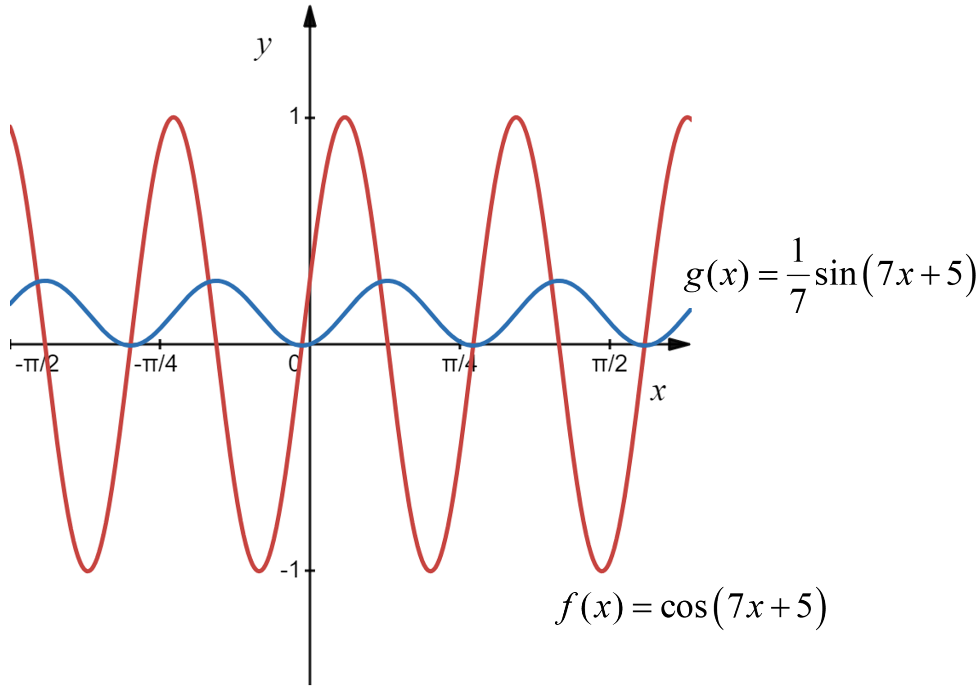
$g(x) = \int_0^x \cos(7t+5)\,dt$

Since the integrand has a symbolic antiderivative, previous example, we can write $g(x)$ as

$g(x) = \frac{1}{7}\sin(7x+5) + C$

Letting $C = 0$ we get

$g(x) = \frac{1}{7}\sin(7x+5)$



Example Evaluate the indefinite integral.

$$\int x^2 \cdot \sin\left(x^3\right)\, dx$$

Solution

$$\int x^2 \sin\left(x^3\right) dx = \int \sin u \cdot \frac{1}{3}\, du$$

$$= \frac{1}{3}\int \sin u\; du$$

$$= -\frac{1}{3}\cos u + C$$

$$= -\frac{1}{3}\cos x^3 + C$$

Let $u = x^3$

$$du = 3x^2\; dx$$

$$\frac{1}{3}du = \underbrace{x^2\; dx}$$

We solve for $x^2\; dx$ because we can find it in the integrand.

Example Evaluate the indefinite integral.

$$\int \sin^4 x \cdot \cos x \, dx$$

Solution

$$\int \sin^4 x \cdot \cos x \, dx = \int (\sin x)^4 \cos x \, dx$$

Let $u = \sin x$

$du = \cos x \, dx$

$$= \int u^4 \, du$$

$$= \frac{1}{5} u^5 + C$$

$$= \frac{1}{5} \sin^5 x + C$$

Example Evaluate the definite integral.

$$\int_0^{\frac{\pi}{4}} \tan x \cdot \sec^2 x \, dx$$

FTC Part 2

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

Solution

$$\int_{0=x}^{\frac{\pi}{4}=x} \tan x \cdot \sec^2 x \, dx = \cancel{\int_0^{\frac{\pi}{4}} u \, du}$$

Let $u = \tan x$

$du = \sec^2 x \, dx$

new upper limit

$$= \int_0^1 u \, du$$

new lower limit

Must Change the original limits!

Lower limit let $x = 0$

$u(0) = \tan 0 = 0$

Upper limit let $x = \dfrac{\pi}{4}$

$u\left(\dfrac{\pi}{4}\right) = \tan \dfrac{\pi}{4} = 1$

$$= \left[\frac{1}{2}u^2\right]_0^1$$

$$= \frac{1}{2}\left[u^2\right]_0^1$$

$$= \frac{1}{2}$$

Example Evaluate the definite integral.

$$\int_{-1}^{1} 5x^2\sqrt{x^3+1}\ dx$$

Solution

$$\int_{-1}^{1} 5x^2\sqrt{x^3+1}\ dx = 5\cdot\int_0^2 u^{\frac{1}{2}}\cdot\frac{1}{3}\ du$$

Let $u = x^3 + 1$

$du = 3x^2\ dx$

$\frac{1}{3}du = x^2\ dx$

New Limits

$u(-1) = (-1)^3 + 1 = 0$

$u(1) = (1)^3 + 1 = 2$

$$= \frac{5}{3}\cdot\int_0^2 u^{\frac{1}{2}}\ du$$

$$= \frac{5}{3}\cdot\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^2$$

Don't forget to use the new limits.

$$= \frac{10}{9}\cdot\left[u^{\frac{3}{2}}\right]_0^2$$

$$= \frac{10}{9}\cdot 2^{\frac{3}{2}} = \frac{10}{9}\cdot 2\sqrt{2} = \frac{20\sqrt{2}}{9}$$

7th grade math

$2^{\frac{3}{2}} = \sqrt{2^3}$

$= \sqrt{8}$

$= 2\sqrt{2}$

Example Evaluate the definite integral.

$$\int_0^1 x \cdot e^{-x^2}\, dx$$

Solution

$$\int_0^1 x \cdot e^{-x^2}\, dx = \int_0^{-1} e^u \cdot \left(-\frac{1}{2}\right) du$$

$$= -\frac{1}{2}\int_0^{-1} e^u\, du$$

$$= \frac{1}{2} \cdot \int_{-1}^{0} e^u\, du$$

$$= \frac{1}{2} \cdot \left[e^u\right]_{-1}^{0}$$

$$= \frac{1}{2}\left(1 - \frac{1}{e}\right) = \frac{1}{2}\left(\frac{e-1}{e}\right) = \frac{e-1}{2e}$$

Let $u = -x^2$

$du = -2x\, dx$

$-\frac{1}{2}\, du = x\, dx$

New Limits

$u(0) = 0$

$u(1) = -1$

Example Evaluate the definite integral.

$$\int_1^2 \frac{\sin(\pi/x)}{x^2}\,dx$$

Solution

$$\int_1^2 \frac{\sin(\pi/x)}{x^2}\,dx = \int_1^2 \frac{1}{x^2}\cdot\sin(\pi/x)\,dx$$

$$= \int_{\pi}^{\pi/2} \sin(u)\cdot\left(-\frac{1}{\pi}\right)du$$

$$= -\frac{1}{\pi}\cdot\int_{\pi}^{\pi/2}\sin(u)\,du$$

$$= \frac{1}{\pi}\cdot\int_{\pi/2}^{\pi}\sin(u)\,du$$

$$= \frac{1}{\pi}\cdot\left[-\cos(u)\right]_{\pi/2}^{\pi}$$

$$= -\frac{1}{\pi}\cdot\left[\cos(u)\right]_{\pi/2}^{\pi} = -\frac{1}{\pi}(-1-0) = \frac{1}{\pi}$$

Let $u = \dfrac{\pi}{x}$

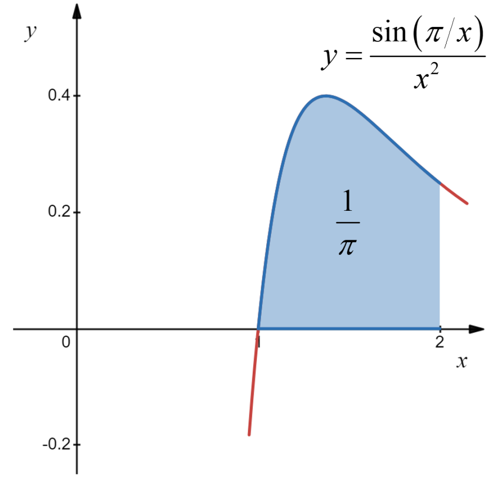
$$du = -\frac{\pi}{x^2}\,dx$$

$$-\frac{1}{\pi}du = \frac{1}{x^2}\,dx$$

New Limits

$u(1) = \pi$

$u(2) = \dfrac{\pi}{2}$



Example Evaluate the indefinite integral.

$$\int \frac{\cos(\ln t)}{t}\,dt$$

Solution

$$\int \frac{\cos(\ln t)}{t}\,dt = \int \cos u\,du$$

Let $u = \ln t$

$$du = \frac{1}{t}\,dt$$

$$= \sin u + C$$

$$= \sin(\ln t) + C$$

Example Evaluate the indefinite integral.

$$\int 6x\sqrt{3x+2}dx$$

Solution

$$\int 6x\sqrt{3x+2}\;dx = 6\cdot\int \frac{u-2}{3}\cdot u^{1/2}\cdot\frac{1}{3}du$$

$$= \frac{6}{9}\cdot\int\left(u^{3/2}-2u^{1/2}\right)du$$

$$= \frac{2}{3}\cdot\left(\frac{2}{5}u^{5/2}-\frac{4}{3}u^{3/2}\right)+C$$

$$= \frac{4}{15}(3x+2)^{5/2}-\frac{8}{9}(3x+2)^{3/2}+C$$

Let $u = 3x+2 \longrightarrow x = \dfrac{u-2}{3}$

$$du = 3dx$$

$$\frac{1}{3}du = dx$$