5.3 The Fundamental Theorem of Calculus

Leonhard Euler
1707 – 1783

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Leonhard Euler’s Work

<u>Quantity</u>:

Euler’s collected works, the *Opera Omnia*,
run to over 25,000 pages.

In 1910 Gustav Eneström catalogued Euler’s
866 books and papers in a document that itself
ran to 388 pages.

After his death, Euler published …

… 228 papers.

Quality:

Mathematical terms (from *MATHWORLD Wolfram*):

Euler – 96 entries

Gauss – 70 entries

Cauchy – 33 entries

Bach – 0 entries

The number e (1748)

Quodsi iam ex hac basi logarithmi construantur, ii vocari solent logarithmi *naturales* seu *hyperbolici,* quoniam quadratura hyperbolae per istiusmodi logarithmos exprimi potest. Ponamus autem brevitatis gratia pro numero hoc 2,71828 18284 59 etc. constanter litteram

$$e,$$

quae ergo denotabit basin logarithmorum naturalium seu hyperbolicorum [1]), cui respondet valor litterae $k = 1$; sive haec littera e quoque exprimet summam huius seriei

$$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc. in infinitum.}$$

Euler's Identity (1748)

$$e^{ix} = \cos x + i \sin x$$

$$e^{+v\sqrt{-1}} = \text{coſ.}\, v + \sqrt{-1}.\,\text{ſin.}\, v$$

$$\text{For} \quad x = \pi \ : \quad e^{i\pi} + 1 = 0$$

Consider the velocity function:

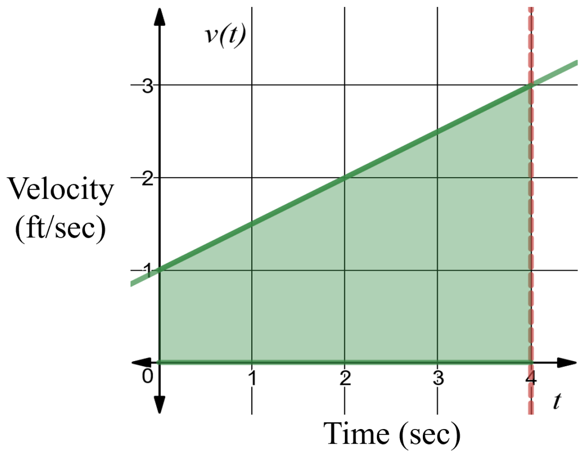
$$v(t) = \frac{1}{2}t + 1$$

Taking the antiderivative, the position function is:

$$s(t) = \int v(t)\, dt$$

$$s(t) = \frac{1}{4}t^2 + t$$

(*C* = *0* since *s* = *0* at *t* = *0*)



The displacement, in feet, after 4 seconds is equal to the area under the velocity curve.

After 4 seconds, the particle has been displaced:

$$DP = \underbrace{\frac{1}{4}\cdot(4)^2 + 4}_{\text{Position at 4 sec}} - \underbrace{\left(\frac{1}{4}\cdot(0) + 0\right)}_{\text{Position at 0 sec}} \longleftrightarrow \text{DP} = \int_0^4 \left(\frac{1}{2}t + 1\right) dt$$

$$\int_a^b v(t)\, dt = s(b) - s(a)$$

$$DP = 8 \text{ feet}$$

$$\text{DP} = \frac{1}{2}(1+3)4 = 8 \text{ feet}$$

WOW!!

The Fundamental Theorem of Calculus, Part 2

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function F such that $F' = f$.

x is variable of integration

Antiderivative of f evaluated at a and b

Limits of integration

$$\int_a^b \underline{f(x)}\,dx = F(b) - F(a) = \underline{F(x)\Big|_a^b}$$

Integrand

Shorthand notation

Example Use Part 2 of the Fundamental Theorem of Calculus to evaluate the integral.

(a) $\int_0^2 \left(\frac{4}{5}t^3 - \frac{3}{4}t^2 + \frac{2}{5}t\right) dt$

(b) $\int_1^4 \frac{2 + x^2}{\sqrt{x}}\, dx$

(c) $\int_{-1}^2 (3u - 2)(u + 1)\, du$

(d) $\int_5^5 \sqrt{t^2 + \sin t}\, dt$

(e) $\int_1^{18} \sqrt{\frac{3}{z}}\, dz$

(f) $\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1 - x^2}}\, dx$

(g) $\int_1^3 \frac{(3x + 1)^2}{x^3}\, dx$

FTC Part 2

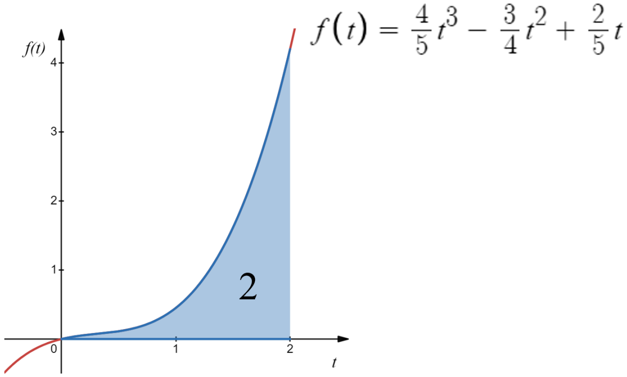
$$\int_a^b f(x)\, dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Solution

(a) $\int_0^2 \left(\frac{4}{5}t^3 - \frac{3}{4}t^2 + \frac{2}{5}t\right) dt$

$$= \left[\tfrac{1}{5}t^4 - \tfrac{1}{4}t^3 + \tfrac{1}{5}t^2\right]_0^2$$

$$= \left(\tfrac{16}{5} - 2 + \tfrac{4}{5}\right) - 0 = 2$$



Example Use Part 2 of the Fundamental Theorem of Calculus to evaluate the integral.

(b) $\displaystyle\int_1^4 \frac{2 + x^2}{\sqrt{x}}\, dx$

FTC Part 2

$$\int_a^b f(x)\, dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Solution

$$\int_1^4 \frac{2+x^2}{\sqrt{x}}\, dx = \int_1^4 \overbrace{\left(\frac{2}{\sqrt{x}} + \frac{x^2}{\sqrt{x}}\right)}^{\text{Not an obvious derivative}} dx$$

$$= \int_1^4 (2x^{-1/2} + x^{3/2})\, dx$$

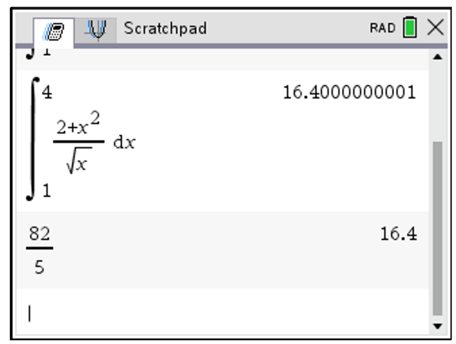
$$= \left[4x^{1/2} + \tfrac{2}{5}x^{5/2}\right]_1^4$$

$$= \left[4(2) + \tfrac{2}{5}(32)\right] - \left(4 + \tfrac{2}{5}\right)$$

$$= 8 + \tfrac{64}{5} - 4 - \tfrac{2}{5}$$

$$= \tfrac{82}{5}$$

With CAS we get



Example Use Part 2 of the Fundamental Theorem of Calculus to evaluate the integral.

(c) $\int_{-1}^{2} (3u - 2)(u + 1)\, du$

FTC Part 2

$$\int_a^b f(x)\, dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Solution

$$\begin{aligned}
\int_{-1}^{2} (3u - 2)(u + 1)\, du &= \int_{-1}^{2} (3u^2 + u - 2)\, du \\
&= \left[u^3 + \tfrac{1}{2}u^2 - 2u\right]_{-1}^{2} \\
&= (8 + 2 - 4) - \left(-1 + \tfrac{1}{2} + 2\right) \\
&= 6 - \tfrac{3}{2} \\
&= \tfrac{9}{2}
\end{aligned}$$

Example Use Part 2 of the Fundamental Theorem of Calculus to evaluate the integral.

(d) $\int_5^5 \sqrt{t^2 + \sin t}\, dt$

FTC Part 2

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Solution

Recall the theorem

$$\boxed{\int_a^a f(x)\,dx = 0}$$

$$\int_5^5 \sqrt{t^2 + \sin t}\, dt = F(5) - F(5) = 0$$

Example Use Part 2 of the Fundamental Theorem of Calculus to evaluate the integral.

(e) $\int_1^{18} \sqrt{\frac{3}{z}}\, dz$

FTC Part 2

$$\int_a^b f(x)\, dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Solution

$$\int_1^{18} \sqrt{\frac{3}{z}}\, dz = \int_1^{18} \sqrt{3} z^{-1/2}\, dz$$

$$= \sqrt{3} \cdot \int_1^{18} z^{-1/2} dz$$

$$= \sqrt{3} \cdot \Big[2z^{1/2}\Big]_1^{18}$$

$$= 2\sqrt{3} \cdot \Big[z^{1/2}\Big]_1^{18}$$

$$= 2\sqrt{3}(18^{1/2} - 1^{1/2})$$

$$= 2\sqrt{3}(3\sqrt{2} - 1)$$

Example Use Part 2 of the Fundamental Theorem of Calculus to evaluate the integral.

(f) $\displaystyle\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}}\,dx$

FTC Part 2

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Solution

$$\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}}\,dx = 4\cdot\int_{1/2}^{1/\sqrt{2}} \frac{1}{\sqrt{1-x^2}}\,dx$$

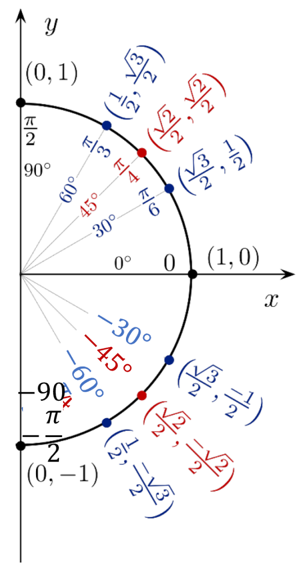
$$= 4\cdot\left[\sin^{-1}x\right]_{1/2}^{1/\sqrt{2}}$$

$$= 4\cdot\left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= 4\cdot\left[\frac{\pi}{4} - \frac{\pi}{6}\right]$$

$$= 4\cdot\left[\frac{3\pi}{12} - \frac{2\pi}{12}\right] = 4\cdot\left[\frac{\pi}{12}\right] = \frac{\pi}{3}$$

$y = \sin^{-1} x$



Example Use Part 2 of the Fundamental Theorem of Calculus to evaluate the integral.

(g) $\displaystyle\int_1^3 \frac{(3x+1)^2}{x^3}\,dx$

FTC Part 2

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Solution

$$\int_1^3 \frac{(3x+1)^2}{x^3}\,dx = \int_1^3 \frac{9x^2+6x+1}{x^3}\,dx$$

$$= \int_1^3 \left(\frac{9}{x} + \frac{6}{x^2} + \frac{1}{x^3}\right) dx$$

$$= \int_1^3 \left(\frac{9}{x} + \frac{6}{x^2} + \frac{1}{x^3}\right) dx$$

$$= 9 \cdot \int_1^3 \frac{1}{x}\,dx + 6 \cdot \int_1^3 x^{-2}\,dx + \int_1^3 x^{-3}\,dx$$

$$= \left[9\ln|x| - \frac{6}{x} - \frac{1}{2x^2}\right]_1^3$$

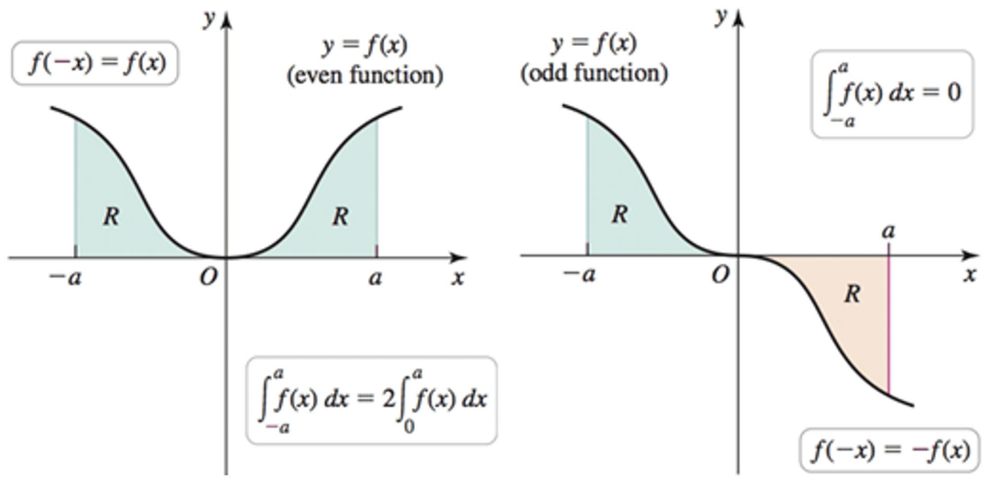
$$= \left(9\ln 3 - 2 - \tfrac{1}{18}\right) - \left(9\ln 1 - 6 - \tfrac{1}{2}\right)$$

$$= 9\ln 3 - 2 - \frac{1}{18} - 0 + 6 + \frac{1}{2}$$

$$= 9\ln 3 + \frac{40}{9}$$

Theorem

- If $f(x)$ is even, then $\int_{-a}^{a} f(x)\,dx = 2\cdot\int_{0}^{a} f(x)\,dx$.
- If $f(x)$ is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.



Theorem

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- If $f(x)$ is odd, then $\int_{-a}^{a} f(x)\,dx = 0.$

Example Evaluate the integrals.

(a) $\int_{-\pi/2}^{\pi/2} 6\cos x\,dx$ (b) $\int_{-6}^{6} \frac{-3x^5+2x}{\sqrt{x^4+1}}\,dx$

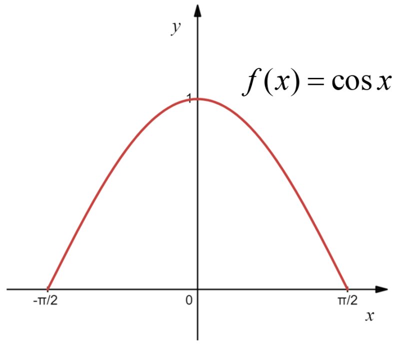
Solution

(a) $$\int_{-\pi/2}^{\pi/2} 6\cos x\,dx = 6\cdot\int_{-\pi/2}^{\pi/2} \cos x\,dx$$

$$= 6\cdot 2\cdot\int_{0}^{\pi/2} \cos x\,dx$$ $f(x)=\cos x$ is an even function.

$$= 12\cdot\left[\sin x\right]_0^{\pi/2}$$

$$= 12\cdot\left[\sin\left(\frac{\pi}{2}\right)-\sin(0)\right]_0^{\pi/2} = 12\cdot(1) = 12$$



Theorem

- If $f(x)$ is even, then $\int_{-a}^{a} f(x)\,dx = 2\cdot\int_{0}^{a} f(x)\,dx.$
- If $f(x)$ is odd, then $\int_{-a}^{a} f(x)\,dx = 0.$

Example Evaluate the integrals.

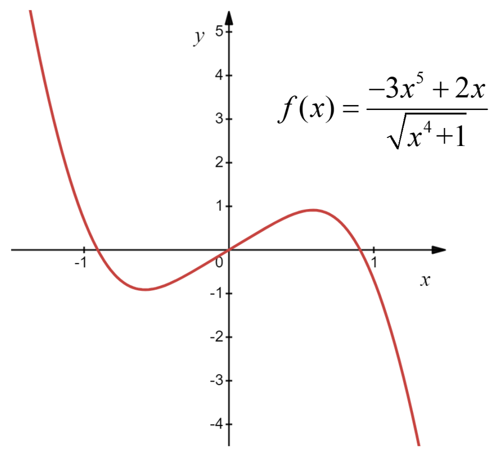
(a) $\int_{-\pi/2}^{\pi/2} 6\cos x\,dx$ (b) $\int_{-6}^{6} \frac{-3x^5+2x}{\sqrt{x^4+1}}\,dx$

Solution

(b) $\int_{-6}^{6} \frac{-3x^5+2x}{\sqrt{x^4+1}}\,dx = 0$

$f(x) = \frac{-3x^5+2x}{\sqrt{x^4+1}}$ is an odd function since

$$f(-x) = \frac{-3(-x)^5+2(-x)}{\sqrt{(-x)^4+1}} = \frac{3x^5-2x}{\sqrt{x^4+1}} = -f(x)$$

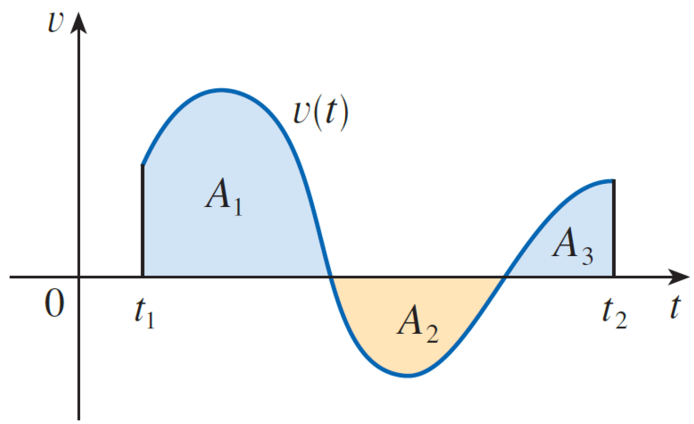


Recal the relationship between velocity, acceleration and the behavior of the particle.

Velocity	Acceleration	Behavior of Particle
Positive	Positive	Speeding Up
Positive	Negative	Slowing Down
Negative	Positive	Slowing Down
Negative	Negative	Speeding Up
Zero	Positive or Negative	Stopped
Positive or Negative	Zero	Constant Speed

The particle speeds up when the velocity is positive and increasing (v and a are both positive) and also when the velocity is negative and decreasing (v and a are both negative). In other words, the particle speeds up when the velocity and acceleration have the same sign. The particle is slowing down when velocity and acceleration have opposite sign.

The figure below shows how both displacement and total distance traveled can be interpreted in terms of areas under a velocity curve.



$$\text{displacement} = \int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3$$

$$\text{total distance} = \int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3$$

Example

T Consider the velocity function $v(t) = t \cdot \sin(t^2)$ for a particle moving horizontally, $t \geq 0$. Velocity measured in feet per second.

(a) Write an expression for the position function $s(t)$ of the particle if at time $t = 0$ seconds, the position of the particle is $s(0) = 12$ feet.

(b) Using a definite integral, find the displacement of the particle from $t = 0$ to $t = 4$ seconds.

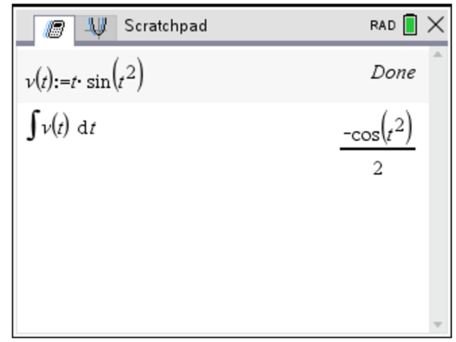
(c) Using a definite integral, find the total distance traveled by the particle from $t = 0$ to $t = 4$ seconds.

(d) Write an expression for the acceleration of the particle in terms of t.

(e) Is the particle speeding up or slowing down at $t = 3.293$ seconds. Justify your anwser.

Solution

(a) Write an expression for the position function $s(t)$ of the particle if at time $t = 0$ seconds, the position of the particle is $s(0) = 12$ feet.



$$s(t) = \int v(t)\,dt$$

$$= \int t \cdot \sin(t^2)\,dt$$

$$s(t) \overset{CAS}{=} -\frac{\cos(t^2)}{2} + C$$

We are given the initial conditions that $s(0) = 12,$ so we get

$$12 = -\frac{\cos(0)}{2} + C$$

$$12 = -\frac{1}{2} + C$$

$$C = \frac{25}{2}$$

Hence, $s(t) = -\dfrac{\cos(t^2)}{2} + \dfrac{25}{2}$

Solution

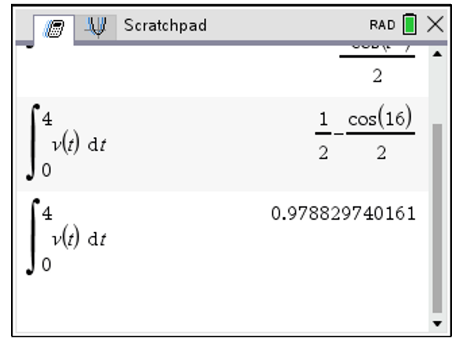
(b) Using a definite integral, find the displacement of the particle from $t = 0$ to $t = 4$ seconds.

$$\text{displacement} = \int_{t_1}^{t_2} v(t)\,dt$$

$$\text{displacement} = \int_0^4 t \cdot \sin(t^2)\ dt$$

$$\overset{CAS}{=} \frac{1-\cos(16)}{2} \text{ feet}$$

$$\overset{CAS}{=} 0.979 \text{ feet}$$

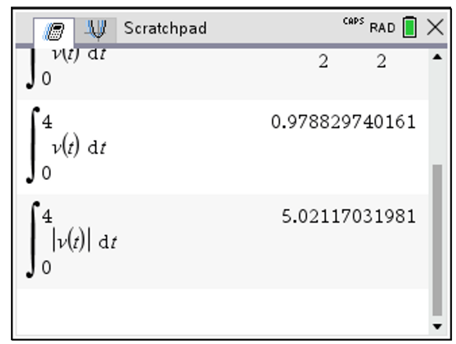


(c) Using a definite integral, find the total distance traveled by the particle from $t = 0$ to $t = 4$ seconds.

$$\text{total distance} = \int_{t_1}^{t_2} |v(t)|\,dt$$

$$\text{total distance} = \int_0^4 \left|t \cdot \sin(t^2)\right|\ dt$$

$$\overset{CAS}{=} 5.021 \text{ feet}$$



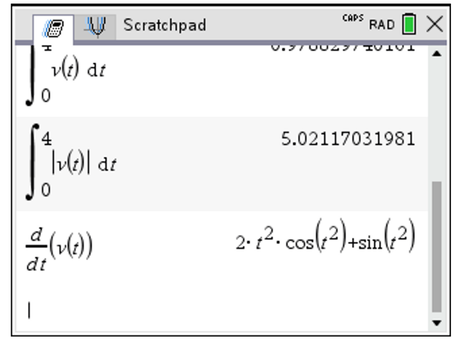
Solution

(d) Write an expression for the acceleration of the particle in terms of t.

$$a(t) = v'(t) = \frac{d}{dt}\left[v(t)\right]$$

$$a(t) = \frac{d}{dt}\left[t \cdot \sin(t^2)\right]$$

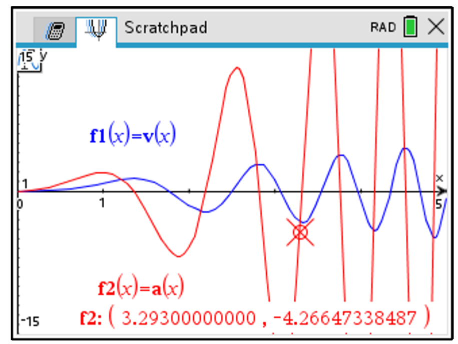
$$a(t) \overset{CAS}{=} 2 \cdot t^2 \cdot \cos(t^2) + \sin(t^2)$$



(e) Is the particle speeding up or slowing down at $t = 3.293$ seconds. Justify your anwser.

The particle is speeding up, since $v(3.293) \overset{CAS}{\approx} -3.255$ ft/sec and

$a(3.293) \overset{CAS}{\approx} -4.266$ ft/sec^2 have the same sign.



The Fundamental Theorem of Calculus, Part 1

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\, dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

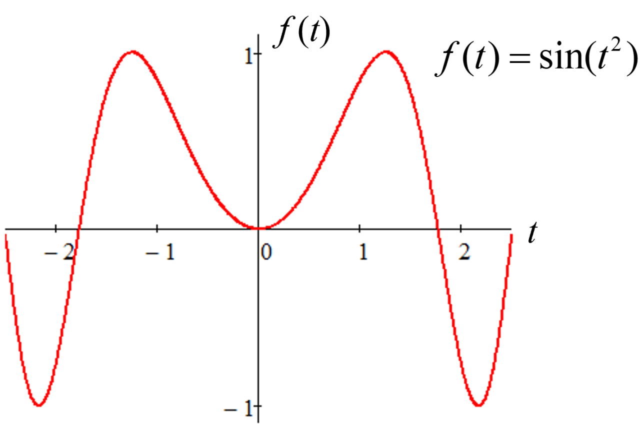
The Fundamental Theorem of Calculus, Part 1

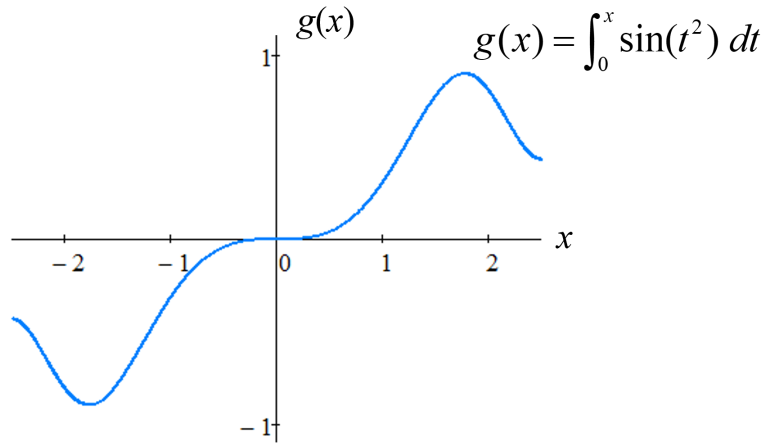
Example

Given $f(t) = \sin(t^2)$, the derivative of some function $g(x)$.

By the FTC Part 1 we can define a function $g(x)$ as follows

$$g(x) = \int_0^x \sin(t^2)dt.$$





$$g'(x) = \frac{d}{dx}\int_0^x \sin(t^2)dt = f(x)$$

Use the **First Fundamental Theorem of Calculus** to find the derivative of the function:

$$g(x)=\int_a^x f\left(t\right)dt$$

$$g'(x)=\frac{d}{dx}g(x)=\frac{d}{dx}\int_a^x f\left(t\right)dt=f\left(x\right)$$

Let’s see how it works.

First Fundamental Theorem:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

1. Derivative of an integral.

First Fundamental Theorem:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

1. Derivative of an integral.

2. Derivative matches upper limit of integration.

First Fundamental Theorem:

$$\frac{d}{dx}\int_{a}^{x} f\left(t\right)dt = f\left(x\right)$$

1. Derivative of an integral.

2. Derivative matches upper limit of integration.

3. Lower limit of integration is a constant.

First Fundamental Theorem:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

New variable.

1. Derivative of an integral.

2. Derivative matches upper limit of integration.

3. Lower limit of integration is a constant.

Example

The long way:

Find $\displaystyle \frac{d}{dx}\int_{-\pi}^{x} \cos t \; dt$

$$= \frac{d}{dx}\left(\sin t\Big|_{-\pi}^{x}\right)$$

$$= \frac{d}{dx}\left(\sin x - \cancelto{0}{\sin(-\pi)}\right)$$

$$= \frac{d}{dx}\sin x$$

$$= \cos x$$

Example

The long way:

Find $\dfrac{d}{dx}\displaystyle\int_{-\pi}^{x}\cos t\ dt = \cos x$

$$= \frac{d}{dx}\left(\sin t\Big|_{-\pi}^{x}\right)$$

$$= \frac{d}{dx}\left(\sin x - \underbrace{\cancel{\sin(-\pi)}}_{0}\right)$$

$$= \frac{d}{dx}\sin x$$

$$= \cos x$$

First Fundamental Theorem:

1. Derivative of an integral.
2. Derivative matches upper limit of integration.
3. Lower limit of integration is a constant.

Example

Let $g(x) = \int_0^x \frac{1}{1+t^2}\, dt$, find $\frac{d}{dx} g(x)$

Solution

$$\frac{d}{dx}\int_0^x \frac{1}{1+t^2}\, dt = \frac{1}{1+x^2}$$

1. Derivative of an integral.
2. Derivative matches upper limit of integration.
3. Lower limit of integration is a constant.

Example

Let $g(x)=\int_0^{x^2}\cos t\ dt$, find $\dfrac{d}{dx}g(x)$.

Solution

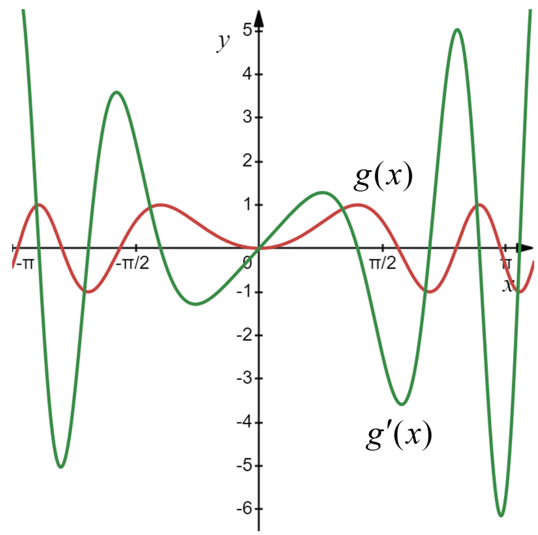
The upper limit of integration does <u>not</u> match the derivative, but we could use the **chain rule**.

$$\frac{d}{dx}g(x)=\frac{d}{dx}\int_0^{x^2}\cos t\ dt$$

$$=\cos\left(x^2\right)\cdot\frac{d}{dx}x^2$$

$$=\cos\left(x^2\right)\cdot 2x$$

$$=2x\cos\left(x^2\right)$$



Example

Let $g(x)=\int_{\ln x}^{5} 3t\sin t\ dt$, find $\frac{d}{dx}g(x)$.

Solution

$$\frac{d}{dx}g(x)=\frac{d}{dx}\int_{\ln x}^{5} 3t\sin t\ dt$$

The lower limit of integration is not a constant, but the upper limit is.

$$=-\frac{d}{dx}\int_{5}^{\ln x} 3t\sin t\ dt$$

We can **change the sign** of the integral and **reverse the limits**.

$$=-3\ln x\cdot\sin(\ln x)\cdot\frac{d}{dx}[\ln x]$$

$$=-3\ln x\cdot\sin(\ln x)\cdot\frac{1}{x}$$

$$=-\frac{3\ln x\cdot\sin(\ln x)}{x}$$

Example

Let $g(x)=\int_{2x}^{x^2}\frac{1}{2+e^t}\,dt$, find $\frac{d}{dx}g(x)$.

Solution

$$\frac{d}{dx}g(x)=\frac{d}{dx}\int_{2x}^{x^2}\frac{1}{2+e^t}\,dt$$

Neither limit of integration is a constant.

$$=\frac{d}{dx}\left(\int_{0}^{x^2}\frac{1}{2+e^t}\,dt+\int_{2x}^{0}\frac{1}{2+e^t}\,dt\right)$$

We **split the integral into two parts**.

It does not matter what constant we use! How about zero!

$$=\frac{d}{dx}\left(\int_{0}^{x^2}\frac{1}{2+e^t}\,dt-\int_{0}^{2x}\frac{1}{2+e^t}\,dt\right)$$

(Limits are reversed.)

$$=\frac{1}{2+e^{x^2}}\cdot 2x-\frac{1}{2+e^{2x}}\cdot 2$$

(Chain Rule.)

$$=\frac{2x}{2+e^{x^2}}-\frac{2}{2+e^{2x}}$$

The Fundamental Theorem of Calculus, Part 1

$$g(x) = \int_a^{u(x)} f(t)dt \quad \text{then} \quad \frac{d}{dx}\, g(x) = f(u(x)) \cdot \frac{d}{dx} u(x)$$

or

$$g(x) = \int_a^{u(x)} f(t)dt \quad \text{then} \quad g'(x) = f(u(x)) \cdot u'(x)$$

CHAIN RULE

We end this section by bringing together the two parts of the Fundamental Theorem.

The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

We noted that Part 1 can be rewritten as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

which says that if f is integrated and then the result is differentiated, we arrive back at the original function f.

Part 2 is a formula for computing the definite integral, the area under the curve.