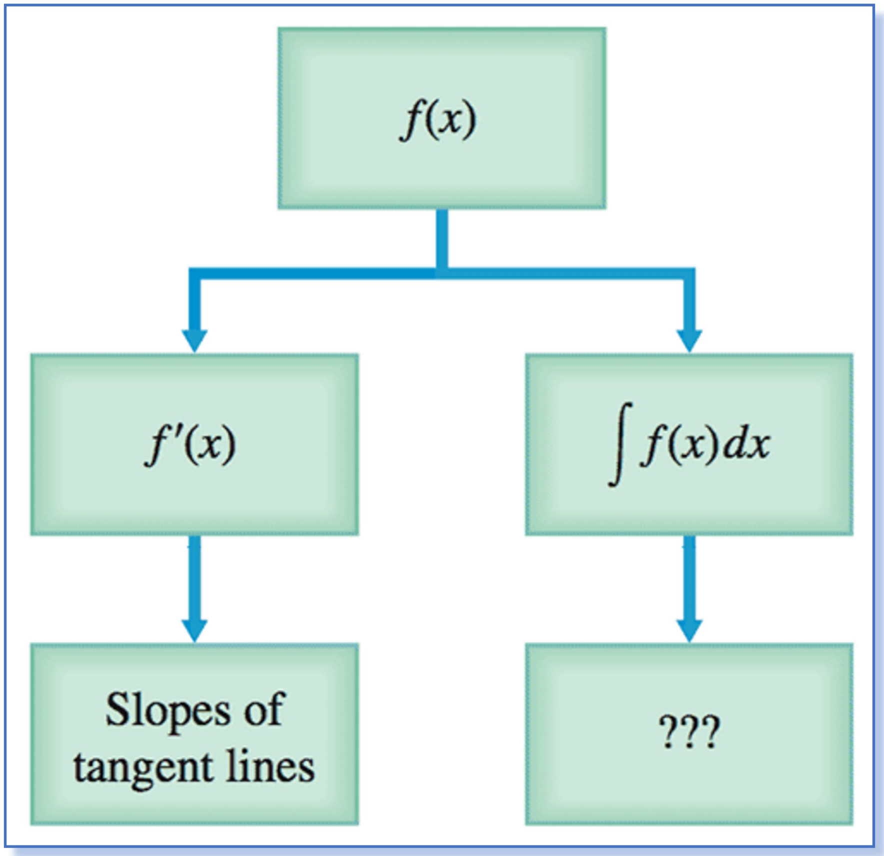
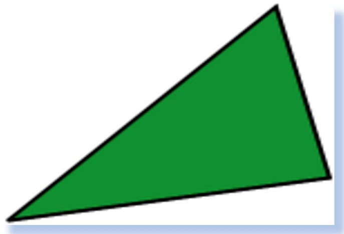
5.2 The Definite Integral

Johann Bernoulli
1667 – 1748

Johann Bernoulli was a Swiss mathematician and was one of the many prominent mathematicians in the Bernoulli family. He is known for his contributions to infinitesimal calculus and educated Leonhard Euler in his youth.

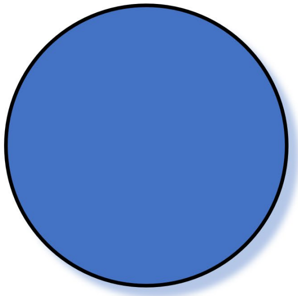

$f(x)$
$f'(x)$
$\int f(x)dx$
Slopes of tangent lines
???



$$A = \frac{1}{2}bh$$

$$A = bh$$

$$A = \frac{1}{2}(b_1 + b_2)h$$



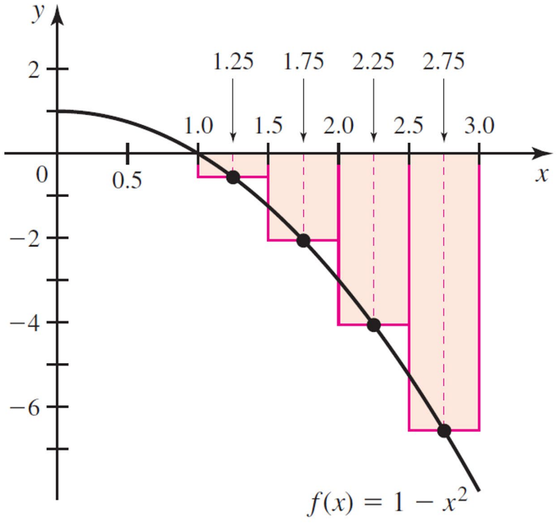
$$A = \pi r^2$$

Net Area

Example

Evaluate and interpret the following Riemann sums for $f(x) = 1 - x^2$ on the interval $[a, b]$ with n equally spaced subintervals. A midpoint Riemann sum with $[a, b] = [1, 3]$ and $n = 4$

Solution



$$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = 0.5$$

$$x_i^* = \bar{x}_i = a + (i - 1/2) \cdot \Delta x = 1 + (i - 1/2) \cdot (0.5)$$

$$\sum_{i=1}^{4} f\left(x_i^*\right) \cdot \Delta x = \sum_{i=1}^{4} f\left(1 + (i - 1/2) \cdot (0.5)\right) \cdot 0.5$$

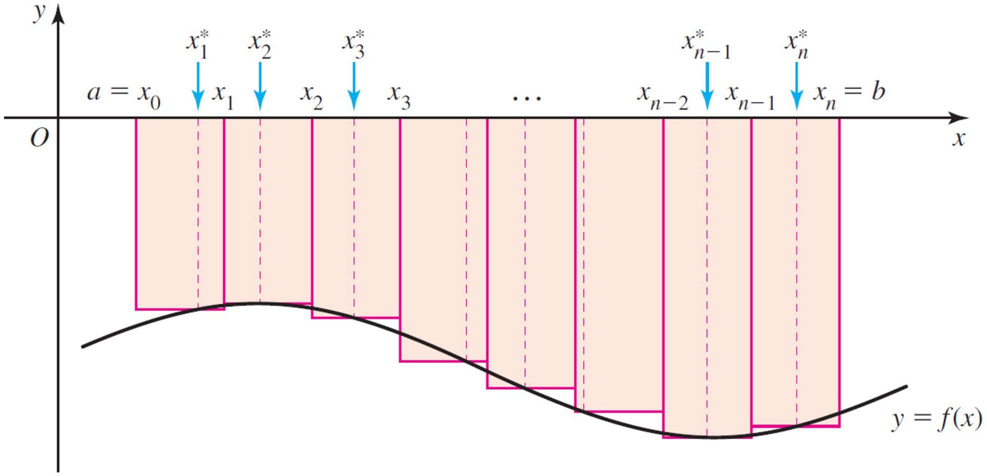
$$= f(1.25)\,(0.5) + f(1.75)\,(0.5) + f(2.25)\,(0.5) + f(2.75)\,(0.5)$$

$$= (-0.5625 - 2.0625 - 4.0625 - 6.5625)\,(0.5)$$

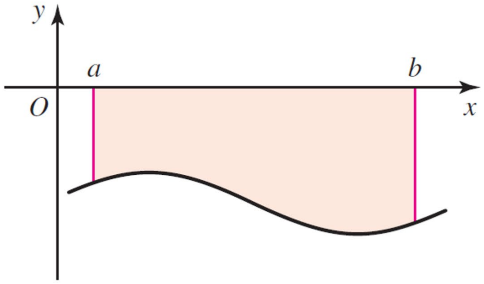
$$= -6.625$$

Negative area? The area is below the *x*-axis.

Net Area



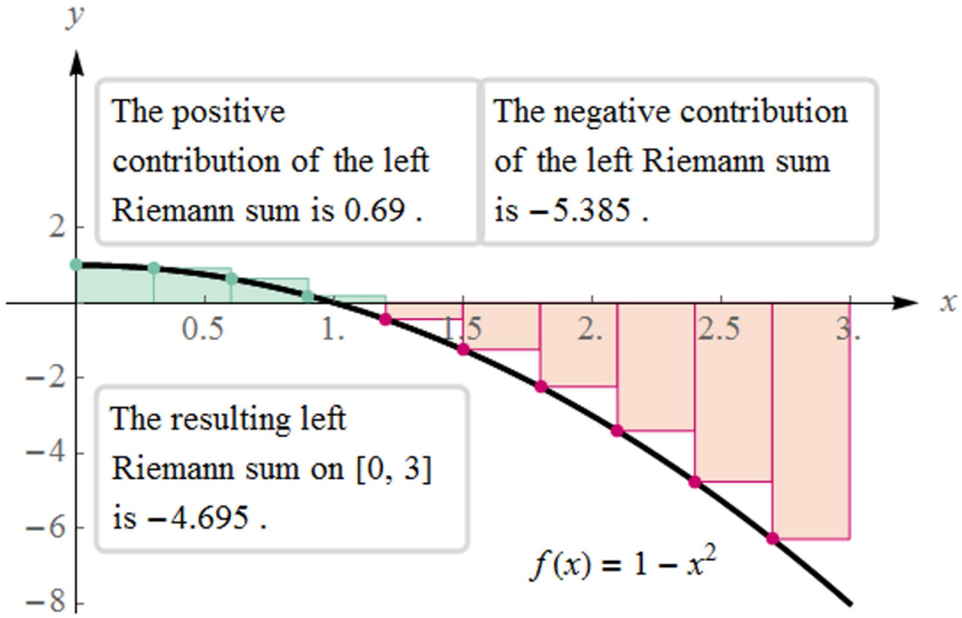
The Riemann sum $\sum_{k=1}^{n} f(x_k^*)\Delta x$ approximates the negative of the area of the region bounded by the x-axis and the curve.



Net Area

Consider the function $f(x) = 1 - x^2$ on the $[0,3]$.

Using a Left-sum with 10 subintervals:



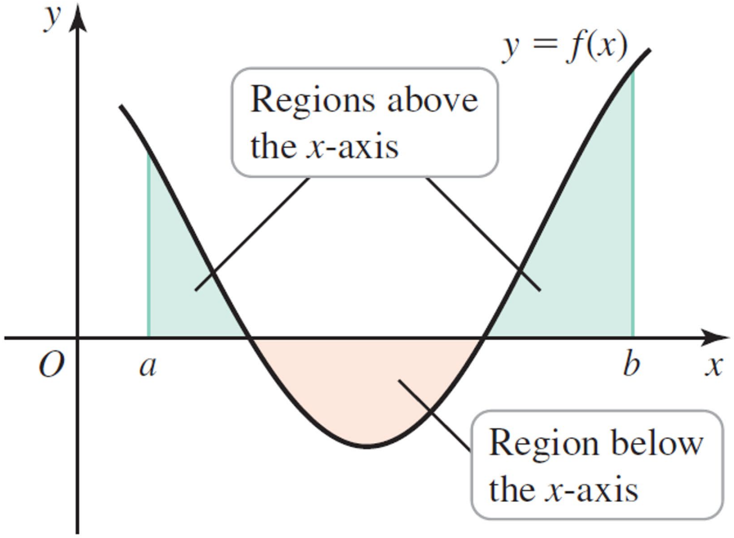
The positive contribution of the left Riemann sum is 0.69 .

The negative contribution of the left Riemann sum is −5.385 .

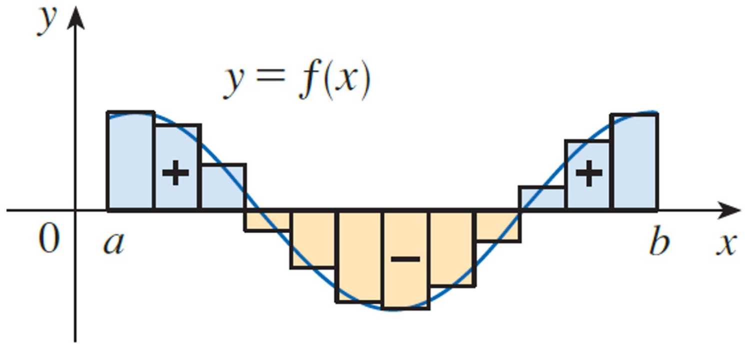
The resulting left Riemann sum on [0, 3] is −4.695 .

Definition **Net Area**

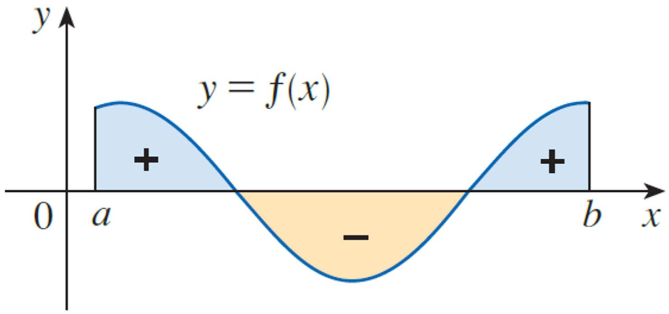
Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the areas of the parts of R that lie above the x-axis *minus* the sum of the areas of the parts of R that lie below the x-axis on $[a, b]$.



Net Area



$\sum f(x_i^*)\,\Delta x$ is an approximation to the net area.



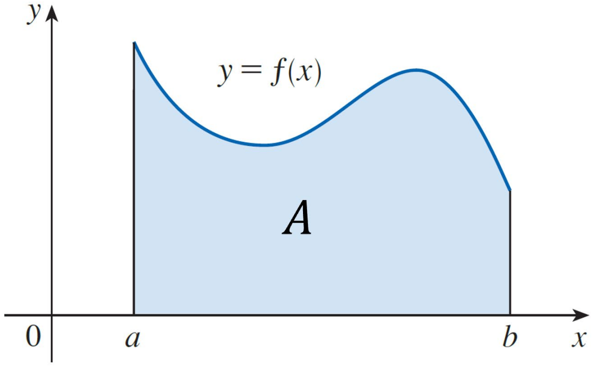
$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$ is the net area.

We get the following definition.

2 Definition of a Definite Integral If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 \,(= a), x_1, x_2, \ldots, x_n \,(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

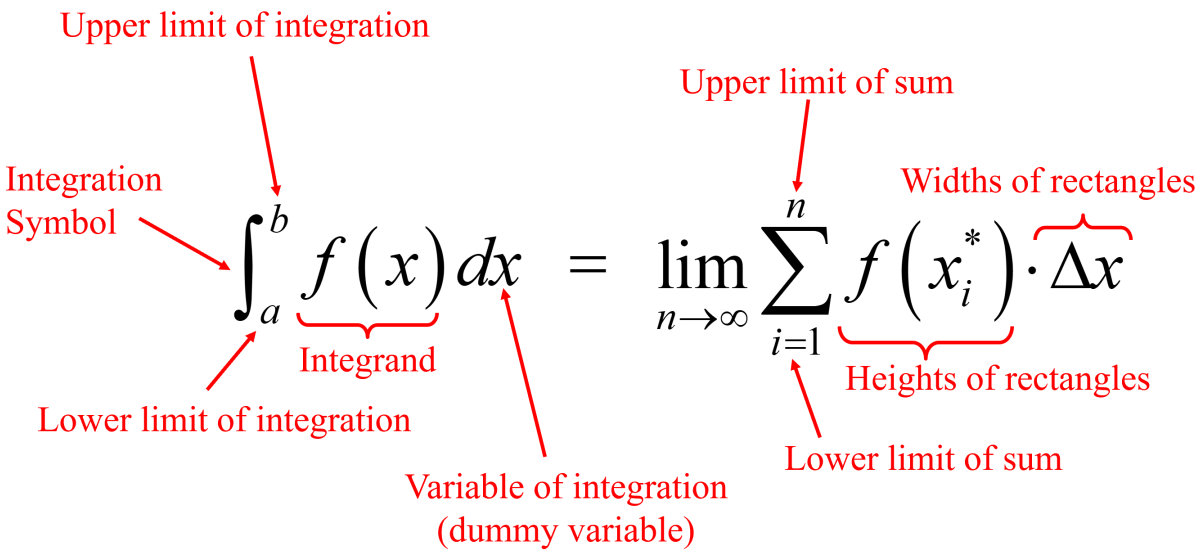
provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.



$$A = \int_a^b f(x)\,dx$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

The Definite Integral



Upper limit of integration

Upper limit of sum

Integration Symbol

$$\int_a^b \underbrace{f(x)}_{\text{Integrand}}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\underbrace{f\left(x_i^*\right)}_{\text{Heights of rectangles}}\cdot\overbrace{\Delta x}^{\text{Widths of rectangles}}$$

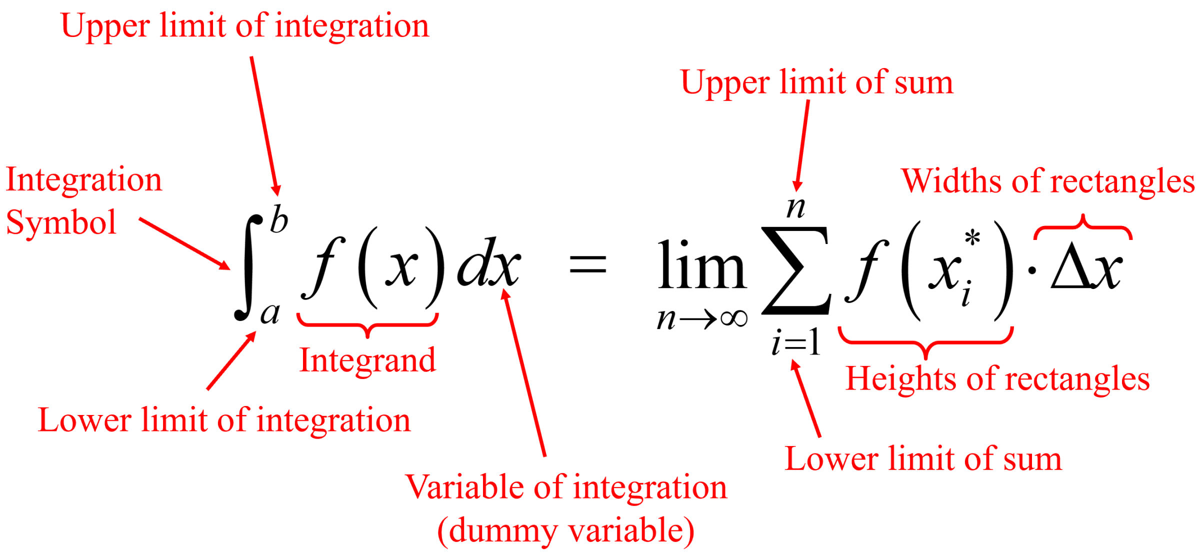
Lower limit of integration

Lower limit of sum

Variable of integration
(dummy variable)

Note 1 The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. It is an elongated *S* and was chosen because an integral is a limit of sums. For now, the symbol *dx* has no meaning by itself; $\int_a^b f(x)dx$ is all one symbol. The *dx* simply indicates that the independent variable is *x*. The procedure of calculating an integral is called **integration**.

The Definite Integral



Upper limit of integration

Upper limit of sum

Integration Symbol

Widths of rectangles

$$\int_a^b \underbrace{f(x)}_{\text{Integrand}}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} \underbrace{f\left(x_i^*\right)}_{\text{Heights of rectangles}}\cdot \overbrace{\Delta x}^{\text{Widths of rectangles}}$$

Lower limit of integration

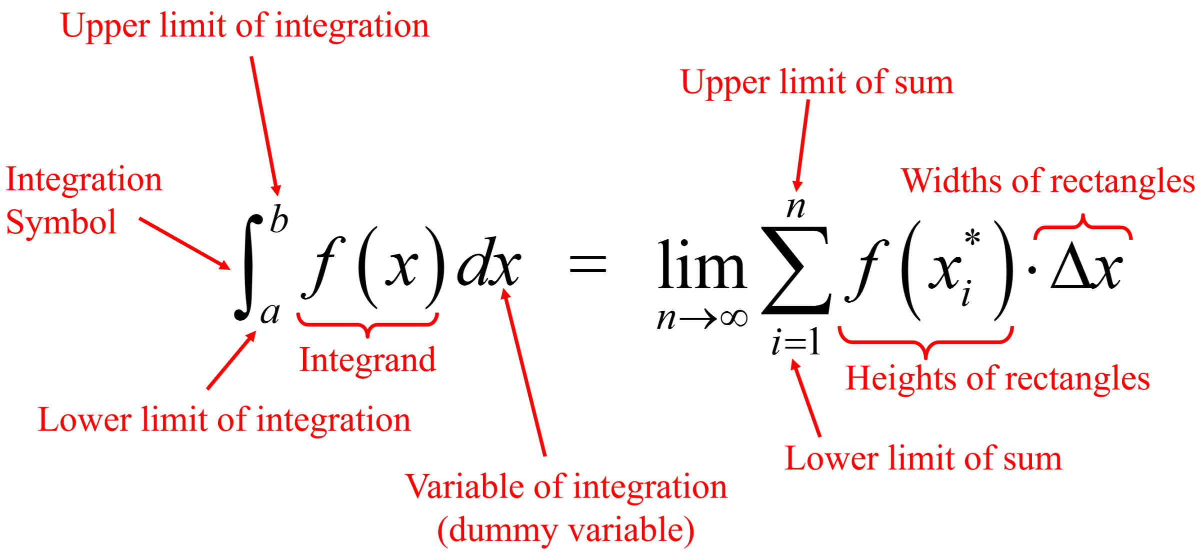
Lower limit of sum

Variable of integration
(dummy variable)

Note 2 The definite integral $\int_a^b f(x)dx$ is a number; it does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

The Definite Integral



$$\int_a^b \underbrace{f(x)}_{\text{Integrand}}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} \underbrace{f\left(x_i^*\right)}_{\text{Heights of rectangles}}\cdot \overbrace{\Delta x}^{\text{Widths of rectangles}}$$

Upper limit of integration

Integration Symbol

Lower limit of integration

Variable of integration (dummy variable)

Upper limit of sum

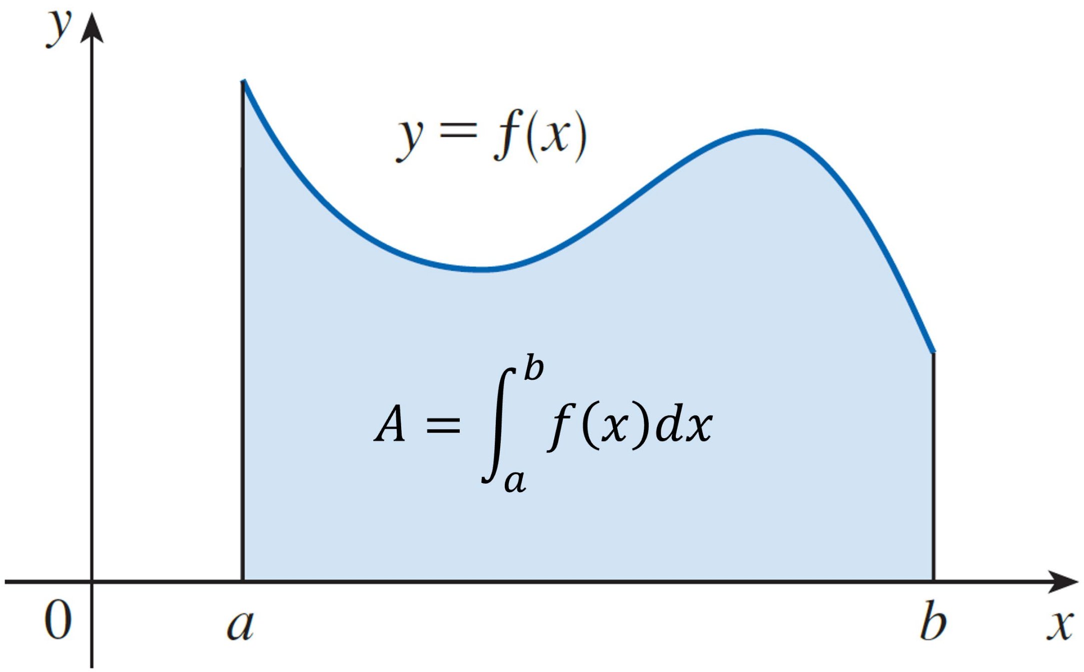
Lower limit of sum

Note 3 The sum

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

that occurs in Definition 2 is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826 –1866). So, Definition 2 says that the definite integral of an integrable function can be approximated to within any desired degree of accuracy by a Riemann sum.

The Definite Integral



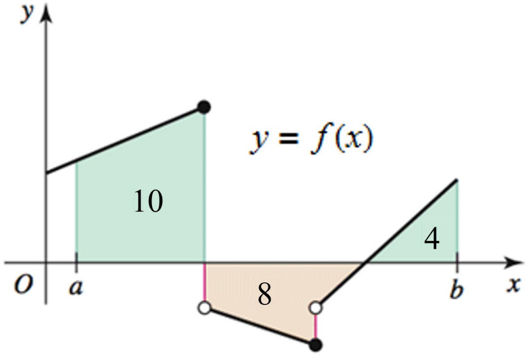
$$A = \int_a^b f(x)dx$$

How do we evaluate the definite integral?

3 Theorem If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

Evaluating Definite Integrals Using Geometry

Example Find $\int_a^b f(x)\,dx$



Solution

$$\int_a^b f(x)\,dx = 10 - 8 + 4 = 6$$

Evaluating Definite Integrals Using Geometry

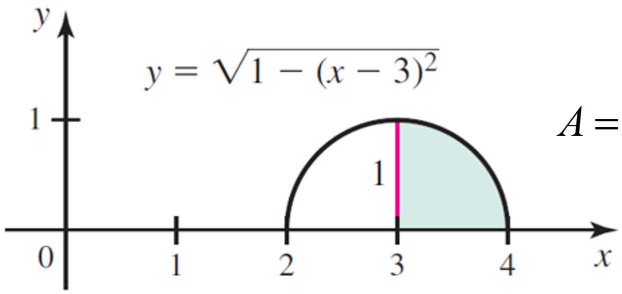
Example Evaluate the following integrals by interpreting each in terms of areas.

(a) $\int_3^4 \sqrt{1-(x-3)^2}\,dx$

(b) $\int_0^3 (x-1)\,dx$

Solution

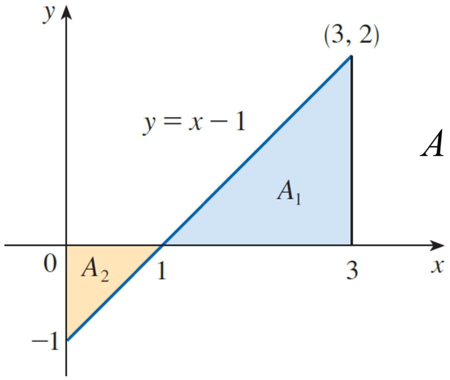
(a)



$A = \pi r^2$

$$\int_3^4 \sqrt{1-(x-3)^2}\,dx = \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$$

(b)



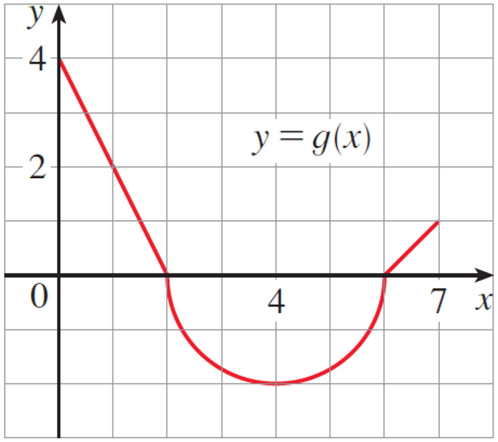
$A = \frac{1}{2}bh$

$$\begin{aligned}\int_0^3 (x-1)\,dx &= A_1 - A_2\\ &= \tfrac{1}{2}(2\cdot 2) - \tfrac{1}{2}(1\cdot 1)\\ &= 1.5\end{aligned}$$

Evaluating Definite Integrals Using Geometry

Example The graph of $g(x)$ consists of two straight lines and a semi-circle. Evaluate the integral by interpreting it in terms of areas.

$$\int_0^7 g(x)\,dx$$



Solution

$\int_0^2 g(x)\,dx = \frac{1}{2} \cdot 4 \cdot 2 = 4$ [area of a triangle]

$\int_2^6 g(x)\,dx = -\frac{1}{2}\pi(2)^2 = -2\pi$ [negative of the area of a semicircle]

$\int_6^7 g(x)\,dx = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$ [area of a triangle]

$\int_0^7 g(x)\,dx = \int_0^2 g(x)\,dx + \int_2^6 g(x)\,dx + \int_6^7 g(x)\,dx = 4 - 2\pi + \frac{1}{2} = \frac{9}{2} - 2\pi$

Evaluating Definite Integrals Using Algebra

4 Theorem If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\Delta x$

Example Express $\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\,\Delta x$ as an integral on the interval $[0, \pi]$.

Solution

Comparing the given limit with the limit in the theorem above, we see that they will be identical if we choose $f(x) = x^3 + x \sin x$. We are given that $a = 0$ and $b = \pi/2$. Therefore, by the above theorem we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\,\Delta x = \int_0^{\pi} (x^3 + x \sin x)\,dx$$

Evaluating Definite Integrals Using Algebra

Sums of Powers

Let n be a positive integer and c a real number.

1) $\displaystyle\sum_{i=1}^{n} 1 = n$

2) $\displaystyle\sum_{i=1}^{n} c = cn$

3) $\displaystyle\sum_{i=1}^{n} i = \frac{n}{2} + \frac{n^2}{2} = \frac{n(n+1)}{2}$

4) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3} = \frac{n(n+1)(2n+1)}{6}$

5) $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2}{4} + \frac{n^3}{2} + \frac{n^4}{4} = \left(\frac{n(n+1)}{2}\right)^2$

Properties of Sums

1) $\displaystyle\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

2) $\displaystyle\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

3) $\displaystyle\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a\,\boxed{6} + \boxed{7}\sum_{i=1}^{n} b_i$

From section 5.1 recall the function $f(x)=\frac{1}{8}x^2+1$ on the $[0,4]$.

Using a Right-sum with 8 subintervals:

$$\text{Let } \Delta x=\frac{b-a}{n}=\frac{4-0}{8}=\frac{1}{2}$$

Right sample point:

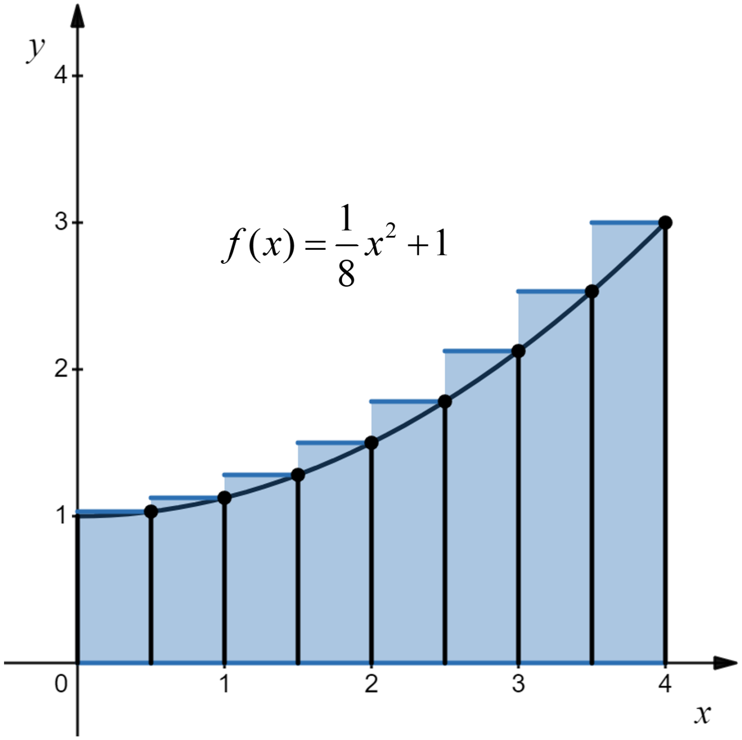
$$x_i=a+i\cdot\Delta x=0+i\cdot\frac{1}{2}=i\cdot\frac{1}{2}$$

$$R_8=\sum_{i=1}^{8}f(x_i)\cdot\Delta x$$

$$R_8=\sum_{i=1}^{8}\left[\frac{1}{8}\left(i\cdot\frac{1}{2}\right)^2+1\right]\cdot\frac{1}{2}$$

$$R_8\overset{CAS}{=}7.1875$$

What is the exact area?



Example Find the exact area under the function $f(x)=\frac{1}{8}x^2+1$ on the $[0,4]$.

Solution Using the limit of a Riemann sum:

$$\int_0^4 f(x)dx=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x$$

Let $\Delta x=\frac{b-a}{n}=\frac{4-0}{n}=\frac{4}{n}$

Right sample point:

$$x_i=a+i\cdot\Delta x=0+i\cdot\frac{4}{n}=i\cdot\frac{4}{n}$$

$$\int_0^4\left(\frac{1}{8}x^2+1\right)dx=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{1}{8}\left(i\cdot\frac{4}{n}\right)^2+1\right]\cdot\frac{4}{n}$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{1}{8}\left(i^2\cdot\frac{16}{n^2}\right)+1\right]\cdot\frac{4}{n}$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left[i^2\cdot\frac{2}{n^2}+1\right]\cdot\frac{4}{n}$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left[i^2\cdot\frac{8}{n^3}+\frac{4}{n}\right]$$

$$=\lim_{n\to\infty}\left[\sum_{i=1}^{n} i^2\cdot\frac{8}{n^3}+\sum_{i=1}^{n}\frac{4}{n}\right]$$

$$=\lim_{n\to\infty}\left[\frac{8}{n^3}\cdot\sum_{i=1}^{n} i^2+\frac{4}{n}\cdot\sum_{i=1}^{n} 1\right]$$

$$=\lim_{n\to\infty}\left[\frac{8}{n^3}\cdot\left(\frac{n}{6}+\frac{n^2}{2}+\frac{n^3}{3}\right)+\frac{4}{n}\cdot n\right]$$

$$=\lim_{n\to\infty}\left[\frac{4}{3n^2}+\frac{4}{n}+\frac{8}{3}+4\right]$$

$$=0+0+\frac{8}{3}+4=\frac{20}{3}=6.\overline{6}$$

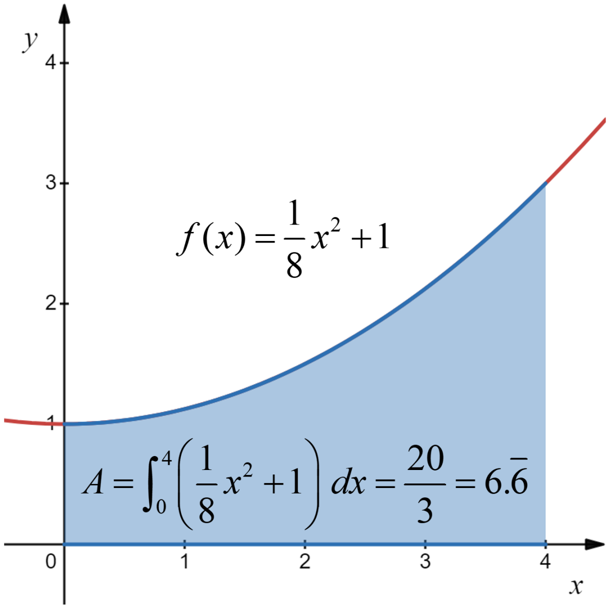
Special Sums

$$\sum_{i=1}^{n} 1=n$$

$$\sum_{i=1}^{n} i^2=\frac{n}{6}+\frac{n^2}{2}+\frac{n^3}{3}$$

So the exact area under the curve is:

$$\int_0^4 \left(\frac{1}{8}x^2 + 1\right) dx = \frac{20}{3} = 6.\overline{6}$$



That was fun!! Let’s do another one!!

Example Use the form of the definition of the integral given in Theorem 4 to evaluate the integral.

$$\int_0^1 (x^3 - 3x^2)\,dx$$

Solution

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

$$x_i = 0 + i\,\Delta x = \frac{i}{n}$$

$$\int_0^1 (x^3 - 3x^2)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\cdot\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^3 - 3\left(\frac{i}{n}\right)^2\right]\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{i^3}{n^3} - 3\cdot\frac{i^2}{n^2}\right]\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{1}{n^4}\cdot i^3 - \frac{3}{n^3}\cdot i^2\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n^4}\cdot\sum_{i=1}^{n} i^3 - \frac{3}{n^3}\cdot\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n^4}\cdot\left(\frac{n^2}{4}+\frac{n^3}{2}+\frac{n^4}{4}\right) - \frac{3}{n^3}\cdot\left(\frac{n}{6}+\frac{n^2}{2}+\frac{n^3}{3}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{2}{n}+\frac{1}{4}-\frac{1}{2n^2}-\frac{3}{2n}-1\right]$$

$$= 0 + 0 + \frac{1}{4} - 0 - 0 - 1 = -\frac{3}{4}$$

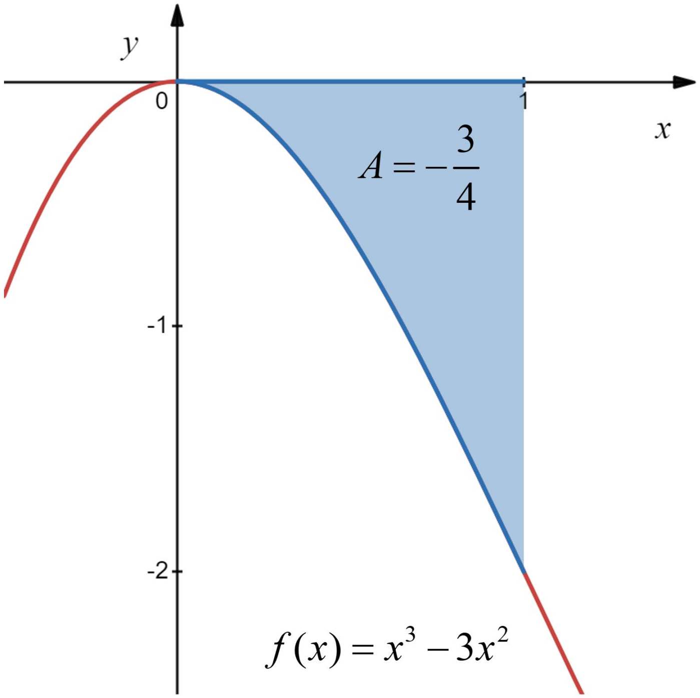
Special Sums

$$\sum_{i=1}^{n} i^2 = \frac{n}{6}+\frac{n^2}{2}+\frac{n^3}{3}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2}{4}+\frac{n^3}{2}+\frac{n^4}{4}$$

So the exact area under the curve is:

$$\int_0^1 (x^3 - 3x^2)\,dx = -\frac{3}{4}$$



Properties of the Definite Integral

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

Properties of the Integral

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant

2. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

4. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Recall the Midpoint Riemann Sum,

$$M_n = \sum_{i=1}^{n} f(\overline{x}_i)\cdot \Delta x \quad \text{where, } \Delta x = \frac{b-a}{n} \text{ and } \overline{x}_i = a + \left(i - \frac{1}{2}\right)\cdot \Delta x$$

Example Use the Midpoint Rule with the given value of n to approximate the integral. Round the answer to four decimal places.

$$\int_0^{\pi} x \sin^2 x \, dx, \quad n = 4$$

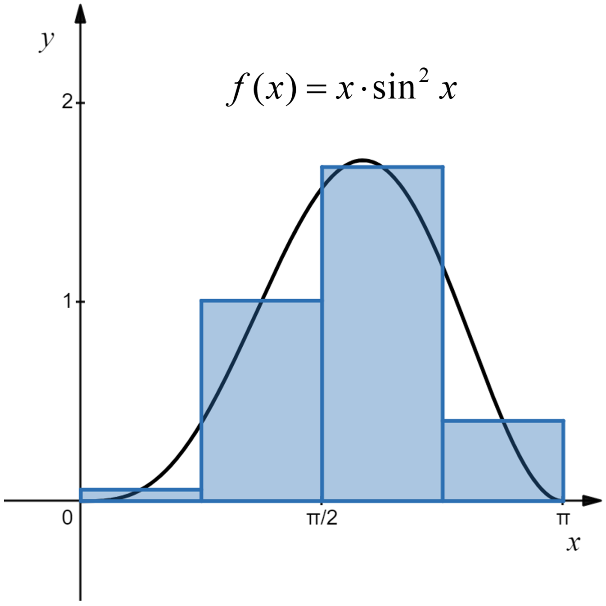
Solution

$$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

$$\overline{x}_i = a + \left(i - \frac{1}{2}\right)\cdot \Delta x = 0 + \left(i - \frac{1}{2}\right)\cdot \frac{\pi}{4} = \left(i - \frac{1}{2}\right)\cdot \frac{\pi}{4}$$

The Midpoint Riemann Sum is

$$M_4 = \sum_{i=1}^{4} f(\overline{x}_i)\cdot \Delta x = \sum_{i=1}^{4}\left(f\left(\left(i - \frac{1}{2}\right)\cdot \frac{\pi}{4}\right)\right)\cdot \frac{\pi}{4} \approx 2.4674$$



Example (a) Set up an expression for $\int_1^3 e^x\,dx$ as a limit of sums.
(b) Use a computer algebra system to evaluate the expression.

Solution

(a) Here we have $f(x) = e^x$, $a = 1$, $b = 3$, and

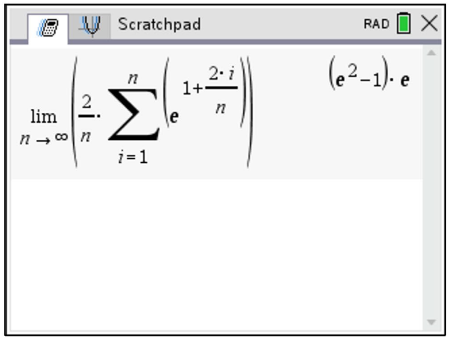
$$\Delta x = \frac{b-a}{n} = \frac{2}{n}$$

$$x_i = a + i\,\Delta x$$

$$x_i = 1 + \frac{2i}{n}$$

$$\begin{aligned}\int_1^3 e^x\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{2i}{n}\right)\frac{2}{n} \\ &= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n} e^{1+2i/n} \\ &= (e^2-1)\cdot e\end{aligned}$$

(b)



$$\lim_{n\to\infty}\left(\frac{2}{n}\cdot\sum_{i=1}^{n}\left(e^{1+\frac{2\cdot i}{n}}\right)\right) \qquad (e^2-1)\cdot e$$

How do we do this by hand? Let's go to section 5.3 and find out!!