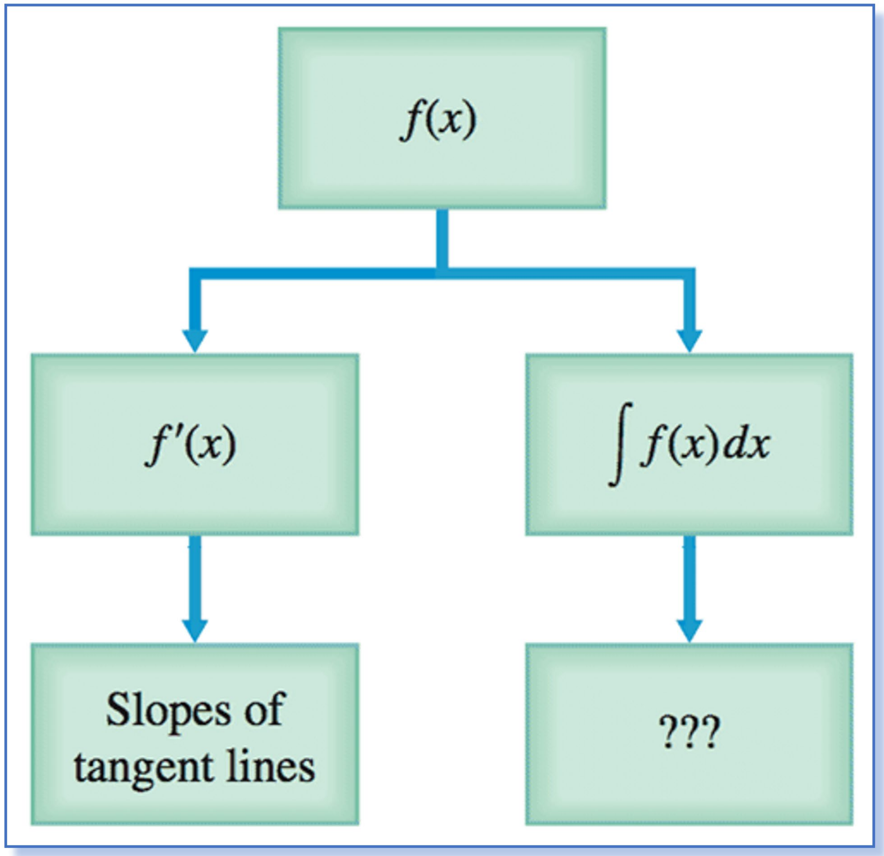
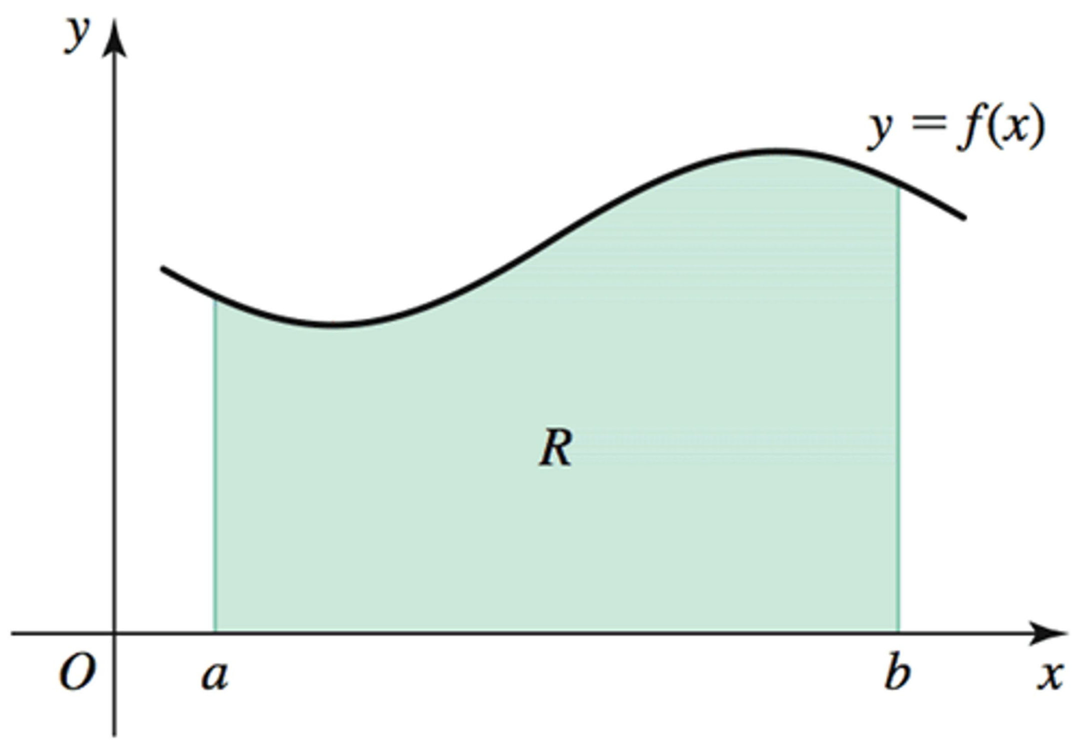
5.1 The Area and Distance Problems

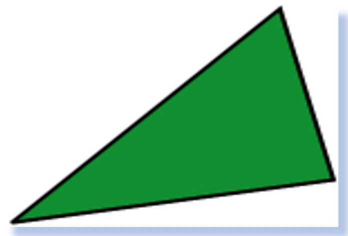
Jacob Bernoulli
1654 – 1705

Jacob Bernoulli was a Swiss mathematician who became familiar with calculus through a correspondence with Gottfried Leibniz, then collaborated with his brother Johann on various applications, notably publishing papers on transcendental curves.

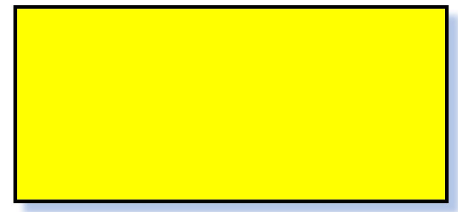

$f(x)$
$f'(x)$
$\int f(x)dx$
Slopes of tangent lines
???

The Area Problem



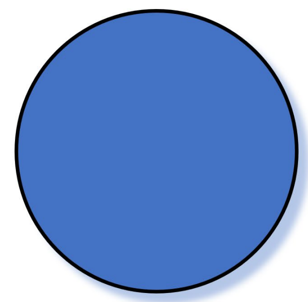


$$A = \frac{1}{2}bh$$



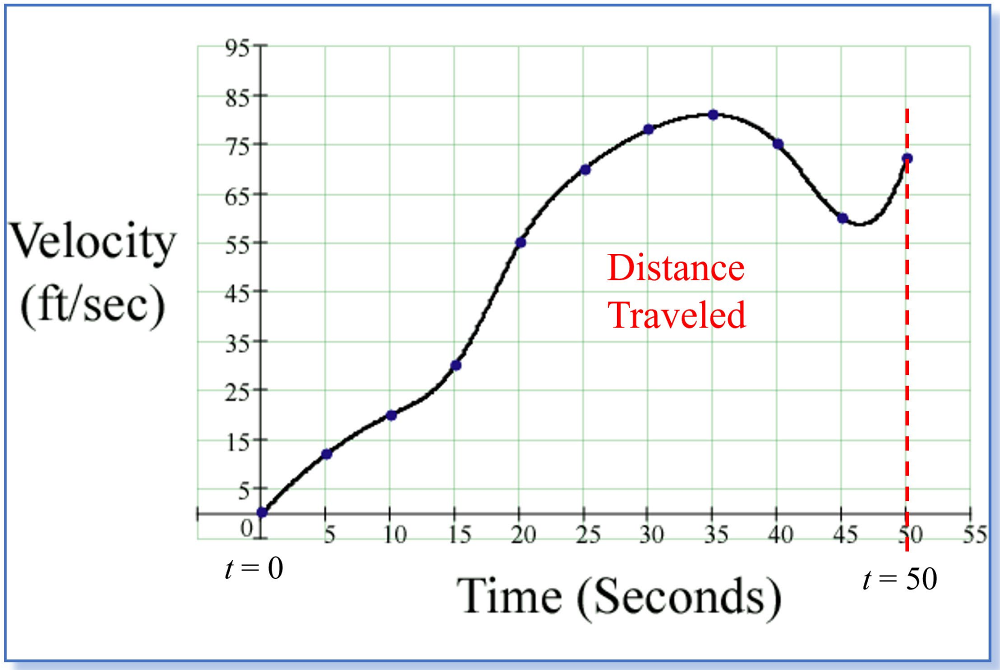
$$A = bh$$

$$A = \frac{1}{2}\left(b_1 + b_2\right)h$$

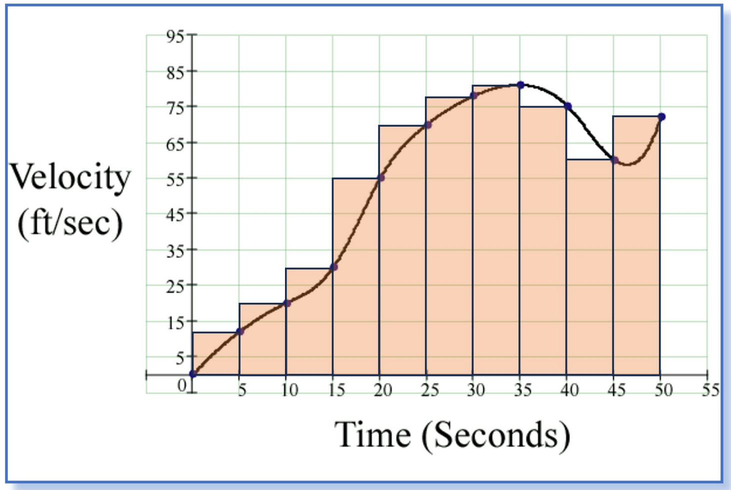


$$A = \pi r^2$$

How about the area under the curve and above the *x*-axis? What does that area represent?



t (seconds)	$v(t)$ (feet per second)
0	0
5	12
10	20
15	30
20	55
25	70
30	78
35	81
40	75
45	60
50	72



Right-Hand Rectangles

$$R_{10} = (12 + 20 + 30 + 55 + 70 + 78 + 81 + 75 + 60 + 72) \cdot 5 = 2765 \text{ feet}$$

t (seconds)	$v(t)$ (feet per second)
0	0
5	12
10	20
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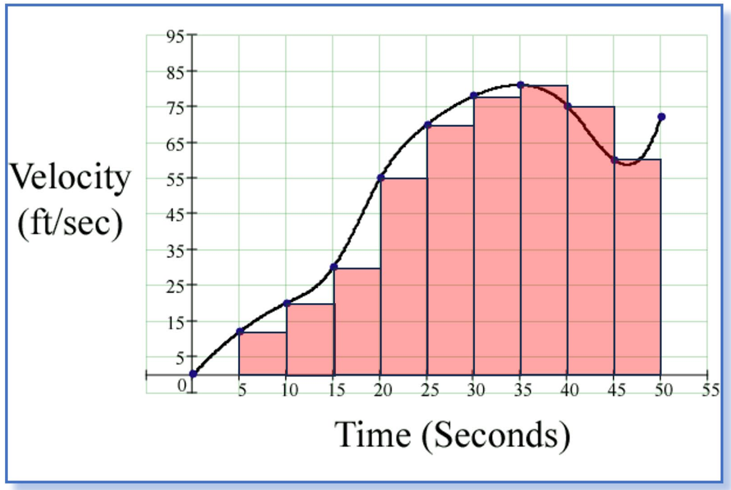


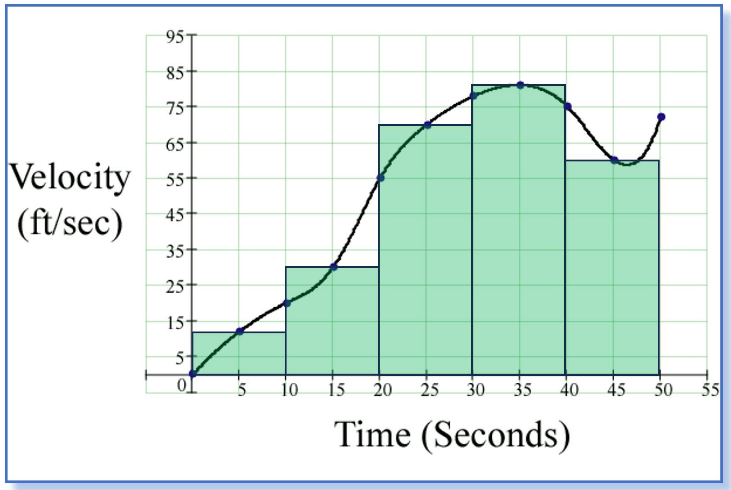
Right-Hand Rectangles

$$R_{10} = (12 + 20 + 30 + 55 + 70 + 78 + 81 + 75 + 60 + 72) \cdot 5 = 2765 \text{ feet}$$

Left-Hand Rectangles

$$L_{10} = (0 + 12 + 20 + 30 + 55 + 70 + 78 + 81 + 75 + 60) \cdot 5 = 2405 \text{ feet}$$

t (seconds)	$v(t)$ (feet per second)
0	0
5	12
10	20
15	30
20	55
25	70
30	78
35	81
40	75
45	60
50	72



Right-Hand Rectangles

$$R_{10} = (12 + 20 + 30 + 55 + 70 + 78 + 81 + 75 + 60 + 72) \cdot 5 = 2765 \text{ feet}$$

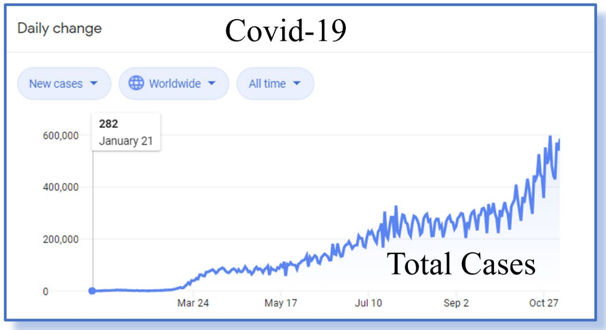
Left-Hand Rectangles

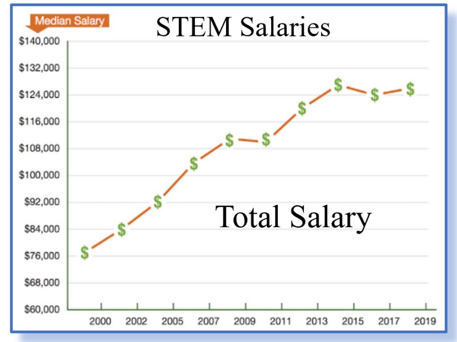
$$L_{10} = (0 + 12 + 20 + 30 + 55 + 70 + 78 + 81 + 75 + 60) \cdot 5 = 2405 \text{ feet}$$

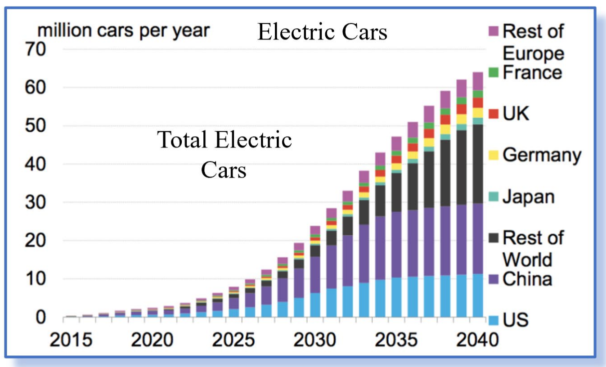
Midpoint Rectangles

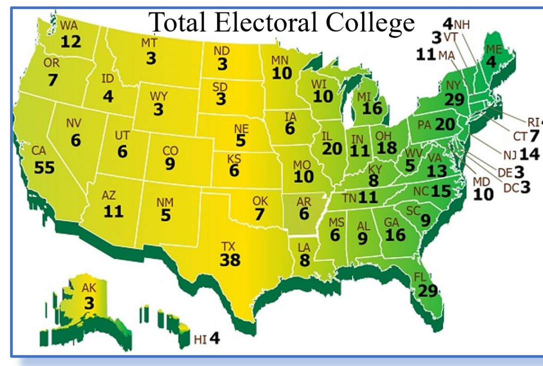
$$M_5 = (12 + 30 + 70 + 81 + 60) \cdot 10 = 2530 \text{ feet}$$

“Area” Under the Curve







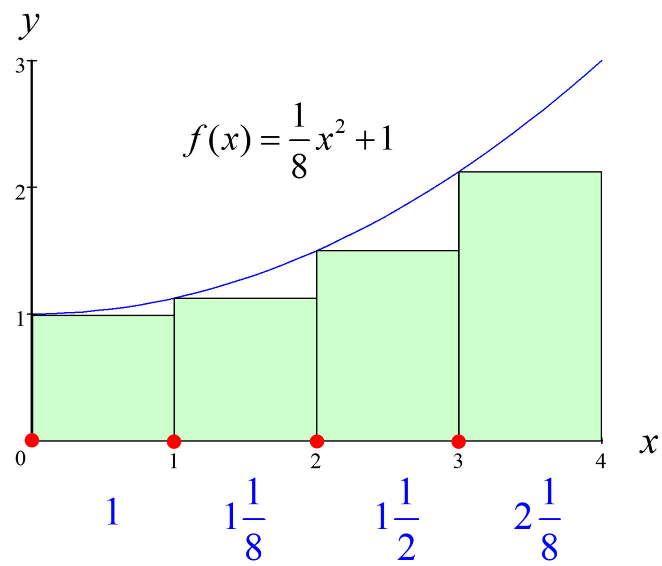


Example Find the area under the curve using the left endpoint of four subintervals over the interval [0, 4]. **Left-hand Rectangles**

$$f(x) = \frac{1}{8}x^2 + 1$$

Solution

x	$f(x)$
0	1
1	$1\frac{1}{8}$
2	$1\frac{1}{2}$
3	$2\frac{1}{8}$



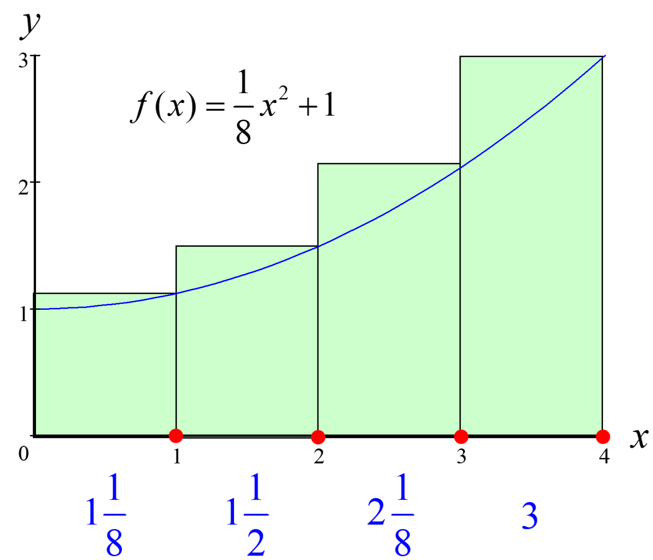
Approximate area: $L_4 = \left(1 + 1\frac{1}{8} + 1\frac{1}{2} + 2\frac{1}{8}\right) \cdot 1 = 5\frac{3}{4} = 5.75$

Example Find the area under the curve using the right endpoint of four subintervals over the interval [0, 4]. <u>**Right-hand Rectangles**</u>

$$f(x) = \frac{1}{8}x^2 + 1$$

Solution

x	$f(x)$
1	$1\frac{1}{8}$
2	$1\frac{1}{2}$
3	$2\frac{1}{8}$
4	3



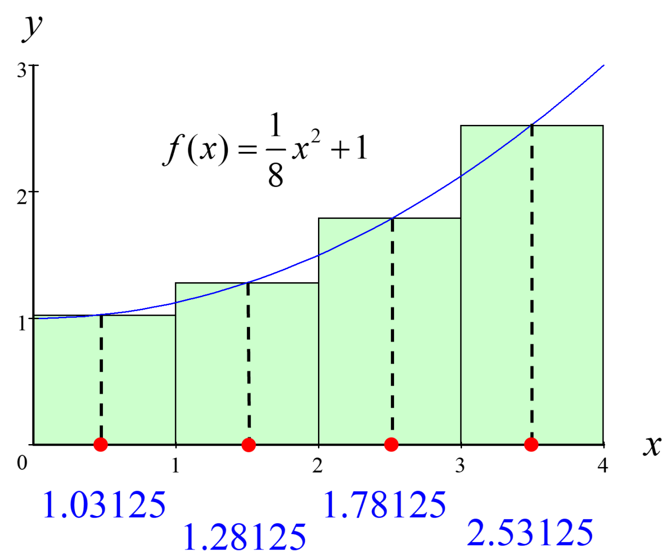
Approximate area: $R_4 = \left(1\frac{1}{8} + 1\frac{1}{2} + 2\frac{1}{8} + 3\right)\cdot 1 = 7\frac{3}{4} = 7.75$

Example Use rectangles generated from the midpoint of each subinterval to find the area under the curve . **Midpoint Rectangles**.

$$f(x) = \frac{1}{8}x^2 + 1$$

Solution

x	$f(x)$
0.5	1.03125
1.5	1.28125
2.5	1.78125
3.5	2.53125



Approximate area:

$$M_4 = (1.03125 + 1.28125 + 1.78125 + 2.53125) \cdot 1 = 6.625$$

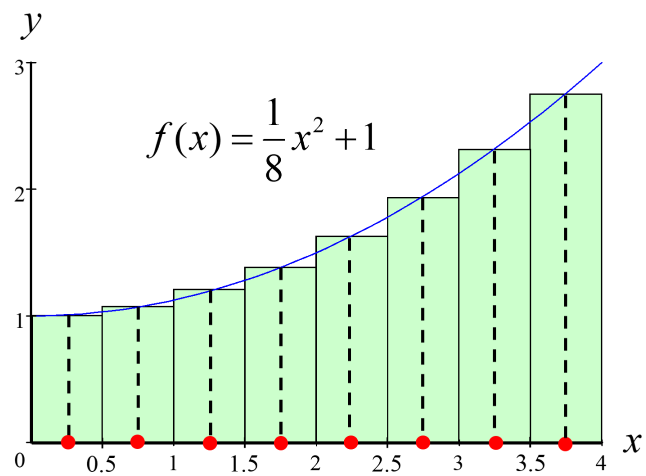
In this example there are four subintervals. As the number of subintervals increases, so does the accuracy.

With 8 subintervals: $f(x) = \frac{1}{8}x^2 + 1$

x	$f(x)$
0.25	1.00781
0.75	1.07031
1.25	1.19531
1.75	1.38281
2.25	1.63281
2.75	1.94531
3.25	2.32031
3.75	2.75781

$13.31248 \times 0.5 = 6.65624$

width of subinterval



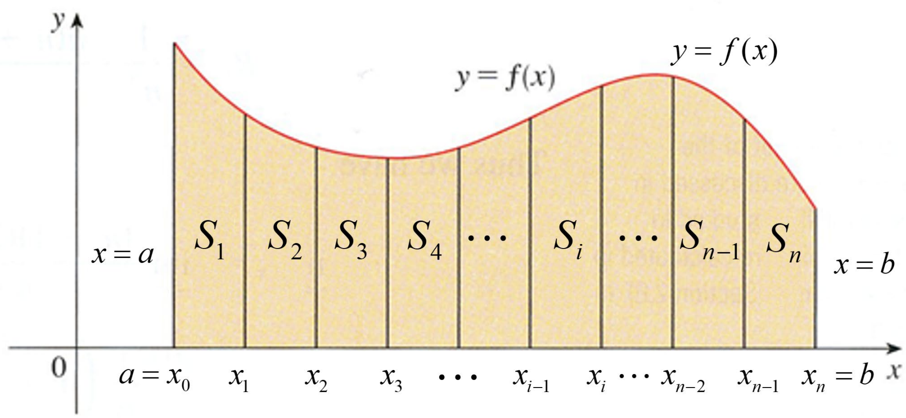
Approximate area:

$M_8 = 6.65624$

What is the exact area?

We need more rectangles!

Approximating Areas by Riemann Sums

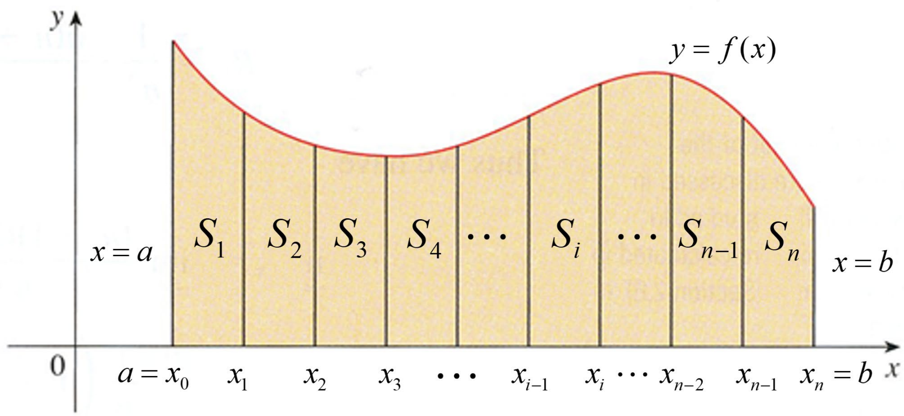


We begin by dividing the interval $[a, b]$ into n subintervals of equal length,

$$[a = x_0, x_1],\ [x_1, x_2],\ [x_2, x_3],\ \ldots,\ [x_{i-1}, x_i],\ \ldots,\ [x_{n-2}, x_{n-1}],\ [x_{n-1}, x_n = b]$$

The length of each subinterval, denoted Δx, is found by dividing the length of the interval by n:

$$\Delta x = \frac{b - a}{n}$$



Definition **Regular Partition**

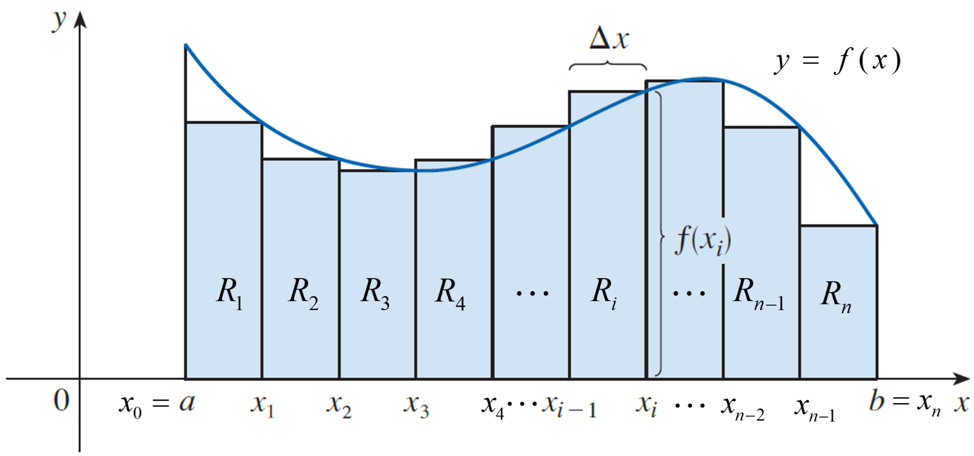
Suppose $[a, b]$ is a closed interval containing n subintervals

$$[a = x_0, x_1],\ [x_1, x_2],\ [x_2, x_3],\ \ldots,\ [x_{i-1}, x_i],\ \ldots,\ [x_{n-2}, x_{n-1}],\ [x_{n-1}, x_n = b]$$

of equal length $\Delta x = \dfrac{b-a}{n}$ with $a = x_0$ and $b = x_n$. The endpoints $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ of the subintervals are called the **grid points** and they create a **regular partition** of the interval $[a, b]$. In general, the i^{th} grid point is

$$x_i = a + i \cdot \Delta x, \text{ for } i = 1, 2, 3, \ldots, n.$$

Using Right-Hand Rectangles (right grid or right sample points) we get,



2 Definition The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}[f(x_1)+f(x_2)+f(x_3)+\cdots+f(x_i)+\cdots+f(x_{n-1})+f(x_n)]\cdot\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x$$

Where, $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\cdot\Delta x$, the right grid or right sample points. Remember i is counting each rectangle.

This summation is called a Right Riemann Sum, it calculates the area under the curve.

Riemann Sum

This tells us to end with $i = n$.

This tells us to add.

This tells us to start with $i = m$.

$$\sum_{i=m}^{n} f(x_i)\,\Delta x$$

Condensed Sum

Expanded Sum

$$\sum_{i=1}^{n} f(x_i)\cdot\Delta x = [f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_i) + \cdots + f(x_{n-1}) + f(x_n)]\cdot\Delta x$$

Heights of Rectangles

Width

The index in a sum is a *dummy variable*. It is internal to the sum, so it does not matter what symbol you choose as an index. For example,

$$\sum_{i=1}^{99} i = \sum_{k=1}^{99} k = \sum_{n=1}^{99} n = \sum_{p=1}^{99} p.$$

Two properties of sums are useful in upcoming work. Suppose that $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ are two sets of real numbers, and suppose that c is a real number. Then we can factor constants out of a sum:

Constant Multiple Rule $$\sum_{k=1}^{n} c\,a_k = c\sum_{k=1}^{n} a_k$$

Addition Rule $$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

Sigma (Summation) Notation

Sigma (or **summation**), **notation** is used to express sums in a compact way. For example, the sum $1 + 2 + 3 + \cdots + 10$ is represented in sigma notation as $\sum_{k=1}^{10} k$. Here is how the notation works.

The symbol $\sum$ (*sigma*, the Greek capital S) stands for *sum*. The **index** k takes on all integer values from the lower limit ($k = 1$) to the upper limit ($k = 10$). The expression that immediately follows $\sum$ (the **summand**) is evaluated for each value of k, and the resulting values are summed. Here are some examples.

$$\sum_{k=1}^{99} k = 1 + 2 + 3 + \cdots + 99 = 4950$$

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n$$

$$\sum_{k=0}^{3} k^2 = 0^2 + 1^2 + 2^2 + 3^2 = 14$$

$$\sum_{k=-1}^{2} \left(k^2 + k\right) = \left((-1)^2 + (-1)\right) + \left(0^2 + 0\right) + \left(1^2 + 1\right) + \left(2^2 + 2\right) = 8$$

Example Find the area under the curve using a Right Riemann sum,

$$f(x) = 4\sin x$$

over the interval [0, π] with 4 subintervals.

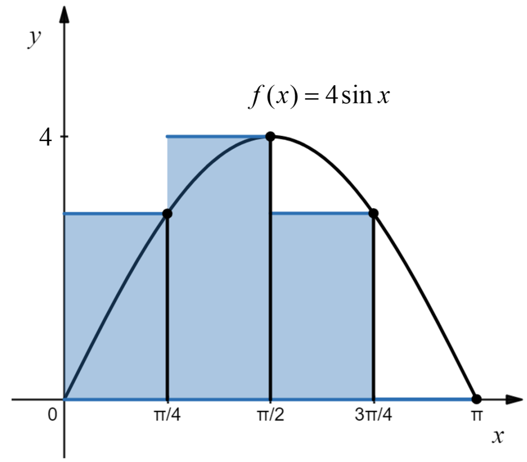
Solution

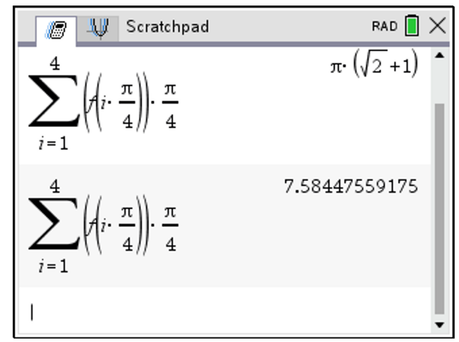
First find Δx.

$$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}.$$

Next we find x_i (the right sample point.)

$$x_i = a + i \cdot \Delta x = 0 + i \cdot \frac{\pi}{4} = i \cdot \frac{\pi}{4}$$





The right Riemann Sum is

$$R_4 = \sum_{i=1}^{4} f(x_i) \cdot \Delta x = \sum_{i=1}^{4} \left(4\sin\left(i \cdot \frac{\pi}{4} \right) \right) \cdot \frac{\pi}{4} = \left[4\sin\left(1 \cdot \frac{\pi}{4}\right) + 4\sin\left(2 \cdot \frac{\pi}{4}\right) + 4\sin\left(3 \cdot \frac{\pi}{4}\right) + 4\sin\left(4 \cdot \frac{\pi}{4}\right) \right] \cdot \frac{\pi}{4}$$

Height of Rectangles Base Width of Rectangles

$$= \left[2\sqrt{2} + 4 + 2\sqrt{2} + 0 \right] \cdot \frac{\pi}{4} = (\sqrt{2} + 1)\pi$$

$$\approx 7.584$$

Example Find the area under the curve using a Right Riemann sum,

$$f(x) = 4\sin x$$

over the interval $[0, \pi]$ with 20 subintervals.

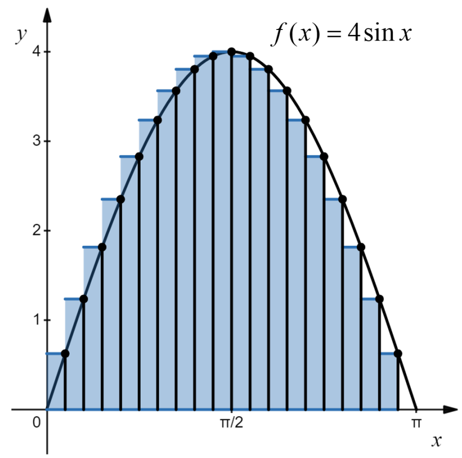
Solution

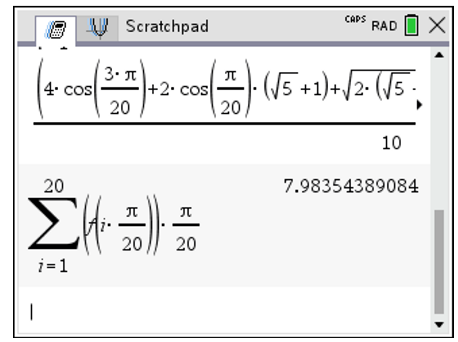
First find Δx.

$$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{20} = \frac{\pi}{20}.$$

Next we find x_i (the right sample point.)

$$x_i = a + i \cdot \Delta x = 0 + i \cdot \frac{\pi}{20} = i \cdot \frac{\pi}{20}$$

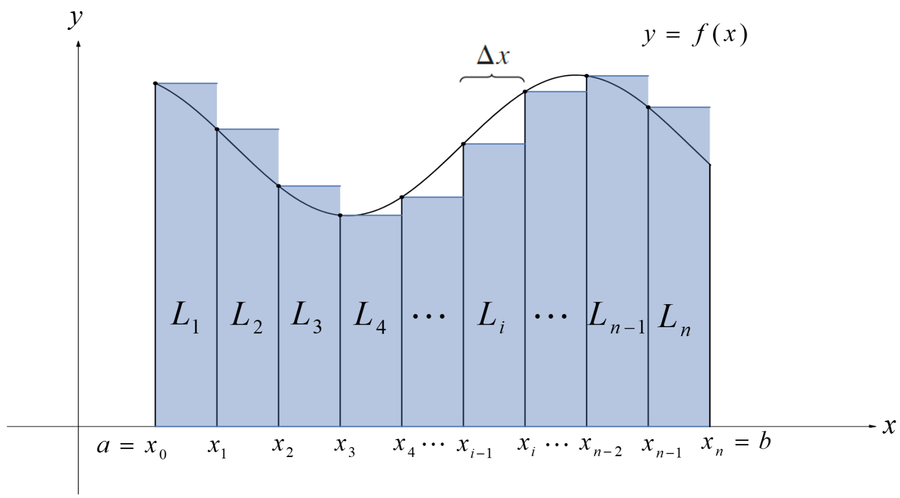




The right Riemann Sum is

$$R_{20} = \sum_{i=1}^{20} f(x_i) \cdot \Delta x = \sum_{i=1}^{20} \left(f\left(i \cdot \frac{\pi}{20} \right) \right) \cdot \frac{\pi}{20} \approx 7.984$$

Height of Rectangles Base Width of Rectangles



Using Left-Hand Rectangles (left grid or sample points) we get,

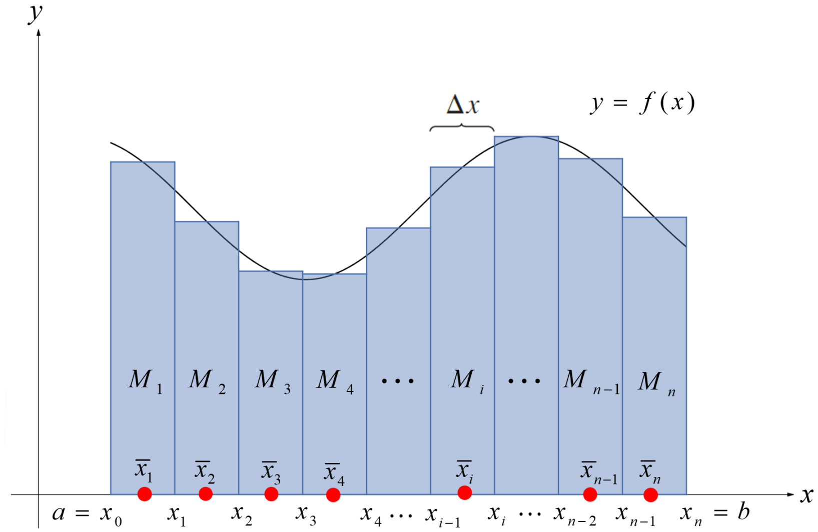
$$L_n = (f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{i-1}) + \cdots + f(x_{n-2}) + f(x_{n-1})) \cdot \Delta x$$

Using sigma notation we get,

$L_n = \sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x$, sometimes called a left sum.

Where, $\Delta x = \dfrac{b-a}{n}$ and $x_{i-1} = a + (i-1) \cdot \Delta x$, the left grid or sample points. Remember i is counting each rectangle.

This summation is called a Left Riemann Sum, it calculates the area under the curve.



Using Midpoint Rectangles (midpoint grid or sample points) we get,

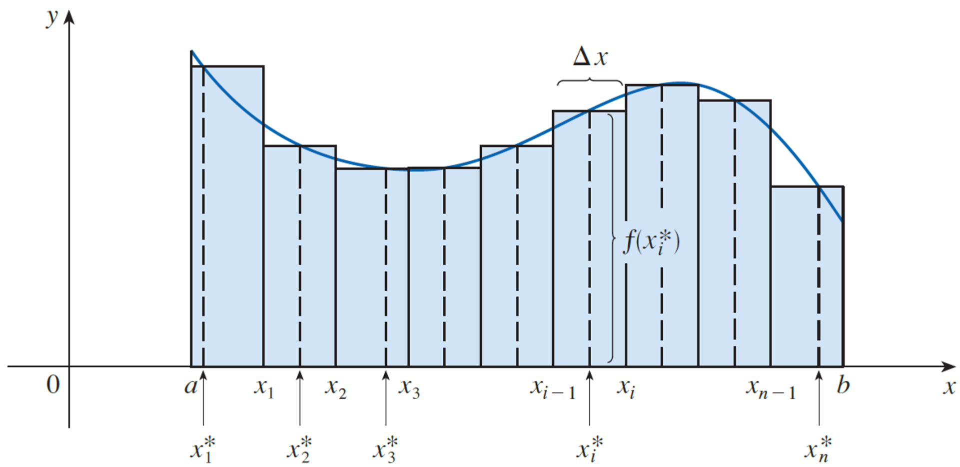
$M_n = (f(\overline{x}_1) + f(\overline{x}_2) + f(\overline{x}_3) + \cdots + f(\overline{x}_i) + \cdots + f(\overline{x}_{n-1}) + f(\overline{x}_n)) \cdot \Delta x$

Using sigma notation we get,

$M_n = \sum_{i=1}^{n} f(\overline{x}_i) \cdot \Delta x$, sometimes called a midpoint sum.

Where, $\Delta x = \dfrac{b-a}{n}$ and $\overline{x}_i = a + \left(i - \dfrac{1}{2}\right) \cdot \Delta x$, the midpoint grid or sample points. Remember i is counting each rectangle.

This summation is called a Midpoint Riemann Sum, it calculates the area under the curve.



Using Rectangles taking any grid or sample point in each subinterval we get,

$$A_n = (f(x_1^*) + f(x_2^*) + f(x_3^*) + \cdots + f(x_k^*) + \cdots + f(x_{n-1}^*) + f(x_n^*)) \cdot \Delta x$$

Using sigma notation we get,

$$A_n = \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x$$

Where, $\Delta x = \dfrac{b-a}{n}$ and $x_i^* \in \left[x_{i-1}, x_i \right]$, remember i is counting each rectangle.

This summation is called a Riemann Sum, it calculates the area under the curve.

NOTE To approximate the area under the graph of f we can form **lower sums** (or **upper sums**) by choosing the sample points x_i^* so that $f(x_i^*)$ is the minimum (or maximum) value of f on the ith subinterval (see Figure 14). [Since f is continuous, we know that the minimum and maximum values of f exist on each subinterval by the Extreme Value Theorem.] It can be shown that an equivalent definition of area is the following: *A is the unique number that is smaller than all the upper sums and bigger than all the lower sums.*

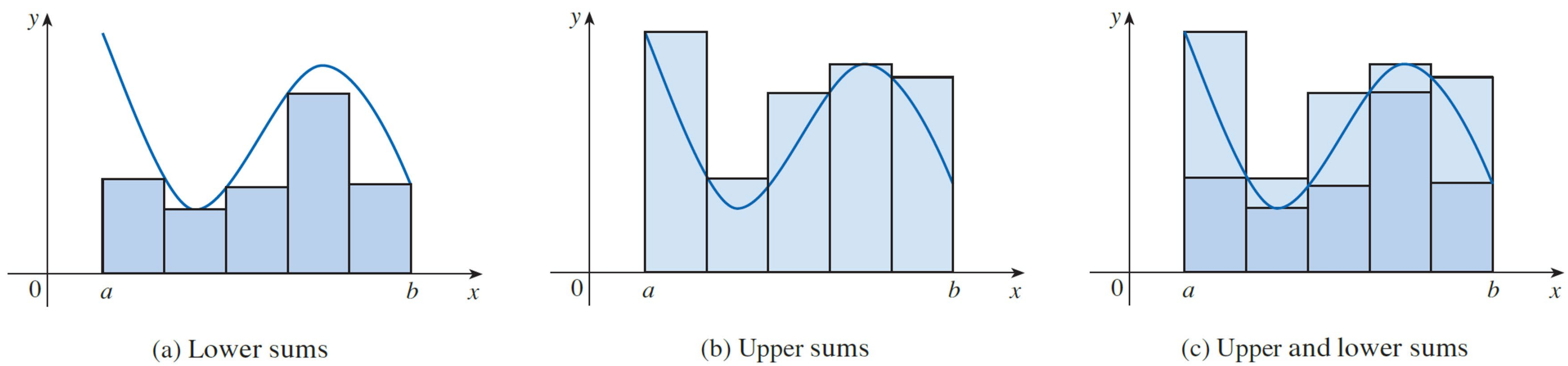


FIGURE 14

Example Consider the function $f(x)=\dfrac{x}{x^2+1}$. Find the area under the curve using a Riemann Sum of $f(x)$ from 1 to 3 using 50 subintervals, with right, left, and midpoint sample points.

Solution

Right Riemann Sum

First find Δx.

$$\Delta x=\frac{b-a}{n}=\frac{3-1}{50}=\frac{2}{50}=\frac{1}{25}$$

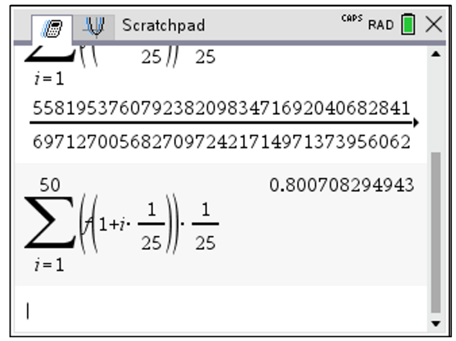
Next we find x_i (the right sample point.)

$$x_i=a+i\cdot\Delta x=1+i\cdot\frac{1}{25}$$

The right Riemann Sum is

$$R_{50}=\sum_{i=1}^{50} f(x_i)\cdot\Delta x=\sum_{i=1}^{50}\left(f\left(1+i\cdot\frac{1}{25}\right)\right)\cdot\frac{1}{25}\approx 0.801$$

Height of Rectangles   Base Width of Rectangles



Example Consider the function $f(x) = \dfrac{x}{x^2+1}$. Find the area under the curve using a Riemann Sum of $f(x)$ from 1 to 3 using 50 subintervals, with right, left, and midpoint sample points.

Solution

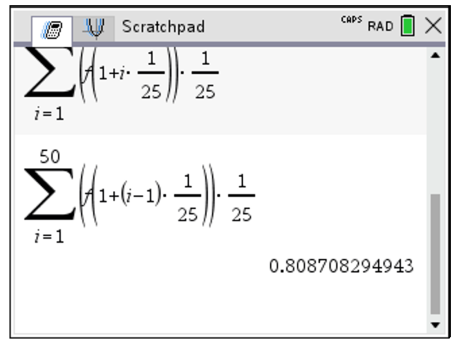
Left Riemann Sum

First find Δx.

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{50} = \frac{2}{50} = \frac{1}{25}$$

Next we find x_{i-1} (the left sample point.)

$$x_i = a + (i-1)\cdot\Delta x = 1 + (i-1)\cdot\frac{1}{25}$$



The left Riemann Sum is

$$L_{50} = \sum_{i=1}^{50} f(x_{i-1})\cdot\Delta x = \sum_{i=1}^{50}\left(f\left(1+(i-1)\cdot\frac{1}{25}\right)\right)\cdot\frac{1}{25} \approx 0.809$$

Height of Rectangles

Base Width of Rectangles

Example Consider the function $f(x)=\dfrac{x}{x^2+1}$. Find the area under the curve using a Riemann Sum of $f(x)$ from 1 to 3 using 50 subintervals, with right, left, and midpoint sample points.

Solution

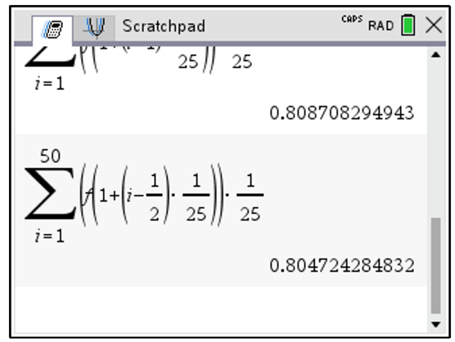
Midpoint Riemann Sum

First find Δx.

$$\Delta x=\frac{b-a}{n}=\frac{3-1}{50}=\frac{2}{50}=\frac{1}{25}$$

Next we find $\overline{x}_i$ (the right sample point.)

$$\overline{x}_i=a+\left(i-\frac{1}{2}\right)\cdot\Delta x=1+\left(i-\frac{1}{2}\right)\cdot\frac{1}{25}$$



The right Riemann Sum is

$$M_{50}=\sum_{i=1}^{50}f(\overline{x}_i)\cdot\Delta x=\sum_{i=1}^{50}\left(f\left(1+\left(i-\frac{1}{2}\right)\cdot\frac{1}{25}\right)\right)\cdot\frac{1}{25}\approx 0.805$$

Height of Rectangles

Base Width of Rectangles

Example Use Definition 2 to find an expression for the area under the graph of f as a limit. Do not evaluate the limit.

$$f(x) = x\sqrt{x^3 + 8}, \quad 1 \leqslant x \leqslant 5$$

Solution

2 Definition The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_i) + \cdots + f(x_{n-1}) + f(x_n)] \cdot \Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

Where, $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i \cdot \Delta x$, the right grid or right sample points. Remember i is counting each rectangle.

$$f(x) = x\sqrt{x^3 + 8}, \quad 1 \leqslant x \leqslant 5$$

First find Δx.

$$\Delta x = (5 - 1)/n = 4/n$$

Next we find x_i (the right sample point.)

$$x_i = 1 + i\,\Delta x = 1 + 4i/n$$

$$A = \lim_{n\to\infty} R_n$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[(1 + 4i/n)\sqrt{(1 + 4i/n)^3 + 8}\right] \cdot \frac{4}{n}$$

How do we compute this limit? Section 5.2!