



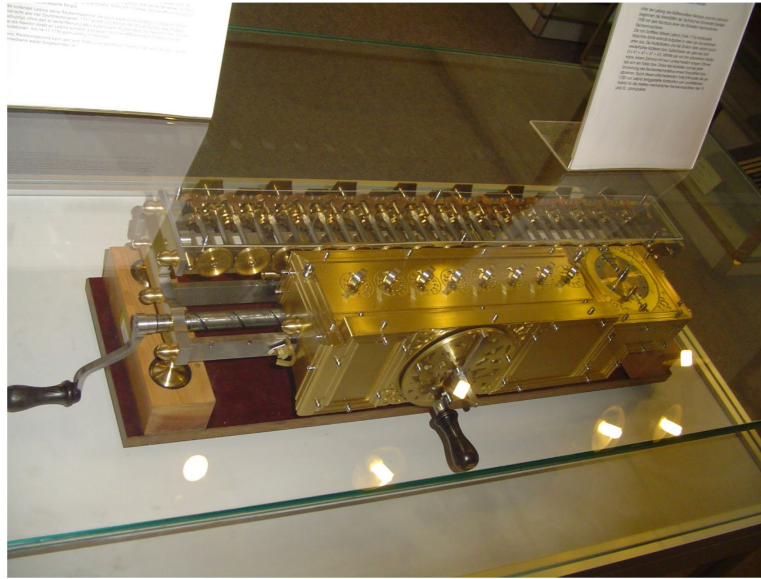
4.9 Antiderivatives



Gottfried Wilhelm von Leibniz
1646 – 1716

Gottfried Leibniz was a German mathematician who developed the present-day notation for the differential and integral calculus though he never thought of the derivative as a limit. With Newton, he was known to be one of the inventors of calculus, he also invented an early calculating machine.

Leibniz Stepped Reckoner Calculator



In 1673, Leibniz built the first true four-function calculator. His unique, drum-shaped gears formed the basis of many successful calculator designs for the next 275 years, an unbroken record for a single underlying calculator mechanism. Leibniz built several versions of the Stepped Reckoner over about 45 years. Only one survives today.

Take the derivative $\frac{dy}{dx} = x^2$

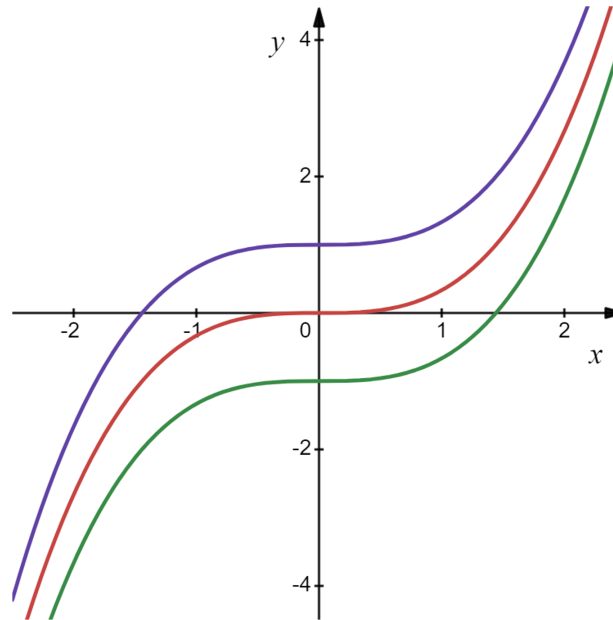
A function you differentiate to get $\frac{dy}{dx}$ is $y = \frac{1}{3}x^3$

or $y = \frac{1}{3}x^3 + 1$

or $y = \frac{1}{3}x^3 - 1$

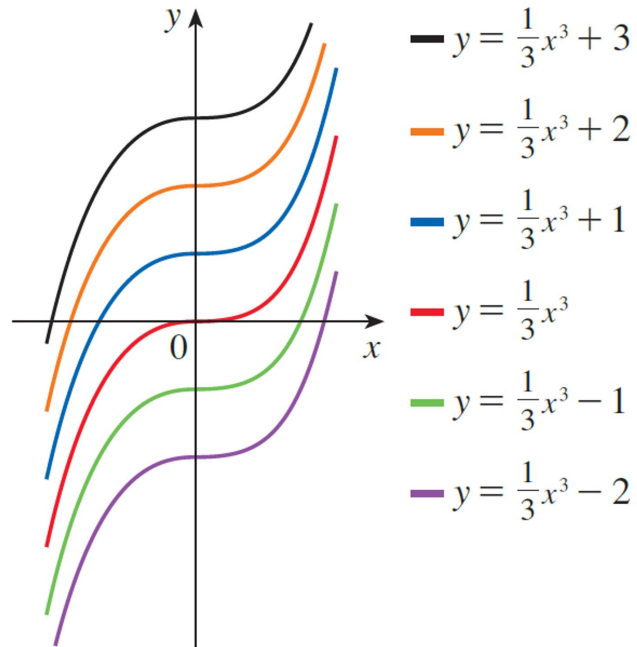
or in general

$$y = \frac{1}{3}x^3 + C$$



So, there is a family of functions $y = \frac{1}{3}x^3 + C$ where $\frac{dy}{dx} = x^2$.

C is called the "Arbitrary Constant."



Members of the family of antiderivatives of $f(x) = x^2$

■ The Antiderivative of a Function

The process of finding the original function from the derivative is so important that it has a name:

Definition A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I .

1 Theorem If F is an antiderivative of f on an interval I , then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Going back to the function $f(x) = x^2$, we see that the general antiderivative of f is $\frac{1}{3}x^3 + C$. By assigning specific values to the constant C , we obtain a family of functions whose graphs are vertical translates of one another. This makes sense because each curve must have the same slope at any given value of x . The process of finding an antiderivative is **antidifferentiation**. We will hear much more about antiderivatives in the future. This section is just an introduction.

Example Find the function $F(x)$ whose derivative is $f(x) = \sin(x)$ and whose graph passes through $(0, 2)$.

Solution

We are given that $\frac{d}{dx} F(x) = \sin(x)$

so, $F(x) = -\cos(x) + C$

$F(x)$ could be $-\cos x$ or could vary by some constant C .

$F(x) = -\cos(x) + C$ **General Solution.**

$2 = -\cos(0) + C$ **Notice that we have the initial conditions $(0, 2)$ to determine the value of C .**

$$2 = -1 + C$$

$$3 = C$$

$$F(x) = -\cos(x) + 3$$

Particular Solution.

2 Table of Antidifferentiation Formulas

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec^2 x$	$\tan x$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$	$\sec x \tan x$	$\sec x$
$\frac{1}{x}$	$\ln x $	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
e^x	e^x	$\frac{1}{1+x^2}$	$\tan^{-1} x$
b^x	$\frac{b^x}{\ln b}$	$\cosh x$	$\sinh x$
$\cos x$	$\sin x$	$\sinh x$	$\cosh x$

To obtain the most general antiderivative from the particular ones in Table 2, we must add an arbitrary constant C , as we did in the previous example.

■ The Antiderivative of a Function

Indefinite Integrals

The diagram shows the equation $\int f(x) dx = F(x) + C$ with several red labels and arrows pointing to its components. The label "Integration Symbol" points to the integral sign. The label "Variable of integration (dummy variable)" points to the dx term. The label "Constant of Integration or Arbitrary Constant" points to the C term. Brackets under the $f(x)$ and $F(x)$ terms are labeled "Integrand" and "Antiderivative" respectively.

$$\int \underbrace{f(x)}_{\text{Integrand}} \underbrace{dx}_{\text{Variable of integration (dummy variable)}} = \underbrace{F(x)}_{\text{Antiderivative}} + \underbrace{C}_{\text{Constant of Integration or Arbitrary Constant}}$$

This is read “The antiderivative of f of x dx ” or “The indefinite integral of f of x dx .”

TABLE OF INTEGRALS

Basic Forms

1. $\int k \, du = ku + C$ (k a constant)

2. $\int u^n \, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$

3. $\int \frac{du}{u} = \ln |u| + C$

4. $\int e^u \, du = e^u + C$

5. $\int b^u \, du = \frac{b^u}{\ln b} + C$

6. $\int \sin u \, du = -\cos u + C$

7. $\int \cos u \, du = \sin u + C$

8. $\int \sec^2 u \, du = \tan u + C$

9. $\int \csc^2 u \, du = -\cot u + C$

10. $\int \sec u \tan u \, du = \sec u + C$

More to come!

Example

Find the velocity $v(t)$ and position $s(t)$ equations for a downward acceleration (gravity) of a m/sec² and an initial velocity of v_0 m/sec.

Solution

$$a(t) = -a$$

$$v(t) = \int a(t) dt = \int (-a) dt$$

$$v(t) = -at + C$$

$$v_0 = -a(0) + C$$

Initial conditions:

When $t = 0$, $v(0) = v_0$.


$$v_0 = C$$

$$v(t) = -at + v_0$$

Since velocity is the derivative of position, position must be the antiderivative of velocity.

$$s(t) = \int v(t) dt$$

$$s(t) = \int (-at + v_0) dt$$

$$s(t) = \frac{-a}{2}t^2 + v_0t + C$$


The power rule in reverse: Increase the exponent by one and multiply by the reciprocal of the new exponent.

Example Find the most general antiderivative of the function. (Check your answer by differentiation.)

(a) $\int (5x^3 + 3x^2 - 7x + 4) dx$ (c) $g(v) = 2 \cos v - \frac{3}{\sqrt{1 - v^2}}$

(b) $\int \frac{2t - 4 + 3\sqrt{t}}{\sqrt{t}} dt$

Solution

(a) $\int \underbrace{(5x^3 + 3x^2 - 7x + 4)}_{f(x)} dx$

$$F(x) = 5 \frac{x^{3+1}}{3+1} + 3 \frac{x^{2+1}}{2+1} - 7 \frac{x^{1+1}}{1+1} + 4 \frac{x^{0+1}}{0+1} + C$$

$$F(x) = \frac{5}{4}x^4 + x^3 - \frac{7}{2}x^2 + 4x + C \quad \text{Most General Antiderivative}$$

$$F'(x) = 5x^4 + 3x^3 - 7x^2 + 4 = f(x)$$

(b) $\int \frac{2t - 4 + 3\sqrt{t}}{\sqrt{t}} dt$

$$F(x) = \int \left(\frac{2t}{t^{1/2}} - \frac{4}{t^{1/2}} + \frac{3t^{1/2}}{t^{1/2}} \right) dt$$

$$F(x) = \int (2t^{1/2} - 4t^{-1/2} + 3) dt$$

$$F(x) = 2 \frac{t^{1/2+1}}{\frac{1}{2}+1} - 4 \frac{t^{-1/2+1}}{-\frac{1}{2}+1} + 3t + C$$

$$F(x) = \frac{4}{3}t^{3/2} - 8t^{1/2} + 3t + C$$

$$F'(x) = \frac{4}{3} \cdot \frac{3}{2}t^{1/2} - 8 \cdot \frac{1}{2}t^{-1/2} + 3$$

$$F'(x) = 2t^{1/2} - 4t^{-1/2} + 3 = f(t)$$

Example Find the most general antiderivative of the function. (Check your answer by differentiation.)

(a) $\int (5x^3 + 3x^2 - 7x + 4) dx$ (c) $g(v) = 2 \cos v - \frac{3}{\sqrt{1 - v^2}}$

(b) $\int \frac{2t - 4 + 3\sqrt{t}}{\sqrt{t}} dt$

Solution

(c) $g(v) = 2 \cos v - \frac{3}{\sqrt{1 - v^2}}$

$$G(x) = \int \left(2 \cos v - 3 \cdot \frac{1}{\sqrt{1 - v^2}} \right) dv$$

$$G(x) = 2 \sin v - 3 \cdot \sin^{-1} v + C$$

$$G'(x) = 2 \cos v - 3 \cdot \frac{1}{\sqrt{1 - v^2}} = g(v)$$

Example Find the antiderivative F of f that satisfies the given condition. Check your answer by comparing the graphs of f and F .

$$f(x) = 4 - 3(1 + x^2)^{-1}, \quad F(1) = 0$$

Solution

$$F(x) = \int (4 - 3(1 + x^2)^{-1}) dx$$

$$F(x) = \int \left(4 - 3 \cdot \frac{1}{1 + x^2} \right) dx$$

$$F(x) = 4x - 3 \cdot \tan^{-1} x + C \quad \text{Most General Antiderivative}$$

We are given the initial conditions $F(1) = 0$, so we get

$$F(1) = 4(1) - 3 \cdot \tan^{-1}(1) + C$$

$$0 = 4(1) - 3 \cdot \tan^{-1}(1) + C$$

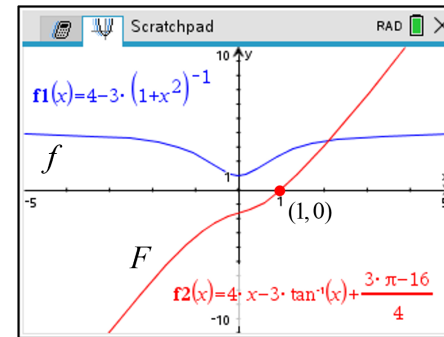
$$0 = 4 - 3 \cdot \frac{\pi}{4} + C$$

$$C = \frac{3\pi}{4} - 4$$

$$C = \frac{3\pi - 16}{4}$$

$$F(x) = 4x - 3 \cdot \tan^{-1} x + \frac{3\pi - 16}{4}$$

Particular Solution.



Note that f is positive and F is increasing on \mathbb{R} . Also, f has smaller values where the slopes of the tangent lines of F are smaller.

Example Find f .

$$f''(x) = e^x - 2 \sin x, \quad f(0) = 3, \quad f(\pi/2) = 0$$

Solution

$$f'(x) = \int f''(x) dx$$

$$f'(x) = \int (e^x - 2 \sin x) dx$$

$$f'(x) = e^x + 2 \cos x + C$$

$$f(x) = \int f'(x) dx$$

$$f(x) = \int (e^x + 2 \cos x + C) dx$$

$$f(x) = e^x + 2 \sin x + Cx + D \quad \text{General Solution.}$$

We are given the initial conditions

$f(0) = 3$ and $f(\pi/2) = 0$, so we get

$$\begin{cases} 3 = e^0 + 2 \sin(0) + C(0) + D \\ 0 = e^{\pi/2} + 2 \sin(\pi/2) + C(\pi/2) + D \end{cases}$$

$$\begin{cases} 3 = 1 + 0 + 0 + D \\ 0 = e^{\pi/2} + 2 \cdot 1 + \frac{\pi}{2}C + D \end{cases}$$

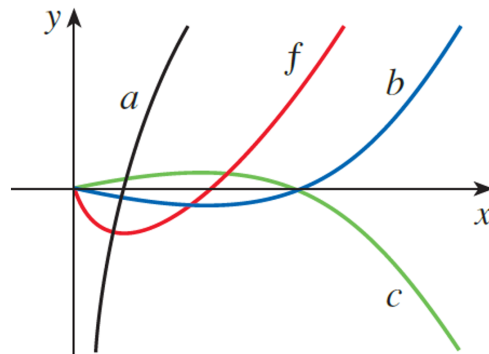
$$\begin{cases} D = 2 \\ 0 = e^{\pi/2} + 2 + \frac{\pi}{2}C + 2 \end{cases}$$

$$-e^{\pi/2} - 4 = \frac{\pi}{2}C \quad C = -\frac{2}{\pi}(e^{\pi/2} + 4)$$

$$f(x) = e^x + 2 \sin x - \frac{2}{\pi}(e^{\pi/2} + 4)x + 2$$

Particular Solution.

Example The graph of a function f is shown. Which graph is an antiderivative of f and why?



Solution

b is the antiderivative of f .

For small x , f is negative, so the graph of its antiderivative must be decreasing.

But both a and c are increasing for small x , so only b can be f 's antiderivative.

Also, f is positive where b is increasing, which supports our conclusion.

Example

A particle is moving with the given data. Find the position of the particle.

$$a(t) = 3 \cos t - 2 \sin t, \quad s(0) = 0, \quad v(0) = 4$$

Solution

$$v(t) = \int a(t) dt$$

$$v(t) = \int (3 \cos t - 2 \sin t) dt$$

$$v(t) = 3 \sin t + 2 \cos t + C$$

We are given the initial conditions

$v(0) = 4$, so we get

$$4 = 3 \sin(0) + 2 \cos(0) + C$$

$$4 = 2 + C$$

$$C = 2$$

So the velocity is

$$v(t) = 3 \sin t + 2 \cos t + 2$$

$$s(t) = \int v(t) dt$$

$$s(t) = \int (3 \sin t + 2 \cos t + 2) dt$$

$$s(t) = -3 \cos t + 2 \sin t + 2t + D$$

We are given the initial conditions

$s(0) = 0$, so we get

$$0 = -3 \cos(0) + 2 \sin(0) + 2(0) + D$$

$$0 = -3 + D$$

$$D = 3$$

So the position function is

$$s(t) = -3 \cos t + 2 \sin t + 2t + 3$$

Example A particle is moving with the given data. Find the position of the particle.

$$a(t) = 3 \cos t - 2 \sin t, \quad s(0) = 0, \quad v(0) = 4$$

Solution

$$s(t) = -3 \cos t + 2 \sin t + 2t + 3$$

$$v(t) = 3 \sin t + 2 \cos t + 2$$

$$a(t) = 3 \cos t - 2 \sin t$$

