4.7 Optimization Problems

Sir Isaac Newton
1643 – 1727

Isaac Newton was the greatest English mathematician of his generation. He laid the foundation for differential and integral calculus. His work on optics and gravitation make him one of the greatest scientists the world has known.

Westminster Abbey

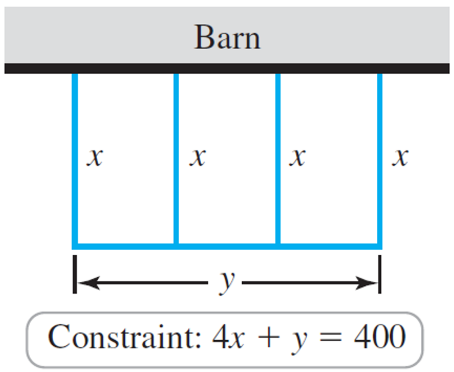
Example

A Classic Problem

You have 400 feet of fence to enclose a rectangular corral along the side of a barn to house some chickens. Three exterior fences and two interior fences partition the corral into three rectangular regions. What dimensions of the corral maximize the enclosed area? What is the area of that corral?

Solution

I **Objective**: Maximize Area of corral



$x =$ Width (feet)

$y = 400 - 4x =$ Length (feet)

II **Objective Function**

$$A = lw$$

$$A(x) = x(400 - 4x)$$

$$A(x) = 400x - 4x^2$$

III **Domain** $x \in [0, 100]$

IV **"Calculus Time"**

$$A(x) = 400x - 4x^2$$

$$A'(x) = 400 - 8x$$

Let $A'(x) = 0$ $\quad 400 - 8x = 0$

So $x = 50$ is the critical number.

Solution

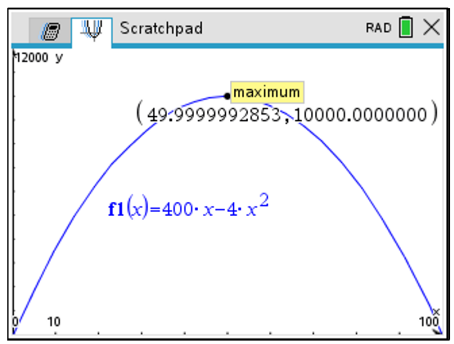
IV "<u>Calculus Time</u>" (continued)

x	$A(x)$
0	0
50	10000
100	0

By The Extreme Value Theorem, the absolute maximum occurs here, since the endpoints give zero area.

$$A(x) = 400x - 4x^2$$
$$A(0) = 0$$
$$A(50) = 400(50) - 4(50)^2$$
$$A(50) = 10000$$
$$A(100) = 0$$

V <u>Conclusion</u>: By EVT, the dimensions of the corral are 50ft by 200ft with an absolute maximum area of 10,000 ft^2.



Optimization Problems
Mini/Max Problems

Optimization concerns the minimization or maximization of functions. It is used to solve complex design problems to improve cost, reliability, and performance in a wide range of applications.

I Introduction The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the objective? What are the constraints? Identify the given quantities, and conditions?

Draw a Diagram In most problems it is useful to draw and identify the given and required quantities on the diagram.

Introduce Notation Assign a symbol to the quantity that is to be aximized or minimized (let's call it Q for now). Also select symbols ($a, b, c, \ldots, x, y, z$) for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols – for example, A for area, h for height, t for time.

II **Write an Objective Function** Express Q in terms of the variables introduced in part I.

Express Your Objective Function in One Variable If Q has been expressed as a function of more than one variable, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the objective equation for Q. Thus , Q will be expressed as a function of one variable, say x, $Q = f(x)$.

III **Write the domain of this function!!**

IV **Find the Extrema of $Q(x)$** Find the absolute maximum or minimum value for your objective function Q. In particular, if the domain of Q is a closed interval use the **Extreme Value Theorem**, if an open interval use **OLE**. Be a name dropper!

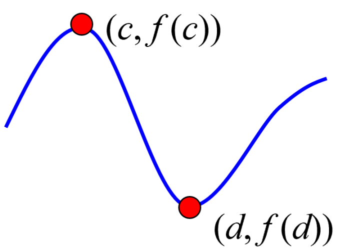
V **Conclusion** State your results.

Extreme Value Theorem (EVT)
Increasing/Decreasing Test (IDT)
First Derivative Test (FDT)
And
One Local Extremum Theorem(OLE)

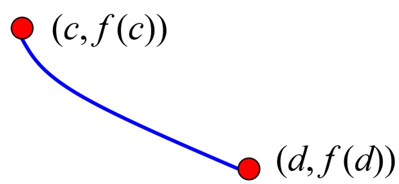
Let's recall.

(EVT)

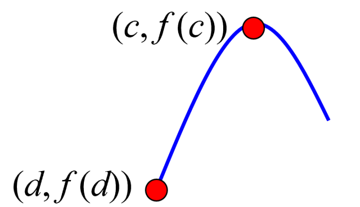
3 The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$. Note: Check the endpoints for extreme values.



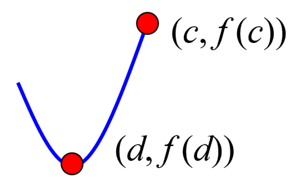
Absolute Maximum and Minimum points are interior points



Absolute Maximum and Minimum points are endpoints



Absolute Maximum point is an interior point and Absolute Minimum point is an endpoint



Absolute Minimum point is an interior point and Absolute Maximum point is an endpoint

Increasing/Decreasing Test **(IDT)**

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The First Derivative Test Suppose that c is a critical number of a continuous function f. **(FDT)**

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

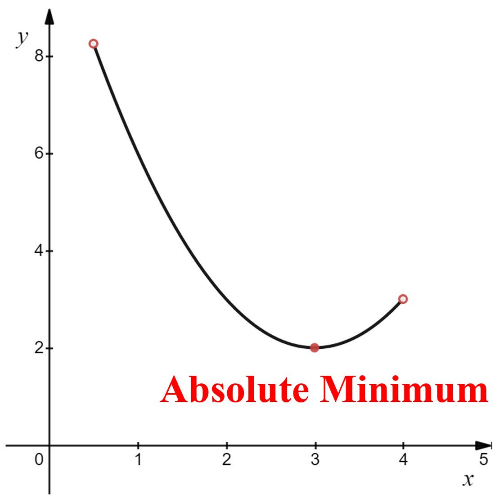
(b) If f' changes from negative to positive at c, then f has a local minimum at c.

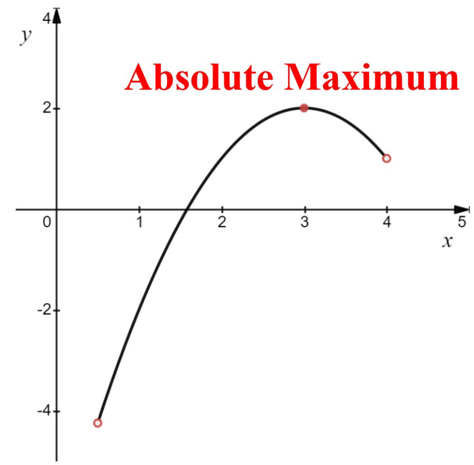
(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

THEOREM **One Local Extremum Implies Absolute Extremum** **(OLE)**

Suppose f is continuous on an interval I that contains exactly one local extremum at c.

- If a local minimum occurs at c, then $f(c)$ is the absolute minimum of f on I.
- If a local maximum occurs at c, then $f(c)$ is the absolute maximum of f on I.





Example A package to be mailed using the US postal service may not measure more than 108 inches in length plus girth. (Length is the longest dimension and girth is the largest distance around the package, perpendicular to the length.) Find the dimensions of the rectangular box with square base of greatest volume that may be mailed.

Solution

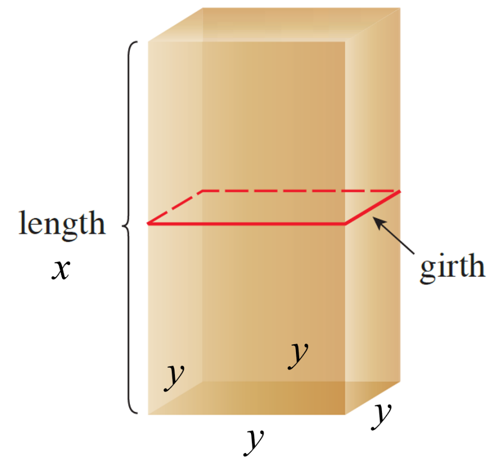
I **Objective**: Maximize Volume of Box

Let $x > 0$ be the length of the package (in)

$y > 0$ be the length of the sides of the square base (in)

We have the constraint equation $x + 4y = 108$

$$x = 108 - 4y$$



II **Objective Function**

$$V = xy^2$$
$$= (108 - 4y)y^2$$
$$= 108y^2 - 4y^3$$

III **Domain**

$$y \in (0, 27)$$

Solution

IV "Calculus Time"

$V(y) = 108y^2 - 4y^3$

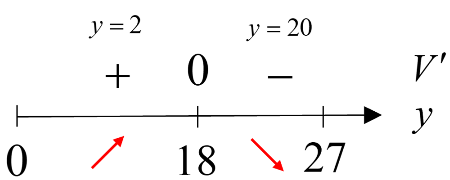
$V'(y) = 216y - 12y^2$

Let $V'(y) = 0$

$216y - 12y^2 = 0$

$12y(18 - y) = 0$

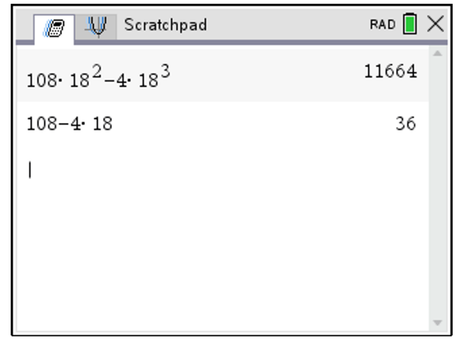
$y = 18$ [since $y \neq 0$]



By OLE, there is an absolute maximum when $y = 18$.

$V(y) = 108y^2 - 4y^3$

$V(18) = 108(18)^2 - 4(18)^3 \overset{CAS}{=} 11664$



$x = 108 - 4y$

$x = 108 - 4(18) = 36$

V <u>Conclusion</u>: Thus, the dimensions that give the greatest volume are 18in × 18in × 36in, giving a greatest possible volume of 11,664 in^3.

Example Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

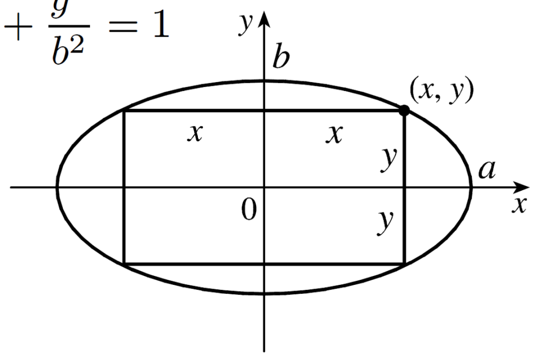
Solution

I **Objective**: Maximize the area of the inscribe rectangle.

Let $x>0$

$y>0$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



II **Objective Function**

The area of the rectangle is

$$A=(2x)(2y)=4xy$$

Constraint equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, solve for y.

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=b^2\left(\frac{a^2-x^2}{a^2}\right)$$

$$y^2=\frac{b^2}{a^2}\left(a^2-x^2\right)$$

$$y=\pm\sqrt{\frac{b^2}{a^2}\left(a^2-x^2\right)}$$

$$y=\frac{b}{a}\sqrt{a^2-x^2}$$

$$A(x)=4\frac{b}{a}x\sqrt{a^2-x^2}$$

Solution

III **<u>Domain</u>** $x \in [0, a]$

IV **"<u>Calculus Time</u>"**

$$A(x) = 4\frac{b}{a}x\sqrt{a^2 - x^2}$$

$$A'(x) \overset{CAS}{=} \frac{4a^2b - 8bx^2}{a\sqrt{a^2 - x^2}}$$

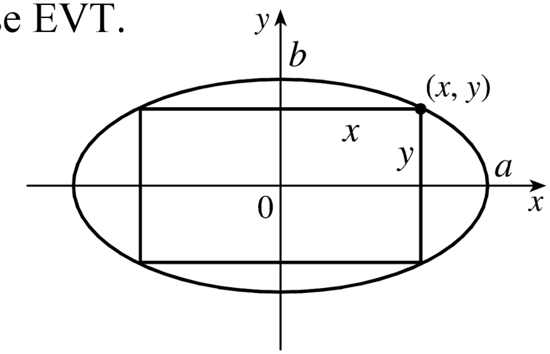
Let $A'(x) = 0$

$$\frac{4a^2b - 8bx^2}{a\sqrt{a^2 - x^2}} = 0$$

So, the critical number is $x \overset{CAS}{=} \frac{a\sqrt{2}}{2}$.

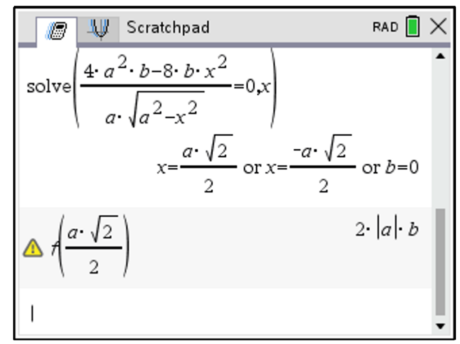
Since $x \in [0, a]$ we can use EVT.

x	$A(x)$
0	0
$\frac{a\sqrt{2}}{2}$	$2ab$
a	0



$$A(x) = 4\frac{b}{a}x\sqrt{a^2 - x^2}$$

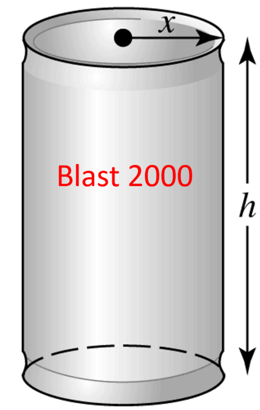
$$A\left(\frac{a\sqrt{2}}{2}\right) \overset{CAS}{=} 2ab$$



V **<u>Conclusion</u>:** By EVT, the absolute maximum area of the inscribed rectangle is $2ab$ units^2.

Example A manufacturer wants to construct cylindrical aluminum cans for a new energy drink with volume 2000 cm^3 (2 liters). What radius and height will minimize the amount of aluminum used? What will be the surface area?

Solution **I** **Objective**: Minimize the of aluminum used, minimize surface area.

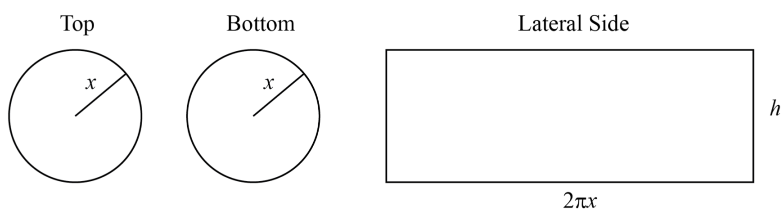


Let x = radius (cm) and h = height (cm).

Volume V is 2000 cm^3, so you have

$V = \pi x^2 h$, Volume of a cylinder

$2000 = \pi x^2 h$. So, $h = \dfrac{2000}{\pi x^2}$, constraint equations.



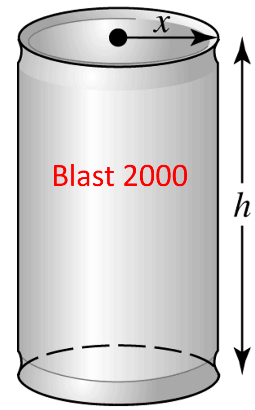
Area πx^2 πx^2 $2\pi x h$ **Total Surface Area** $S = 2\pi x h + 2\pi x^2$

Solution

II Objective Function

Surface area $S = 2\pi xh + 2\pi x^2$, $x > 0$ (since x is the radius), can now be written as a function of x.

Recall, $h = \dfrac{2000}{\pi x^2}$, substituting h into the equation above you have,



$$S(x) = 2\pi x\left(\frac{2000}{\pi x^2}\right) + 2\pi x^2$$

$$= \frac{4000}{x} + 2\pi x^2$$

$$= \frac{4000 + 2\pi x^3}{x}$$ A rational function!!

III Domain $x \in (0, +\infty)$

Solution

IV "**Calculus Time**"

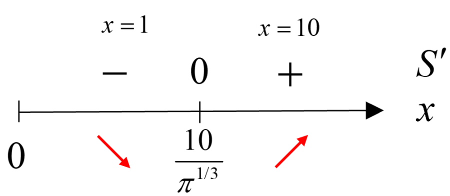
$$S(x) = \frac{4000 + 2\pi x^3}{x}$$

$$S'(x) = \frac{4(\pi x^3 - 1000)}{x^2}$$

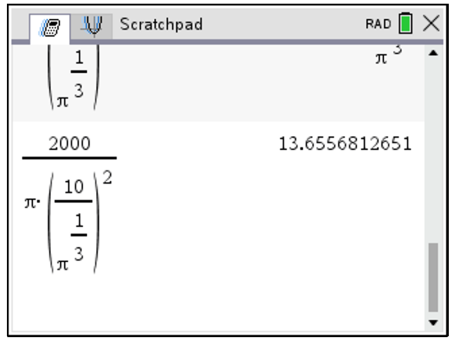
Let $S'(x) = 0$.

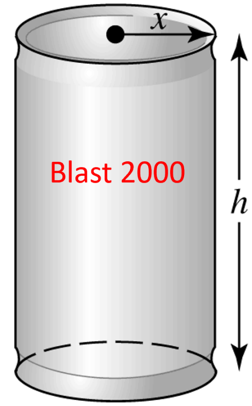
$$\frac{4(\pi x^3 - 1000)}{x^2} = 0$$

So, the critical number is $x = \frac{10}{\pi^{1/3}} \cong 6.828$.



By OLE, there is an absolute minimum when $x = \frac{10}{\pi^{1/3}}$.





$$S\left(\frac{10}{\pi^{1/3}}\right) = 600\pi^{1/3} \approx 878.755$$

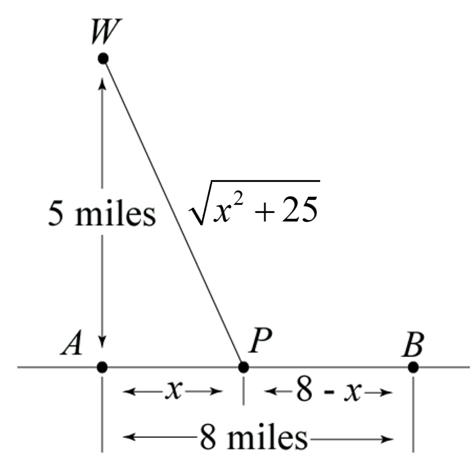
Recall, $h = \frac{2000}{\pi x^2} = \frac{2000}{\pi\left(\frac{10}{\pi^{1/3}}\right)^2} = \frac{20}{\pi^{1/3}} \approx 13.656$

V **Conclusion:**
The absolute minimum surface area is $600\pi^{1/3}$ cm^2 when the radius is $\frac{10}{\pi^{1/3}}$ cm and the height is $\frac{20}{\pi^{1/3}}$ cm.

Example The figure below shows an offshore oil well located at a point W that is 5 miles from the closest point A on a straight shoreline. Oil is to be piped from W to a shore point B that is 8 miles from point A by piping it on a straight line under water from W to some point P between A and B and then on to B via pipe along the shoreline. If the cost of laying pipe is $1,500,000 per mile under water and $650,000 over land, where should the point P be located to minimize the cost of laying the pipe?

Solution **I** **Objective**: Minimize the of cost of the construction of the pipeline.

Let x = distance from point A to P (miles)



II **Objective Function**

$$\text{Cost} = \$1,500,000(\text{under water}) + \$650,000(\text{over land})$$

$$\text{Cost} = 1.5|WP| + 0.65|PB| \text{ (cost in millions of dollars)}$$

$$C(x) = 1.5\sqrt{x^2+25} + 0.65(8-x)$$

III **Domain** $x \in [0,8]$

Solution

IV "<u>Calculus Time</u>"

$$C(x) = 1.5\sqrt{x^2+25} + 0.65(8-x)$$

$$C'(x) = \frac{1.5x}{\sqrt{x^2+25}} - 0.65$$

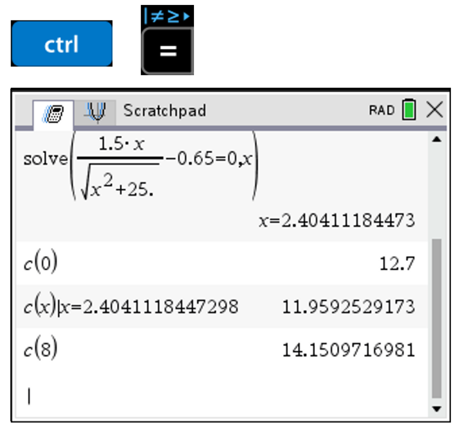
Let $C'(x) = 0$.

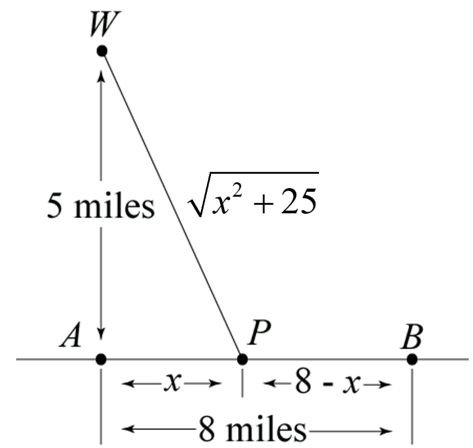
$$\frac{1.5x}{\sqrt{x^2+25}} - 0.65 = 0$$

So, the critical number is $x \cong 2.404$.

Since $x \in [0,8]$ we can use EVT.

x	$C(x)$	
0	12.7	
2.404	11.959	absolute minimum cost
8	14.151	

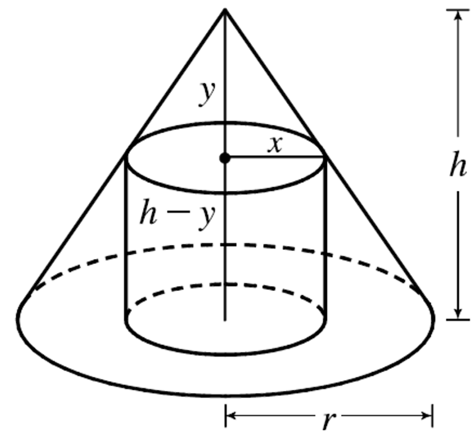




V <u>Conclusion</u>: By EVT, the absolute minimum cost of building the pipeline is $11,959,000.

Example A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest possible volume of such a cylinder.

Solution



I **Objective**: Maximize the volume of the inscribed cylinder.

Let x = radius of inscribed cylinder (units)

By similar triangles, $y/x = h/r$, so $y = hx/r$.

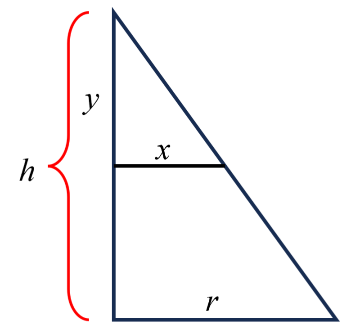
II **Objective Function**

$$V = \pi r^2 h$$

$$V = \pi x^2 (h - y)$$

$$V(x) = \pi x^2 \left(h - \frac{h}{r} x \right)$$

$$V(x) = \pi x^2 h - \frac{\pi h}{r} x^3$$



III **Domain** $x \in [0, r]$

Solution

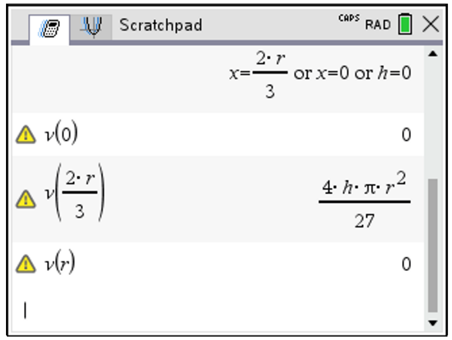
IV "Calculus Time"

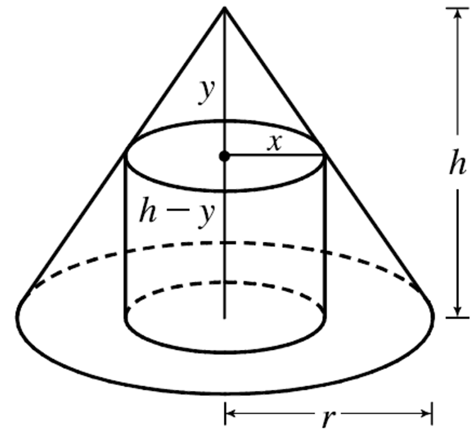
$$V(x)=\pi x^2 h-\frac{\pi h}{r}x^3$$

$$V'(x)=2h\pi x-\frac{3h\pi}{r}x^2$$

Let $V'(x)=0$.

$$2h\pi x-\frac{3h\pi}{r}x^2=0$$





So, the critical number is $x \cong \frac{2r}{3}$.

Since $x\in[0,r]$ we can use EVT.

x	$C(x)$	
0	0	
$\frac{2r}{3}$	$\frac{4h\pi r^2}{27}$	absolute maximum volume
r	0	

V **<u>Conclusion</u>:** By EVT, the absolute minimum volume of the inscribed right circular cylinder is $\frac{4h\pi r^2}{27}$ units^2.