4.4 Indeterminate Forms and l'Hospital's Rule

Marquis de L'Hospital
1661 - 1704

de L'Hospital was a French mathematician who wrote the first textbook on calculus, which consisted of the lectures of his teacher Johann Bernoulli.

Indeterminate Forms (Types $\frac{0}{0}, \frac{\infty}{\infty}$)

Suppose we are trying to analyze the behavior of the function

$$\lim_{x \to 1} \frac{\ln x}{x - 1}$$

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an **indeterminate form of type $\frac{0}{0}$**.

We met some limits of this type in Chapter 2.

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x - 1)}{(x + 1)(x - 1)} = \lim_{x \to 1} \frac{x}{x + 1} = \frac{1}{2}$$

Indeterminate Forms (Types $\frac{0}{0}$, $\frac{\infty}{\infty}$)

We used a geometric argument to show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

But these methods do not work for limits such as

$$\lim_{x \to 1} \frac{\ln x}{x - 1}$$

So, in this section we introduce a systematic method, know as *l'Hospital's Rule*, for the evaluation of indeterminate forms.

Another situation in which a limit is not obvious occurs when we look for a horizontal asymptote

$$\lim_{x \to \infty} \frac{\ln x}{x - 1} = \frac{\infty}{\infty}$$

L'Hospital's Rule

It isn't obvious how to evaluate this limit because both numerator and denominator become large as $x \to \infty$. There is a struggle between numerator and denominator. If the numerator wins, the limit will be ∞ (the numerator was increasing significantly faster than the denominator); if the denominator wins, the answer will be 0. Or there may be some compromise, in which case the answer will be some finite positive number. We now introduce a systematic method, known as *l'Hospital's Rule*, for the evaluation of indeterminate forms of type $\frac{0}{0}$ or type $\frac{\pm\infty}{\pm\infty}$.

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Example Find the limit. Use l'Hospital's Rule where appropriate.

(a) $\displaystyle\lim_{x\to\pi/4}\frac{\sin x-\cos x}{\tan x-1}$ (b) $\displaystyle\lim_{x\to\infty}\frac{\ln\sqrt{x}}{x^2}$ (c) $\displaystyle\lim_{x\to0}\frac{x^2\sin x}{\sin x-x}$

Solution

(a) $\displaystyle\lim_{x\to\pi/4}\frac{\sin x-\cos x}{\tan x-1}$

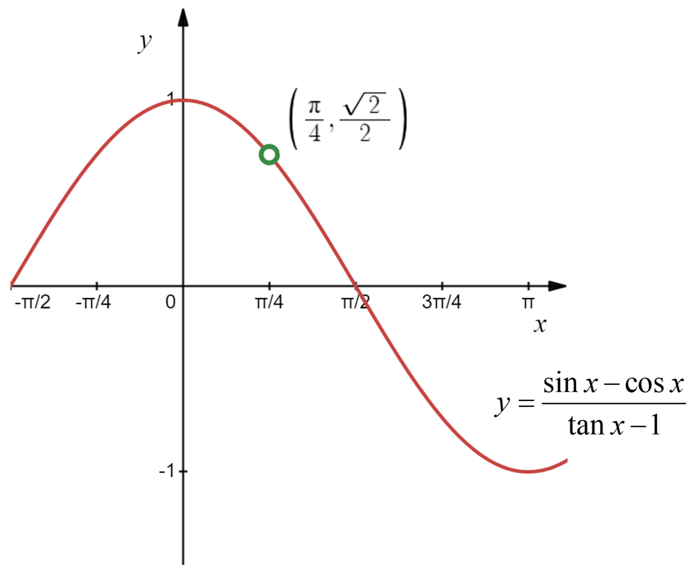
This limit has the form $\frac{0}{0}$.

$$\overset{\text{H}}{=}\lim_{x\to\pi/4}\frac{\cos x+\sin x}{\sec^2 x}$$

$$=\lim_{x\to4}\frac{\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}}{\left(\sqrt{2}\right)^2}$$

$$=\frac{\sqrt{2}}{2}$$

Hole in the graph at $\left(\frac{\pi}{4},\frac{\sqrt{2}}{2}\right)$.



Example Find the limit. Use l'Hospital's Rule where appropriate.

(a) $\lim_{x\to\pi/4} \frac{\sin x - \cos x}{\tan x - 1}$ (b) $\lim_{x\to\infty} \frac{\ln\sqrt{x}}{x^2}$ (c) $\lim_{x\to 0} \frac{x^2 \sin x}{\sin x - x}$

Solution

(b) $\lim_{x\to\infty} \frac{\ln\sqrt{x}}{x^2}$ This limit has the form $\frac{\infty}{\infty}$.

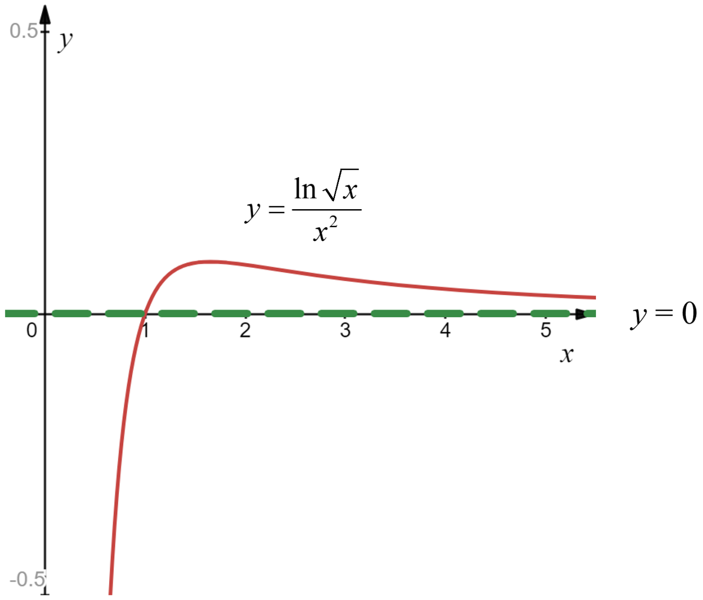
$$= \lim_{x\to\infty} \frac{\frac{1}{2}\ln x}{x^2} \qquad \frac{\infty}{\infty}$$

$$\overset{\text{H}}{=} \lim_{x\to\infty} = \frac{\frac{1}{2x}}{2x}$$

$$= \lim_{x\to\infty} \frac{1}{4x^2}$$

$$= 0$$

Horizontal asymptote at $y = 0$.



Example Find the limit. Use l'Hospital's Rule where appropriate.

(a) $\displaystyle\lim_{x\to\pi/4}\frac{\sin x-\cos x}{\tan x-1}$ (b) $\displaystyle\lim_{x\to\infty}\frac{\ln\sqrt{x}}{x^2}$ (c) $\displaystyle\lim_{x\to0}\frac{x^2\sin x}{\sin x-x}$

Solution

(c) $\displaystyle\lim_{x\to0}\frac{x^2\sin x}{\sin x-x}$ This limit has the form $\frac{0}{0}$.

$$\overset{\text{H}}{=}\lim_{x\to0}\frac{x^2\cos x+2x\sin x}{\cos x-1}$$ This limit has the form $\frac{0}{0}$.

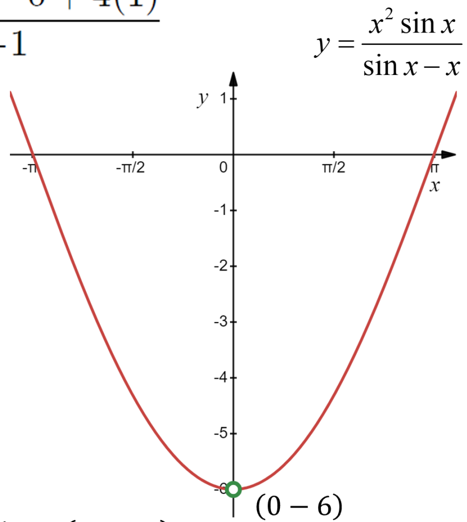
$$\overset{\text{H}}{=}\lim_{x\to0}\frac{-x^2\sin x+2x\cos x+2x\cos x+2\sin x}{-\sin x}$$

$$=\lim_{x\to0}\frac{(2-x^2)\sin x+4x\cos x}{-\sin x}\qquad\frac{0}{0}$$

$$\overset{\text{H}}{=}\lim_{x\to0}\frac{(2-x^2)\cos x-2x\sin x-4x\sin x+4\cos x}{-\cos x}$$

$$=\frac{2(1)-0-0+4(1)}{-1}$$

$$=-6$$



Hole in the graph at (0 − 6).

Indeterminate Products (Type $\mathbf{0 \cdot \infty}$)

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$ (or $-\infty$), then it isn't clear what the value of $\lim_{x \to a} f(x)g(x)$, if any, will be. There is a struggle between f and g. If f wins, the answer will be 0; if g wins, the answer will be ∞ (or $-\infty$). Or there may be a compromise where the answer is a finite nonzero number. This kind of limit is called an **indeterminate form of type $\mathbf{0 \cdot \infty}$**. We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}$$

$$= \frac{0}{1/\infty} \qquad\qquad = \frac{\infty}{1/0}$$

$$= \frac{0}{0} \qquad\qquad = \frac{\infty}{\infty}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or ∞/∞ so that we can use l'Hospital's Rule.

Example Find the limit. Use l'Hospital's Rule where appropriate.

$$\lim_{x\to 1^+} \ln x \cdot \tan(\pi x/2)$$

Solution

$\displaystyle\lim_{x\to 1^+} \ln x \cdot \tan(\pi x/2)$ This limit has the form $0 \cdot (-\infty)$.

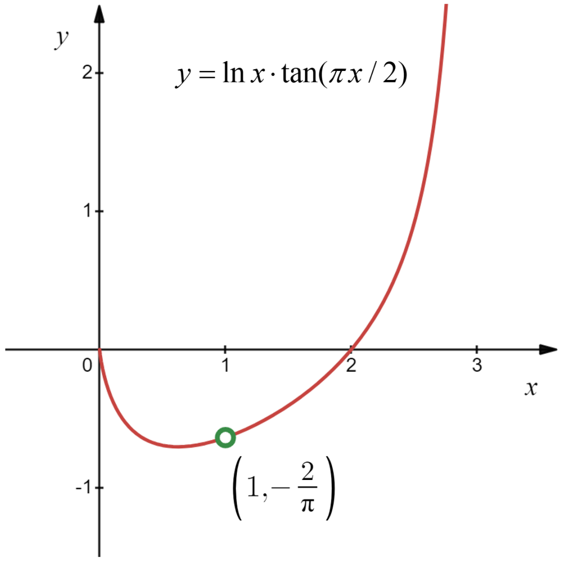
$\displaystyle= \lim_{x\to 1^+} \frac{\ln x}{\cot(\pi x/2)}$ This limit has the form $\frac{0}{0}$.

$$\overset{\text{H}}{=} \lim_{x\to 1^+} \frac{1/x}{(-\pi/2)\csc^2(\pi x/2)}$$

$$= \lim_{x\to 1^+} \frac{\frac{1}{x}}{\frac{-\pi}{2}} \cdot \sin^2(\pi x/2)$$

$$= \lim_{x\to 1^+} \left(-\frac{2}{\pi x}\right) \cdot \sin^2(\pi x/2) = \left(-\frac{2}{\pi(1)}\right) \cdot (1)^2 = -\frac{2}{\pi}$$

Hole in the graph at $\left(1, -\frac{2}{\pi}\right)$.



■ Indeterminate Differences (Type $\infty - \infty$)

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a} [f(x) - g(x)]$$

is called an **indeterminate form of type $\infty - \infty$**. Again there is a contest between f and g. Will the answer be ∞ (f wins) or will it be $-\infty$ (g wins) or will they compromise on a finite number? To find out, we try to convert the difference into a quotient (for instance, by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.

Example Find the limit. Use l’Hospital’s Rule where appropriate.

$$\lim_{x\to 0+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$$

Solution

$$\lim_{x\to 0+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$$ This limit has the form $\infty-\infty$.

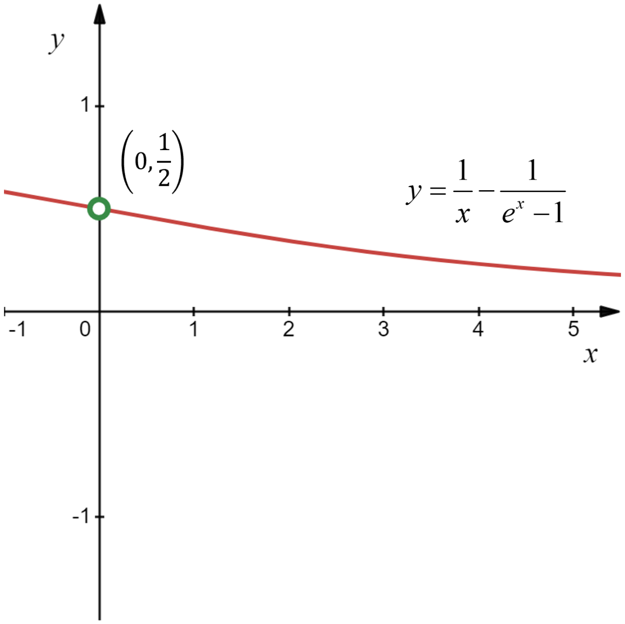
$$=\lim_{x\to 0+}\frac{e^x-1-x}{x(e^x-1)}$$ This limit has the form $\frac{0}{0}$.

$$\overset{\text{H}}{=}\lim_{x\to 0+}\frac{e^x-1}{xe^x+e^x-1}$$ This limit has the form $\frac{0}{0}$.

$$\overset{\text{H}}{=}\lim_{x\to 0+}\frac{e^x}{xe^x+e^x+e^x}$$

$$=\frac{1}{0+1+1}$$

$$=\frac{1}{2}$$ Hole in the graph at $\left(0,\frac{1}{2}\right)$.



Indeterminate Powers (Types 0^0, ∞^0, 1^∞)

Several indeterminate forms arise from the limit

$$\lim_{x\to a} [f(x)]^{g(x)}$$

1. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ type 0^0
2. $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ type ∞^0
3. $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$ type 1^∞

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{let} \quad y = [f(x)]^{g(x)}, \quad \text{then} \quad \ln y = g(x) \ln f(x)$$

Example Find the limit. Use l'Hospital's Rule where appropriate.

(a) $\lim_{x \to \infty} (e^x + 10x)^{1/x}$ (b) $\lim_{x \to 0^+} (1 - \cos x)^{\sin x}$

Solution

(a) $\lim_{x \to \infty} (e^x + 10x)^{1/x}$

Let $y = (e^x + 10x)^{1/x}$

$\ln y = \dfrac{1}{x} \ln(e^x + 10x)$, so

$$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{1}{x} \ln(e^x + 10x)$$

$$= \lim_{x \to \infty} \frac{\ln(e^x + 10x)}{x}$$ This limit has the form $\frac{\infty}{\infty}$.

$$\overset{\text{H}}{=} \lim_{x \to \infty} \frac{\dfrac{1}{e^x + 10x} \cdot (e^x + 10)}{1}$$

Solution

$$\overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{1}{e^x+10x}\cdot(e^x+10)}{1}$$

$$= \lim_{x\to\infty} \frac{e^x+10}{e^x+10x}$$ This limit has the form $\frac{\infty}{\infty}$.

$$\overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x+10}$$ This limit has the form $\frac{\infty}{\infty}$.

$$\overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x}$$

$$= \lim_{x\to\infty} (1)$$

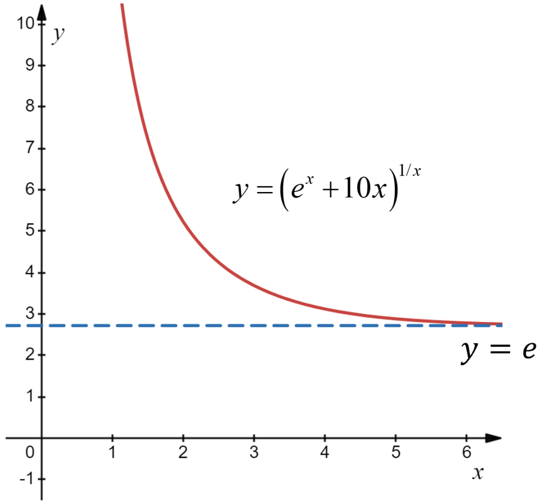
$$= 1$$

Hence $\lim_{x\to\infty} \ln y = 1$

Using the fact that $y = e^{\ln y}$ we get

$$\lim_{x\to\infty} (e^x+10x)^{1/x} = \lim_{x\to\infty} e^{\ln y}$$
$$= e^1$$
$$= e$$

Horizontal asymptote at $y = e$.



Example Find the limit. Use l'Hospital's Rule where appropriate.

(a) $\lim\limits_{x\to\infty}(e^x + 10x)^{1/x}$ (b) $\lim\limits_{x\to 0+}(1-\cos x)^{\sin x}$

Solution

(b) $\lim\limits_{x\to 0+}(1-\cos x)^{\sin x}$ This limit has the form 0^0.

Let $y = (1-\cos x)^{\sin x}$

$\ln y = \sin x \ln(1-\cos x)$, so

$$\lim_{x\to 0+}\ln y = \lim_{x\to 0+}\sin x\ln(1-\cos x) \quad 0\cdot(-\infty)$$

$$= \lim_{x\to 0+}\frac{\ln(1-\cos x)}{\csc x} \quad \frac{-\infty}{+\infty}$$

$$\overset{\text{H}}{=} \lim_{x\to 0+}\frac{\frac{1}{1-\cos x}\cdot\sin x}{-\csc x\cot x}$$

$$= -\lim_{x\to 0+}\frac{\sin x}{(1-\cos x)\csc x\cot x}$$

$$= -\lim_{x\to 0+}\frac{\sin x}{\csc x\cot x - \cot^2 x}\cdot\left(\frac{\sin^2 x}{\sin^2 x}\right)$$

$$= -\lim_{x\to 0+}\frac{\sin^3 x}{\cos x - \cos^2 x}$$

$$= -\lim_{x\to 0+}\frac{\sin^3 x}{(1-\cos x)\cos x} \quad \frac{0}{0}$$

$$\overset{\text{H}}{=} -\lim_{x\to 0+}\frac{3\sin^2 x\cos x}{(1-\cos x)(-\sin x)+\cos x\,(\sin x)}$$

$$= -\lim_{x\to 0+}\frac{3\sin x\cos x}{(\cos x - 1)+\cos x}$$

$$= -\frac{0}{0+1} = 0$$

Hence $\lim\limits_{x\to 0^+}\ln y = 0$.

Solution

Using the fact that $y = e^{\ln y}$ we get

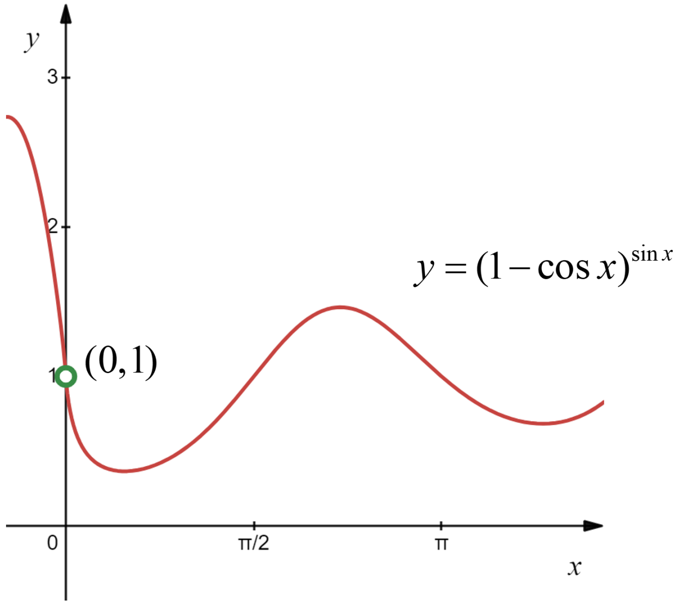
$$\lim_{x\to 0+} (1-\cos x)^{\sin x} = \lim_{x\to 0+} e^{\ln y}$$

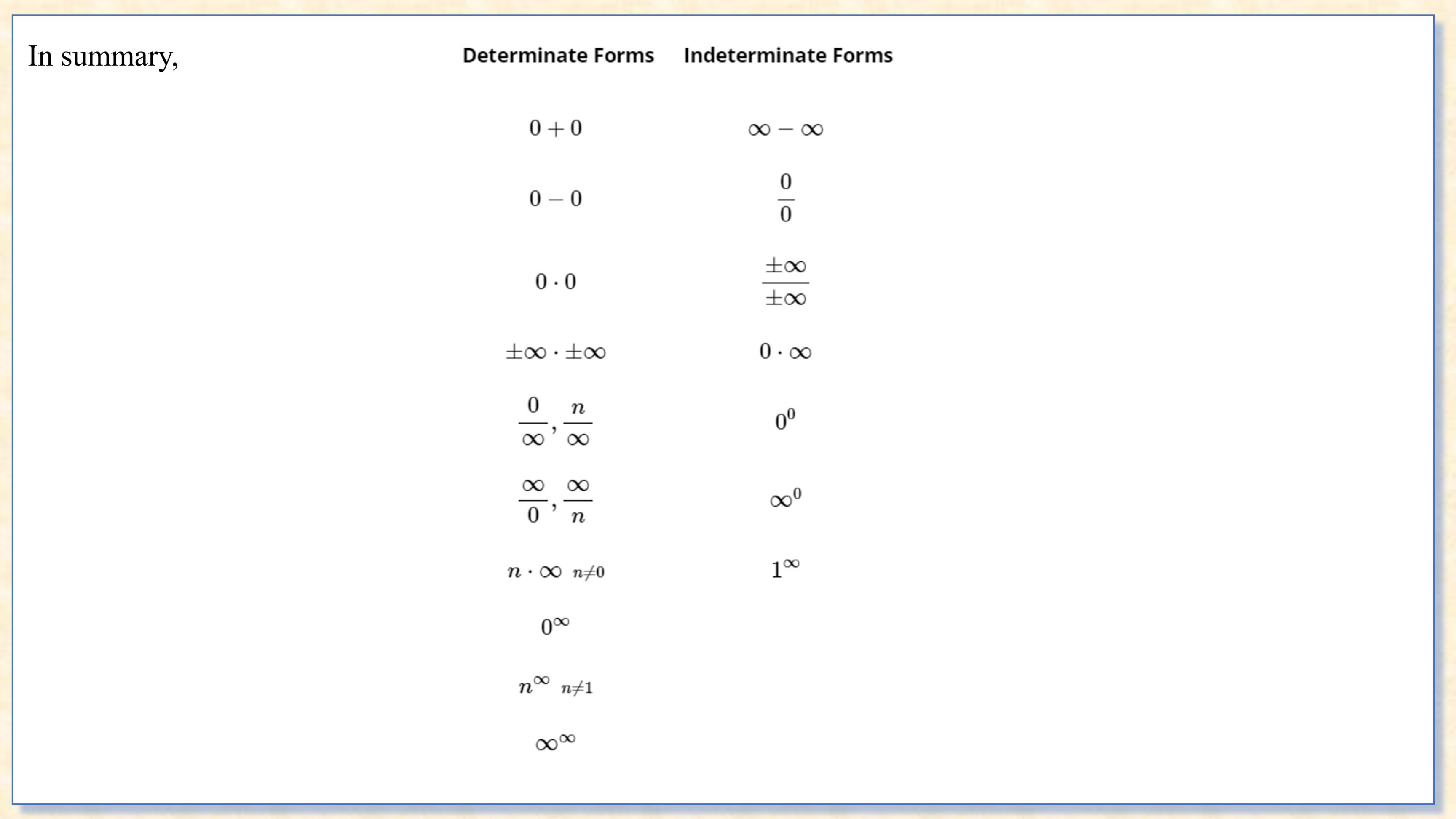
Note: $\lim_{x\to 0^+} \ln y = 0.$

$$= e^0$$

$$= 1$$

Hole in the graph at (0,1).





In summary,

Determinate Forms	Indeterminate Forms
$0+0$	$\infty-\infty$
$0-0$	$\frac{0}{0}$
$0\cdot 0$	$\frac{\pm\infty}{\pm\infty}$
$\pm\infty\cdot\pm\infty$	$0\cdot\infty$
$\frac{0}{\infty}, \frac{n}{\infty}$	0^0
$\frac{\infty}{0}, \frac{\infty}{n}$	∞^0
$n\cdot\infty \quad {\scriptstyle n\neq 0}$	1^∞
0^∞	
$n^\infty \quad {\scriptstyle n\neq 1}$	
∞^∞	