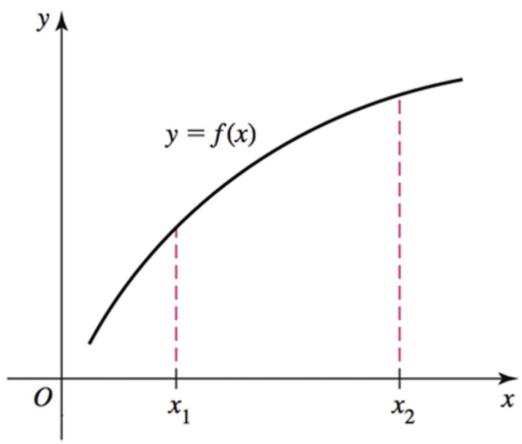
4.3 What Derivatives Tell Us about the Shape of a Graph

Blaise Pascal
1623 – 1662

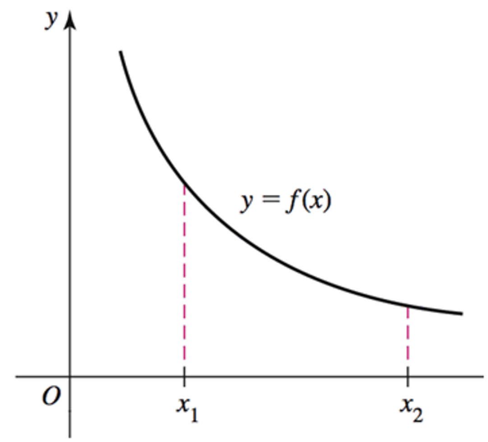
Blaise Pascal was a very influential French mathematician and philosopher who contributed to many areas of mathematics. He worked on conic sections and projective geometry and in correspondence with Fermat he laid the foundations for the theory of probability.

Definition **Increasing and Decreasing Functions**

Suppose a function f is defined on an interval I. We say that f is **increasing** on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$. We say that f is **decreasing** on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.



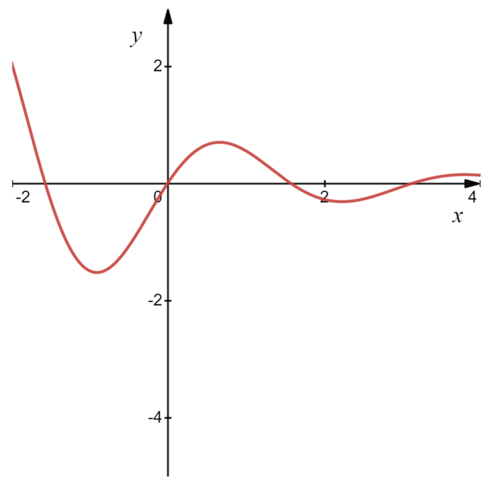
f increasing: $f(x_2) > f(x_1)$ whenever $x_2 > x_1$

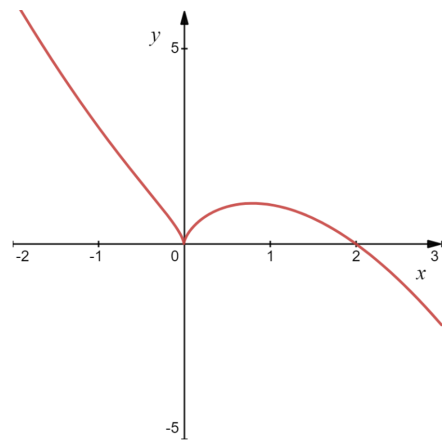


f decreasing: $f(x_2) < f(x_1)$ whenever $x_2 > x_1$

How can we use calculus?

What Does f' Say about f?





Thus, it appears that f increases when $f'(x)$ is positive and decreases when $f'(x)$ is negative. We get the following theorem.

Increasing/Decreasing Test (IDT)

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The First Derivative Test

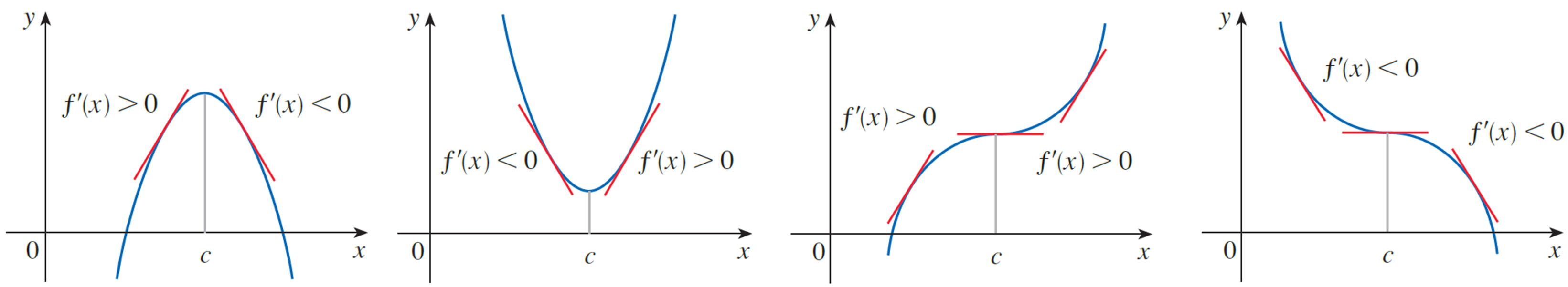
Recall from Section 4.1 that if f has a local maximum or minimum at c, then c must be a critical number of f (by Fermat's Theorem), but not every critical number gives rise to a maximum or a minimum. We therefore need a test that will tell us whether or not f has a local maximum or minimum at a critical number.

The First Derivative Test Suppose that c is a critical number of a continuous function f. **(FDT)**

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.



(a) Local maximum at c (b) Local minimum at c (c) No maximum or minimum at c (d) No maximum or minimum at c

Example Consider $y = 3x^4 - 4x^3 - 12x^2 + 5$ on its entire domain $(-\infty, +\infty)$.

a) Find the critical numbers for y.

b) Find intervals of increasing and decreasing.

c) Identify the coordinates of all the extrema, determine if they are local or absolute.

Solution

a) Find the critical numbers for y.

$y' = 12x^3 - 12x^2 - 24x$

Let $y' = 0$

$12x^3 - 12x^2 - 24x = 0$

$12x(x^2 - x - 2) = 0$

$12x(x-2)(x+1) = 0$

Critical numbers: $x = 0,\ x = 2,\ x = -1$

b) Find intervals of increasing and decreasing.

$x=-2$ $\quad x=-\frac{1}{2}$ $\quad x=1$ $\quad x=3$

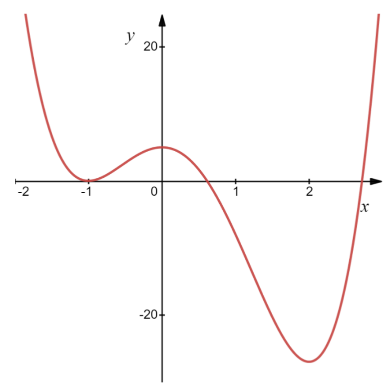
$-\ \ 0\ \ +\ \ 0\ \ -\ \ 0\ \ +\ \ y'$

x: $\searrow\ -1\ \nearrow\ 0\ \searrow\ 2\ \nearrow$

By IDT,

Increasing: $x \in (-1,0) \cup (2,+\infty)$

Decreasing: $x \in (-\infty,-1) \cup (0,2)$



c) Identify the coordinates of all the extrema, determine if they are local or absolute.

Solution

$y = 3x^4 - 4x^3 - 12x^2 + 5$

Let $x = -1$

$3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 5 = 0$

Let $x = 0$

$3(0)^4 - 4(0)^3 - 12(0)^2 + 5 = 5$

Let $x = 2$

$3(2)^4 - 4(2)^3 - 12(2)^2 + 5 = -27$

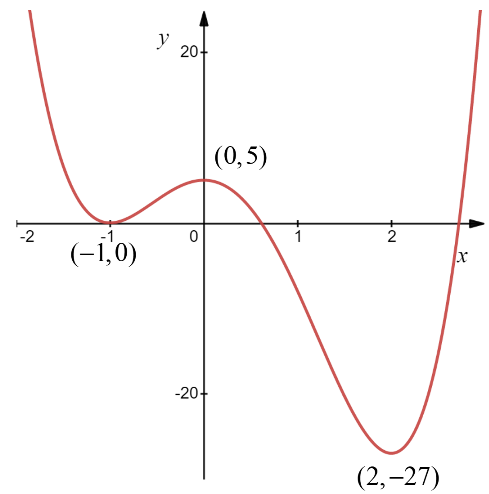
	$x=-2$		$x=-\frac{1}{2}$		$x=1$		$x=3$	
y'	$-$	0	$+$	0	$-$	0	$+$	
x	$\searrow$	-1	$\nearrow$	0	$\searrow$	2	$\nearrow$	

By FDT, the coordinates of the local maximum point is $(0,5)$, and the coordinates of the local minimum points are $(-1,0)$ and $(2,-27)$.

Because of the end-behavior of the function, $(2,-27)$ is an absolute minimum point.

$$\lim_{x\to-\infty} 3x^4 - 4x^3 - 12x^2 + 5 = +\infty$$

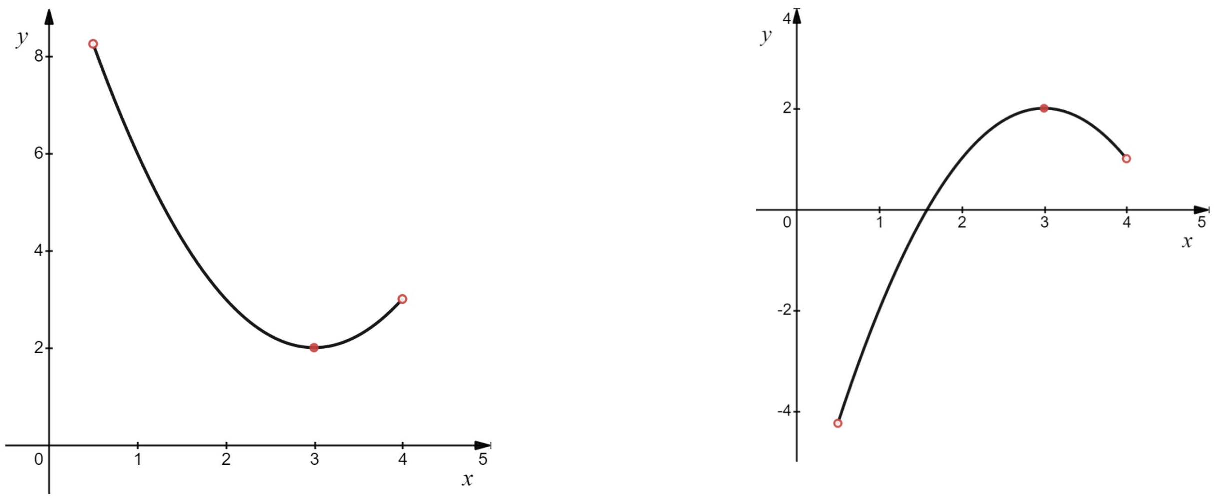
$$\lim_{x\to+\infty} 3x^4 - 4x^3 - 12x^2 + 5 = +\infty$$



THEOREM **One Local Extremum Implies Absolute Extremum** **(OLE)**

Suppose f is continuous on an interval I that contains exactly one local extremum at c.

- If a local minimum occurs at c, then $f(c)$ is the absolute minimum of f on I.
- If a local maximum occurs at c, then $f(c)$ is the absolute maximum of f on I.



Example Consider the function $f(x) = x^2 \ln x$.

(a) Find the intervals on which f is increasing or decreasing.

(b) Find the local maximum and minimum values of f, determine if they are absolute.

Solution

(a) Find the intervals on which f is increasing or decreasing.

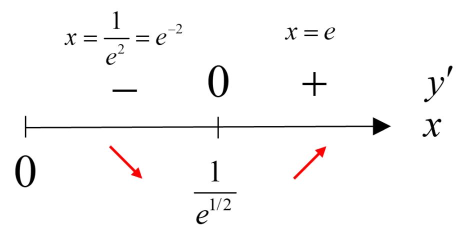
$$f(x) = x^2 \ln x$$

$$\begin{aligned} f'(x) &= x^2(1/x) + (\ln x)(2x) \\ &= x + 2x\ln x \\ &= x(1 + 2\ln x) \end{aligned}$$

The domain of f is $(0, \infty)$, so the sign of f' is determined solely by the factor $1 + 2\ln x$.

$1 + 2\ln x = 0$ $\qquad$ $x = e^{-1/2}$

$\ln x = -\dfrac{1}{2}$ $\qquad$ $x = \dfrac{1}{e^{1/2}}$



By IDT,

Increasing: $x \in \left(e^{-1/2}, +\infty\right)$

Decreasing: $x \in \left(0, e^{-1/2}\right)$

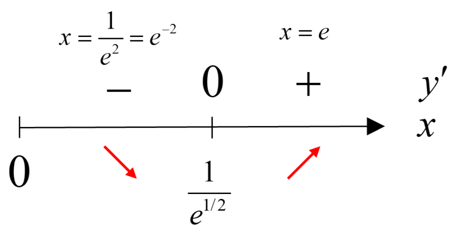
Example Consider the function $f(x) = x^2 \ln x$.

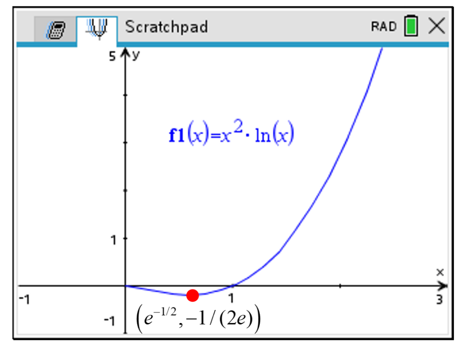
(a) Find the intervals on which f is increasing or decreasing.

(b) Find the local maximum and minimum values of f, determine if they are absolute.

Solution

(b) Find the local maximum and minimum values of f, determine if they are absolute.





Absolute Minimum Point

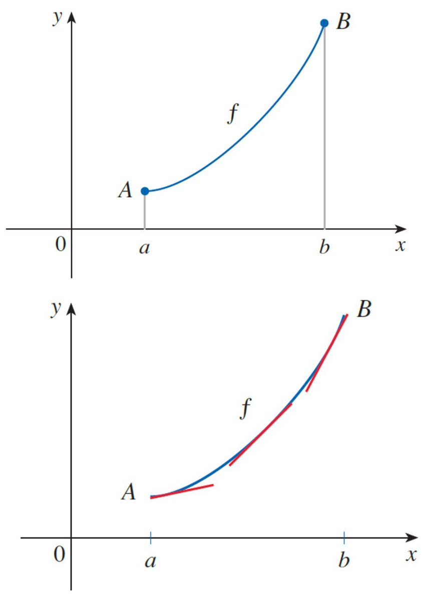
f changes from decreasing to increasing at $x = e^{-1/2}$.

Hence, by FDT f has a local minimum value at $x = e^{-1/2}$.

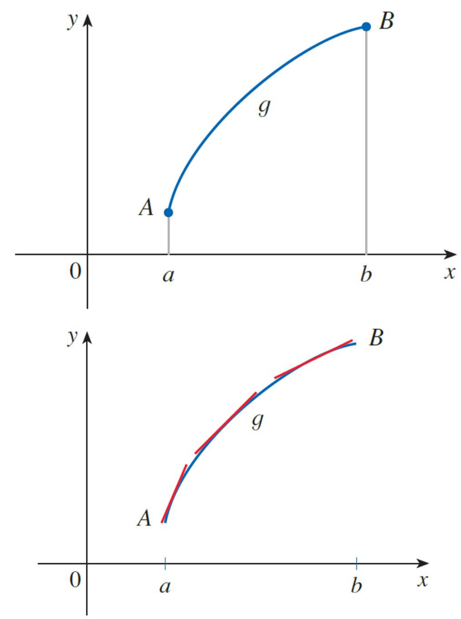
Thus, $f(e^{-1/2}) = (e^{-1/2})^2 \ln(e^{-1/2}) = e^{-1}(-1/2) = -1/(2e)$ is a local minimum value.

But wait! By **OLE** $y = -\frac{1}{2e}$ is the absolute minimum value for f.

What Does f'' Say about f?



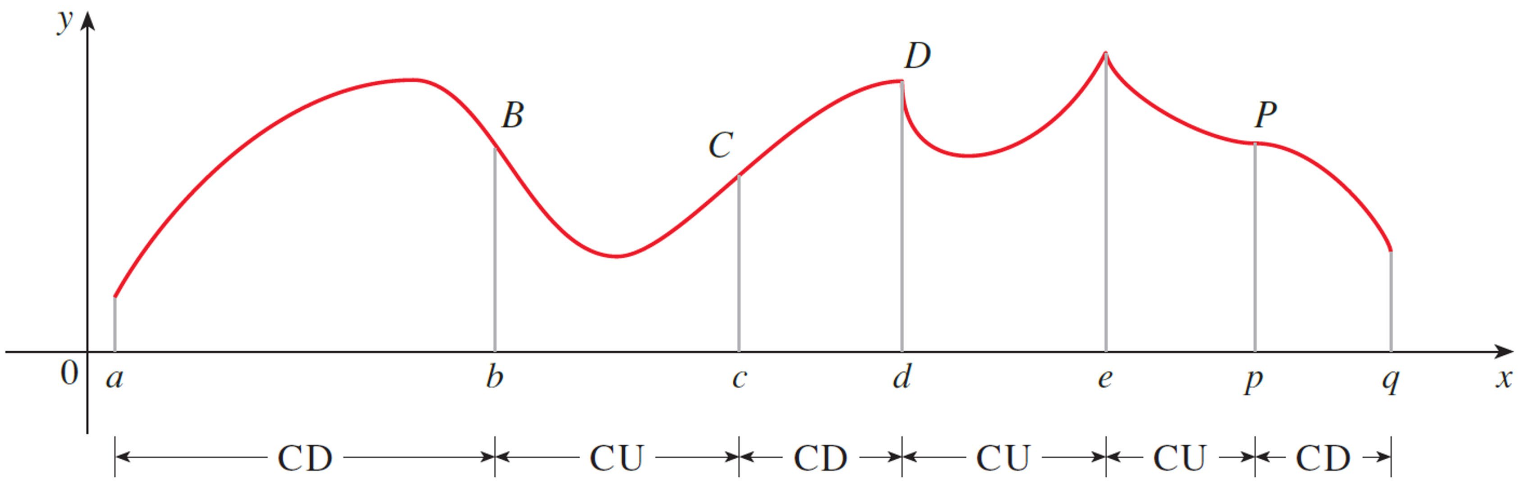
(a) Concave upward



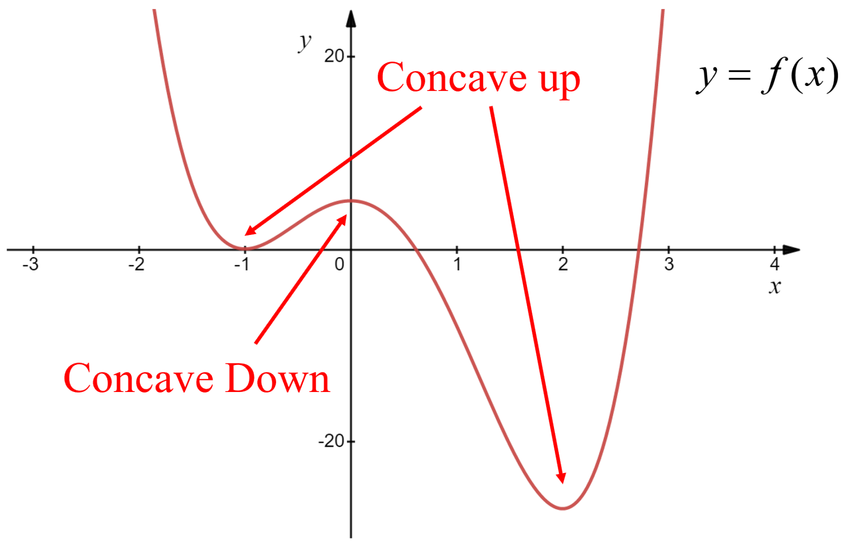
(b) Concave downward

Definition If the graph of f lies above all of its tangents on an interval I, then f is called **concave upward** on I. If the graph of f lies below all of its tangents on I, then f is called **concave downward** on I.

What Does f'' Say about f?



What Does f'' Say about f?



Let's take the second derivative of y.

$$y'' = 36x^2 - 24x - 24.$$

Let $x = -1$, then $y''(-1) = 36$.

Let $x = 0$, then $y''(0) = -24$.

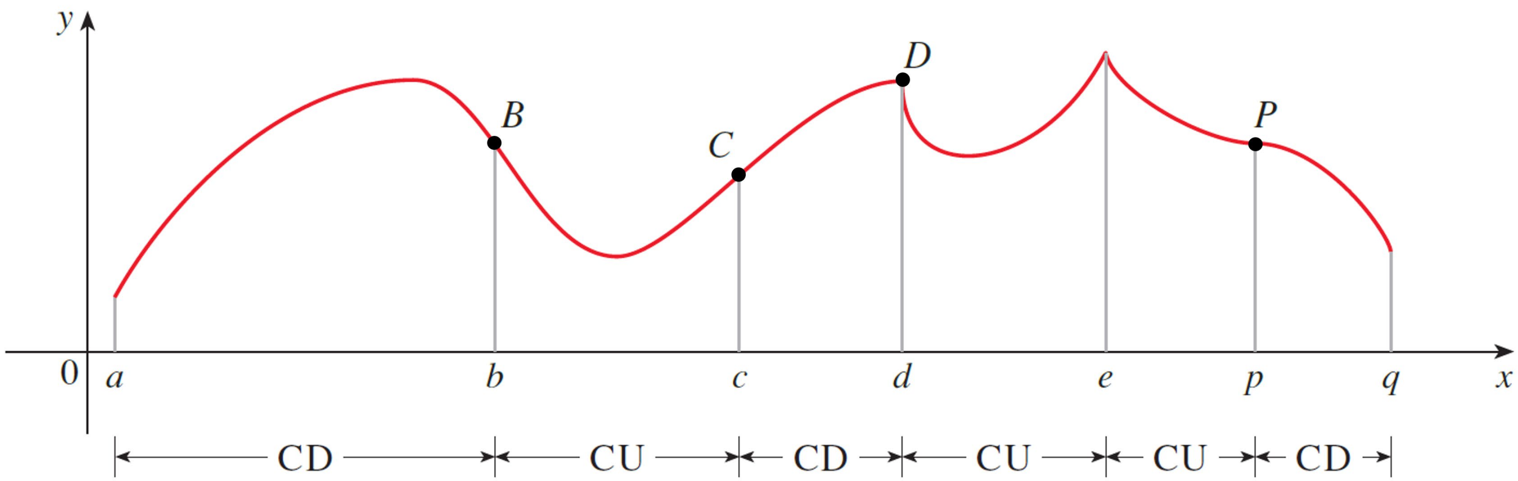
Let $x = 2$, then $y''(2) = 72$.

Concavity Test (CT)

(a) If $f''(x) > 0$ on an interval I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ on an interval I, then the graph of f is concave downward on I.

What Does f'' Say about f?



Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P. **(DIP)**

Example Consider the function $y = x^3 - 3x^2 + 4$.

(a) Find the intervals on which f is increasing or decreasing.
(b) Find the local maximum and minimum points of f.
(c) Find the intervals of concavity and the inflection points.

Solution

(a) Find the intervals on which f is increasing or decreasing.

$$y' = 3x^2 - 6x$$

Set $y' = 0$

$$3x^2 - 6x = 0$$

$$3x(x-2) = 0$$

$$x = 0,\ 2$$

are the critical numbers.

Possible extrema at $x = 0,\ 2$.

y'	+	0	−	0	+
x	↗	0	↘	2	↗

By IDT,

Increasing: $x \in (-\infty, 0) \cup (2, +\infty)$

Decreasing: $x \in (0, 2)$

(b) Find the local maximum and minimum points of f.

By FDT, the coordinates of a local maximum point are $(0,4)$ and the coordinates of a local minimum point are $(2,0)$. Because of end-behavior there are no absolute extrema.

$$\lim_{x \to -\infty} (x^3 - 3x^2 + 4) = \infty$$

$$\lim_{x \to +\infty} (x^3 - 3x^2 + 4) = +\infty$$

Solution

(c) Find the intervals of concavity and the inflection points.

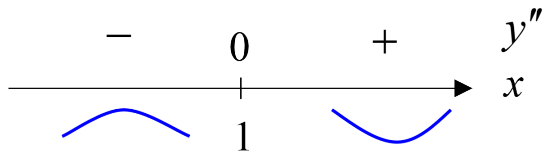
Now let’s examine concavity and look for inflection points by setting the second derivative equal to zero.

$y'' = 6x - 6$ <u>Possible</u> inflection point at $x = 1$.

$0 = 6x - 6$

$6 = 6x$

$x = 1$



$y''(0) = 6 \cdot 0 - 6 = -6$ ➡ negative

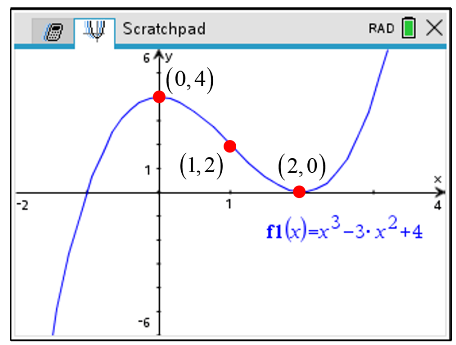
$y''(2) = 6 \cdot 2 - 6 = 6$ ➡ positive

By CT,

Concave Upward: $x \in (1, +\infty)$

Concave Downward: $x \in (-\infty, 1)$

By DIP, there is an inflection point at (1, 2) since the second derivative changes from negative to positive.



Example Consider the function $f(\theta) = 2\cos\theta + \cos^2\theta, \quad 0 \leqslant \theta \leqslant 2\pi$.

(a) Find the intervals of increase or decrease.
(b) Find the local maximum and minimum values.
(c) Find the intervals of concavity and the inflection points.

Solution

(a) Find the intervals of increase or decrease.

$$f(\theta) = 2\cos\theta + \cos^2\theta, \ 0 \le \theta \le 2\pi$$

$$f'(\theta) = -2\sin\theta + 2\cos\theta\,(-\sin\theta)$$

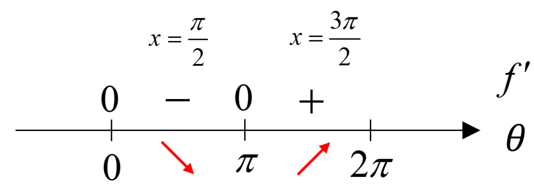
$$= -2\sin\theta\,(1 + \cos\theta)$$

Let $f'(\theta) = 0$

$$-2\sin\theta\,(1 + \cos\theta) = 0$$

$x = 0$ and $x = \pi$

are the critical numbers.



By IDT,

Increasing: $x \in (\pi, 2\pi)$

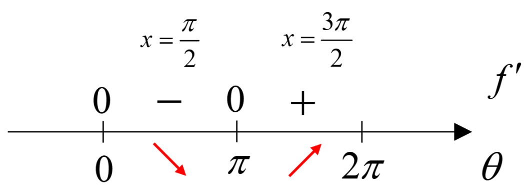
Decreasing: $x \in (0, \pi)$

Solution

(b) Find the local maximum and minimum values.

$f(\theta) = 2\cos\theta + \cos^2\theta,\ 0 \le \theta \le 2\pi$

$f'(\theta) = -2\sin\theta\,(1+\cos\theta)$



Hence, by FDT f has a local minimum value at $x = \pi$.

$f(\pi) = 2\cos(\pi) + \cos^2(\pi) = -2 + 1 = -1$

But wait! By **OLE** $y = -1$ is the absolute minimum value for f.

So, $(\pi, -1)$ is the absolute minimum point on f.

Since f is defined on a closed interval there must be an absolute maximum value for f by **EVT**. Let's check the endpoints.

$f(0) = 2\cos(0) + \cos^2(0) = 2 + 1 = 3$

$f(2\pi) = 2\cos(2\pi) + \cos^2(2\pi) = 2 + 1 = 3$

By **EVT** $y = 3$ is the absolute maximum value for f.

So, $(0,3)$ and $(2\pi, 3)$ are the absolute maximum points on f.

Solution

(c) Find the intervals of concavity and the inflection points.

$$f(\theta) = 2\cos\theta + \cos^2\theta,\ 0 \le \theta \le 2\pi$$

$$f'(\theta) = -2\sin\theta\,(1+\cos\theta)$$

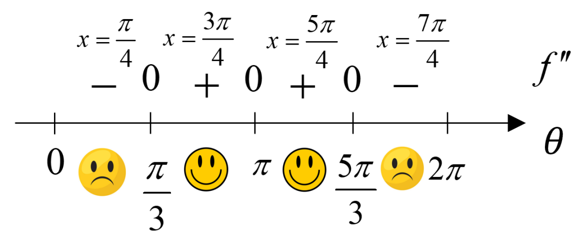
$$\begin{aligned} f''(\theta) &= -2\sin\theta\,(-\sin\theta) + (1+\cos\theta)(-2\cos\theta) \\ &= 2\sin^2\theta - 2\cos\theta - 2\cos^2\theta \\ &= 2(1-\cos^2\theta) - 2\cos\theta - 2\cos^2\theta \\ &= -4\cos^2\theta - 2\cos\theta + 2 \\ &= -2(2\cos^2\theta + \cos\theta - 1) \\ &= -2(2\cos\theta - 1)(\cos\theta + 1) \end{aligned}$$

Let $f(\theta) = 0$.

$$-2(2\cos\theta - 1)(\cos\theta + 1) = 0$$

$$\cos\theta = \frac{1}{2} \quad \text{or} \quad \cos\theta = -1$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3} \quad \text{or} \quad \theta = \pi$$



By CT,

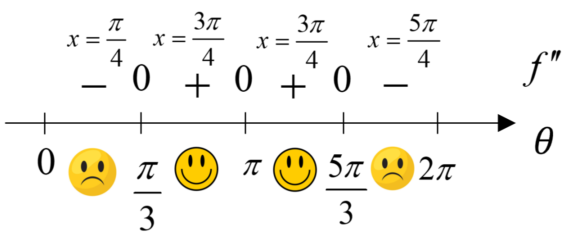
Concave Upward: $x \in \left(\frac{\pi}{3}, \pi\right) \cup \left(\pi, \frac{5\pi}{3}\right)$

Concave Downward: $x \in \left(0, \frac{\pi}{3}\right) \cup \left(\frac{5\pi}{3}, 2\pi\right)$

Solution

(c) Find the intervals of concavity and the inflection points.

$f(\theta) = 2\cos\theta + \cos^2\theta,\ 0 \le \theta \le 2\pi$

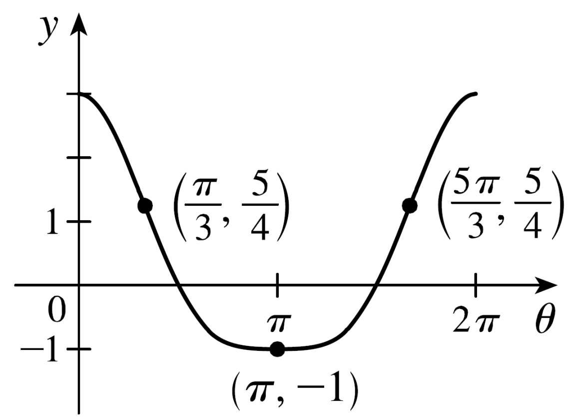


By DIP, inflection points at $\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$.

$$f\left(\frac{\pi}{3}\right) = 2\cos\left(\frac{\pi}{3}\right) + \cos^2\left(\frac{\pi}{3}\right) = 2\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 = \frac{5}{4}$$

$$f\left(\frac{5\pi}{3}\right) = 2\cos\left(\frac{5\pi}{3}\right) + \cos^2\left(\frac{5\pi}{3}\right) = 2\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 = \frac{5}{4}$$

Inflection points are $\left(\dfrac{\pi}{3}, \dfrac{5}{4}\right)$ and $\left(\dfrac{\pi}{3}, \dfrac{5}{4}\right)$.

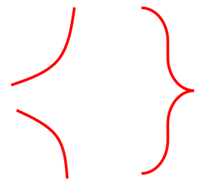


Given a real number c in the domain of a continuous function $y = f(x)$,

First derivative:

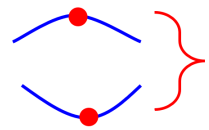
$y'(c)$ is positive ➡ Curve is increasing.

$y'(c)$ is negative ➡ Curve is decreasing.



IDT

$y'(c)$ is zero or undefined ➡ Possible local maximum or minimum points.

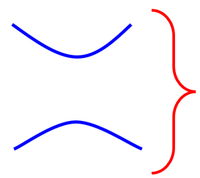


FDT

Second derivative:

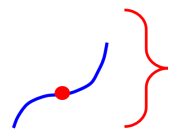
$y''(c)$ is positive ➡ Curve is concave up.

$y''(c)$ is negative ➡ Curve is concave down.



CT

$y''(c)$ is zero or undefined ➡ Possible inflection point (where concavity changes).

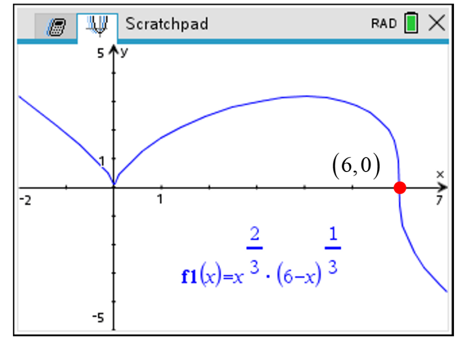


DIP

Example

Consider the function $f(x) = x^{2/3}(6 - x)^{1/3}$. If the function below is it's second derivative, find intervals of concavity and inflection point(s).

$$f''(x) = \frac{-8}{x^{4/3}(6 - x)^{5/3}}$$



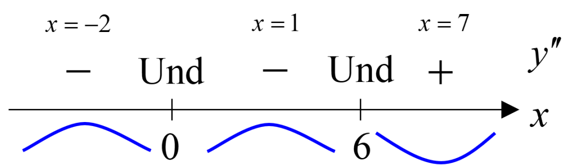
Solution

First note that the domain of f is $\mathbb{R}$.

$$f''(x) = \frac{-8}{x^{4/3}(6 - x)^{5/3}}$$

$x = 0$ and $x = 6$

are the critical numbers for $f''(x)$.



By CT,

Concave Upward: $x \in (6, +\infty)$

Concave Downward: $x \in (-\infty, 0) \cup (0, 6)$

$$f(6) = 6^{2/3}(6-6)^{1/3} = 0$$

By DIP, there is an inflection point at (6,0) since the second derivative changes from negative to positive.

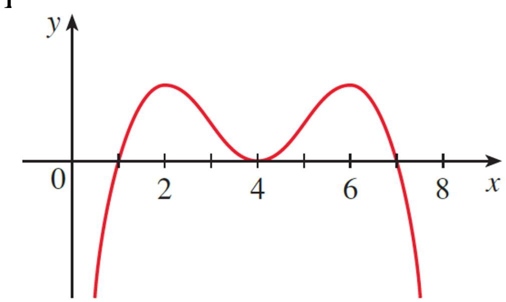
Example

In each part state the x-coordinates of the inflection points of f. Give reasons for your answers.

(a) The curve is the graph of f.

(b) The curve is the graph of f'.

(c) The curve is the graph of f''.



Solution

(a) There is an IP at $x = 3$ because the graph of f changes from CD to CU there. There is an IP at $x = 5$ because the graph of $f(x)$ changes from CU to CD there.

(b) There is an IP at $x = 2$ and at $x = 6$ because $f'(x)$ has a maximum value there, and so f'' changes from positive to negative there. There is an IP at $x = 4$ because $f'(x)$ has a minimum value there and so $f''(x)$ changes from negative to positive there.

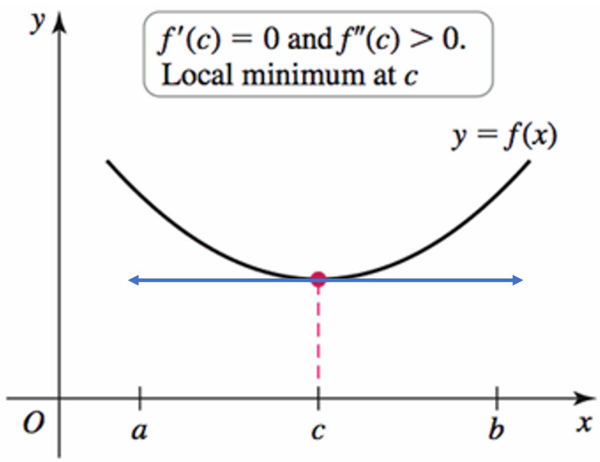
(c) There is an inflection point at $x = 1$ because $f''(x)$ changes from negative to positive there, and so the graph of $f(x)$ changes from concave downward to concave upward. There is an inflection point at $x = 7$ because $f''(x)$ changes from positive to negative there, and so the graph of $f(x)$ changes from concave upward to concave downward.

The Second Derivative Test

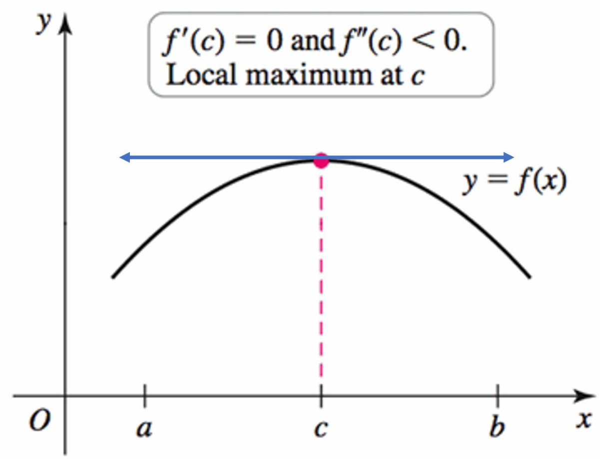
The Second Derivative Test Suppose f'' is continuous near c. **(SDT)**

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.



(a)



(b)

Example Using the Second Derivative Test, find the local extreme points for $y = x^3 - 3x^2 + 4$.

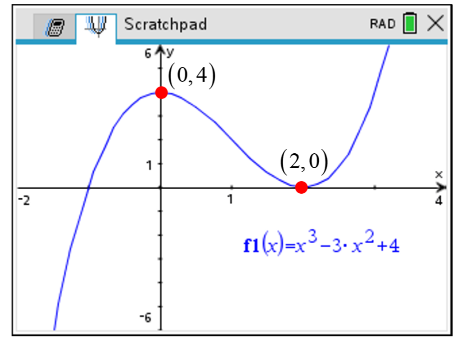
Solution

$y' = 3x^2 - 6x$

$y' = 3x(x-2)$

$y'(0) = 0$ and $y'(2) = 0$

Possible local extrema at $x = 0, 2$.



We can use the Second Derivative Test:

$y'' = 6x - 6$

$y''(0) = 6 \cdot 0 - 6 = -6$

Because the second derivative at $x = 0$ is negative and $y'(0) = 0$, the graph is concave down and therefore (0,4) is a local maximum point.

$y''(2) = 6 \cdot 2 - 6 = 6$

Because the second derivative at $x = 2$ is positive and $y'(2) = 0$, the graph is concave up and therefore (2,0) is a local minimum point.

Example Consider the function $f(x) = 1 + \dfrac{1}{x} - \dfrac{1}{x^2}$.

(a) Find the vertical and horizontal asymptotes.
(b) Find the intervals of increase or decrease.
(c) Find the local maximum and minimum values.
(d) Find the intervals of concavity and the inflection points.
(e) Use the information from parts (a) – (d) to sketch the graph of f.

Solution

(a) Find the vertical and horizontal asymptotes.

$$f(x) = 1 + \frac{1}{x} - \frac{1}{x^2}$$

has domain $(-\infty, 0) \cup (0, \infty)$.

$$\lim_{x\to\pm\infty}\left(1 + \overset{0}{\cancel{\frac{1}{x}}} - \overset{0}{\cancel{\frac{1}{x^2}}}\right) = 1$$

So, $y = 1$ is a HA.

$$\lim_{x\to 0+}\left(1 + \frac{1}{x} - \frac{1}{x^2}\right) = \lim_{x\to 0+}\left(\frac{\overset{0}{\cancel{x^2}} + \overset{0}{\cancel{x}} - 1}{x^2}\right)$$

$$= \lim_{x\to 0^+}\frac{-1}{\left(0^+\right)^2}$$

$$= -\infty$$

A similar argument can be made for $x \to 0^-$.

So, $x = 0$ is a VA.

Solution

(b) Find the intervals of increase or decrease.

$$f(x) = 1 + \frac{1}{x} - \frac{1}{x^2}$$

$$f'(x) = -\frac{1}{x^2} + \frac{2}{x^3}$$

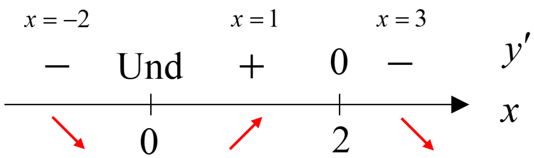
$$= -\frac{1}{x^3}(x-2) \quad x \neq 0$$

Let $f'(x) = 0$

$$-\frac{1}{x^3}(x-2) = 0$$

$x = 0$ and $x = 2$

are the critical numbers.



By IDT,

Increasing: $x \in (0,2)$

Decreasing: $x \in (-\infty,0) \cup (2,\infty)$

Solution

(c) Find the local maximum and minimum values.

	$x=-2$		$x=1$		$x=3$
y'	$-$	Und	$+$	0	$-$
x	↘	0	↗	2	↘

$f(x)$ changes from increasing to decreasing at $x = 2$, so $f(2)$ is a local maximum value. There is no local minimum value.

$$f(x) = 1 + \frac{1}{x} - \frac{1}{x^2}$$

$$f(2) = 1 + \frac{1}{2} - \frac{1}{2^2} = \frac{5}{4}$$

So by FDT, $\left(2, \frac{5}{4}\right)$ is a local maximum point.

Is $\left(2, \frac{5}{4}\right)$ is a absolute maximum point?

Recall that $\lim\limits_{x\to-\infty} f(x) = 1$ and $\lim\limits_{x\to\infty} f(x) = 1$

$$f(x) = 1 + \frac{1}{x} - \frac{1}{x^2} < 1 \text{ as } x \to -\infty$$

$$f(x) = 1 + \frac{1}{x} - \frac{1}{x^2} > 1 \text{ as } x \to +\infty$$

In other words, $f(x)$ is below $y = 1$ as $x \to -\infty$ and above $y = 1$ as $x \to +\infty$.

Thus, $\left(2, \frac{5}{4}\right)$ is an absolute maximum point.

Solution

(d) Find the intervals of concavity and the inflection points.

$$f(x) = 1 + \frac{1}{x} - \frac{1}{x^2}$$

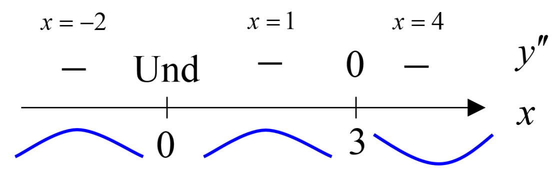
$$f'(x) = -\frac{1}{x^2} + \frac{2}{x^3}$$

$$f''(x) = \frac{2}{x^3} - \frac{6}{x^4}$$

$$= \frac{2}{x^4}(x-3)$$

$x = 0$ and $x = 3$

are the critical numbers for $f''(x)$.



By CT,

Concave Upward: $x \in (3, +\infty)$

Concave Downward: $x \in (-\infty, 0) \cup (0, 3)$

$$f(2) = 1 + \frac{1}{3} - \frac{1}{3^2} = \frac{11}{9}$$

By DIP, there is an inflection point at $\left(3, \frac{11}{9}\right)$ since the second derivative changes from negative to positive.

Solution

(e) Use the information from parts (a) – (d) to sketch the graph of f.

$$f(x) = 1 + \frac{1}{x} - \frac{1}{x^2}$$

$y = 1$ is a HA.

$x = 0$ is a VA.

$f(x)$ is below $y = 1$ as $x \to -\infty$ and above $y = 1$ as $x \to +\infty$

Increasing: $x \in (0, 2)$

Decreasing: $x \in (-\infty, 0) \cup (2, \infty)$

$\left(2, \frac{5}{4}\right)$ is an absolute maximum point

Concave Upward: $x \in (3, +\infty)$

Concave Downward: $x \in (-\infty, 0) \cup (0, 3)$

Inflection point at $\left(3, \frac{11}{9}\right)$