4.2 The Mean Value Theorem

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About 790 - About 850

Al'Khwarizmi was an Islamic mathematician who wrote on Hindu-Arabic numerals and was among the first to use zero as a place holder in positional base notation. The word algorithm derives from his name. His algebra treatise Hisab al-jabr gives us the word algebra and can be considered as the first book to be written on algebra.

Rolle's Theorem

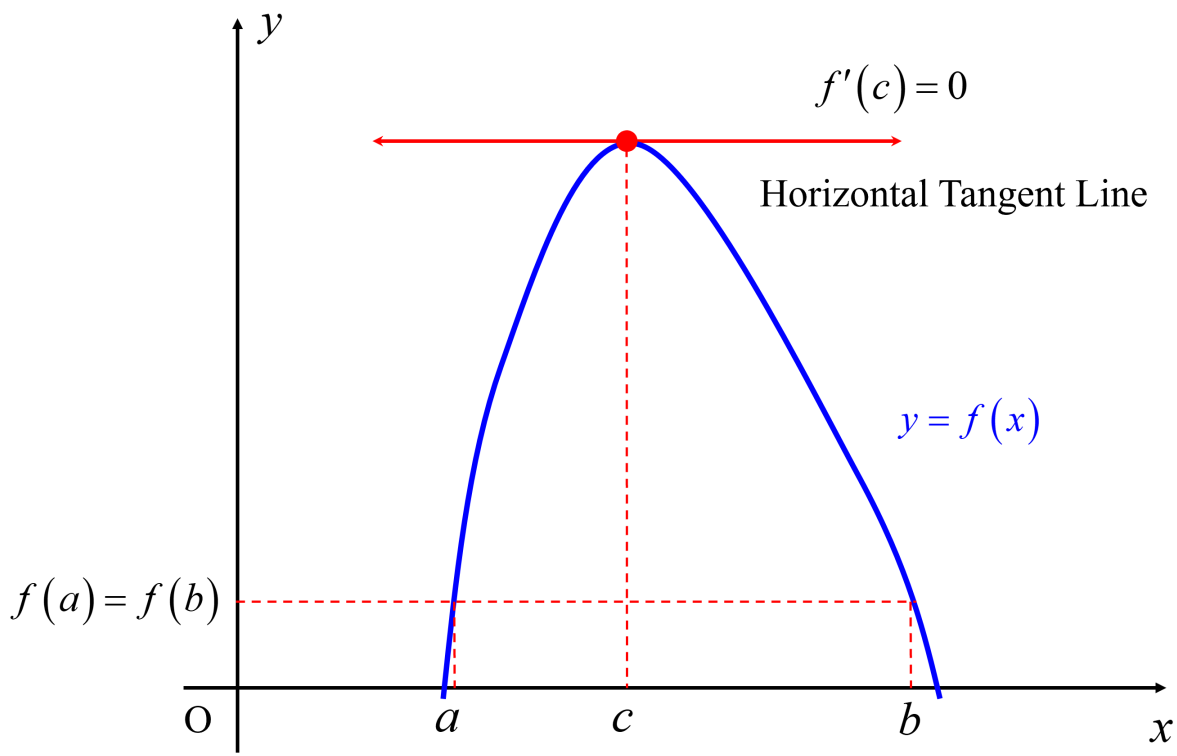
To arrive at the Mean Value Theorem, we first need the following result.

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

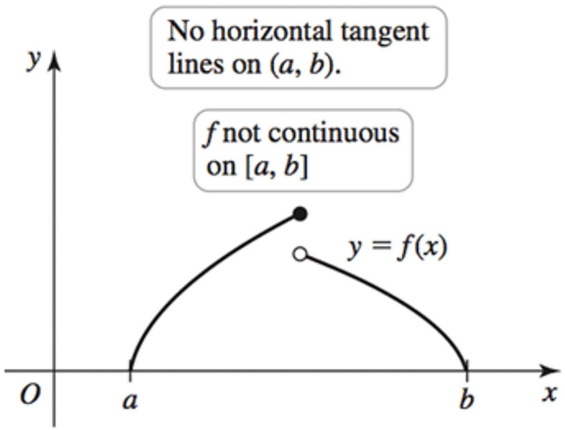
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

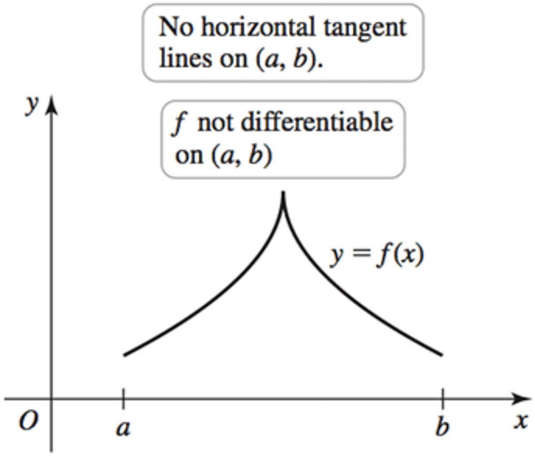
Rolle's Theorem



Why does Rolle's Theorem require continuity? A function that is not continuous on $[a, b]$ may have identical values at both endpoints and still not have a horizontal tangent line at any point on the interval. Similarly, a function that is continuous on $[a, b]$ but not differentiable at a point of (a, b) may also fail to have a horizontal tangent line.



(a)



(b)

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

Example

Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

$$f(x) = x + 1/x, \quad \left[\tfrac{1}{2}, 2\right]$$

Solution

1. f is a rational function that is continuous on its domain, $(-\infty, 0) \cup (0, \infty)$, so it is continuous on $\left[\frac{1}{2}, 2\right]$.

2. $f'(x) = 1 - 1/x^2 = \dfrac{x^2 - 1}{x^2}$

   f' has the same domain and is differentiable on $\left(\frac{1}{2}, 2\right)$.

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

Solution

3. $f(x) = x + 1/x, \quad \left[\frac{1}{2}, 2\right]$

$$f\left(\tfrac{1}{2}\right) = \tfrac{5}{2} = f(2)$$

By Rolle's Theorem we let $f'(c) = 0$

$$f'(x) = \frac{x^2 - 1}{x^2}$$

$$\frac{c^2 - 1}{c^2} = 0$$

$$c^2 - 1 = 0$$

$$c = \pm 1$$

Only 1 is in $\left(\frac{1}{2}, 2\right)$, so 1 satisfies the conclusion of Rolle's Theorem.

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

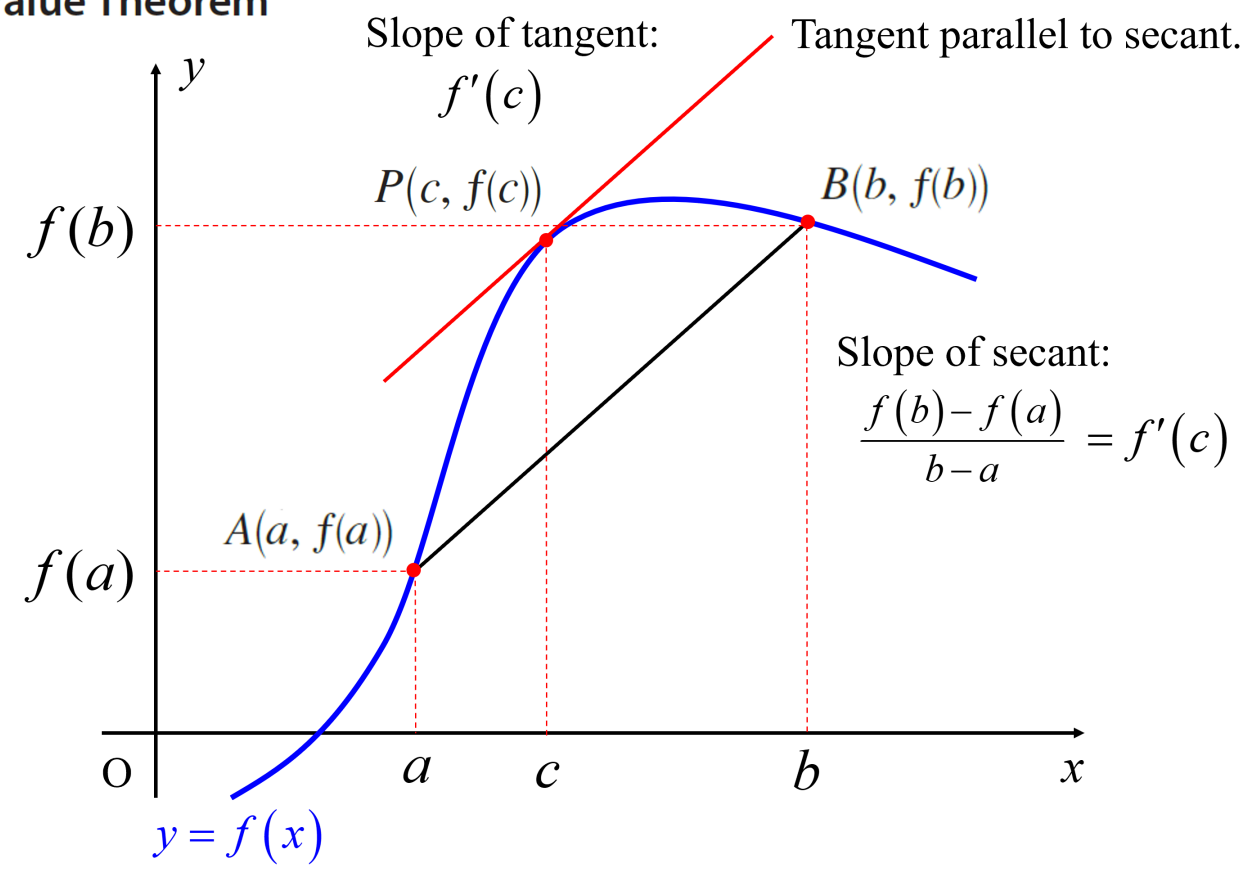
$$\boxed{1} \qquad f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

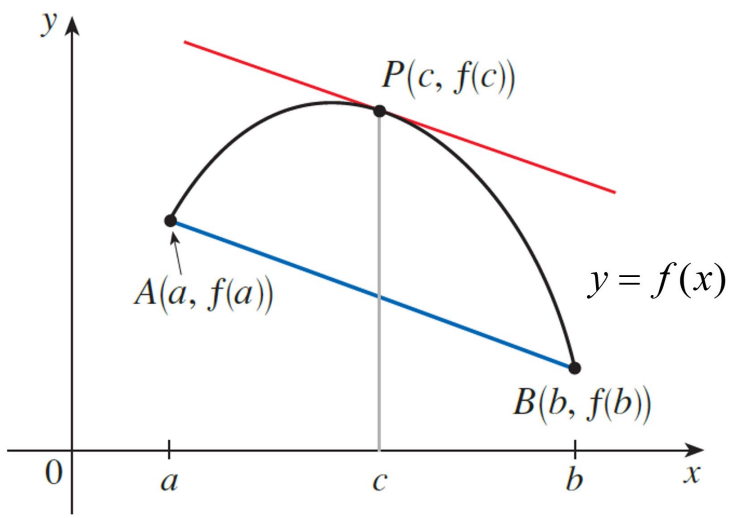
$$\boxed{2} \qquad f(b) - f(a) = f'(c)(b - a)$$

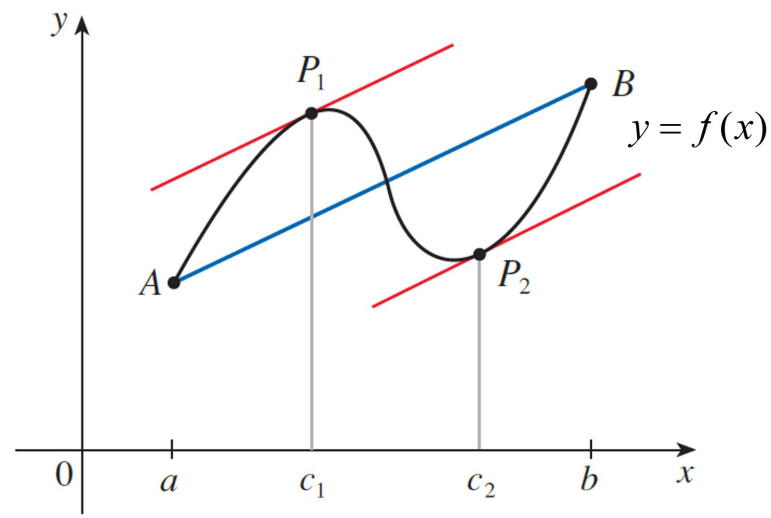
The Mean Value Theorem says that **at some point in the open interval, the slope of the secant line equals the slope of the tangent line**.

The Mean Value Theorem



The Mean Value Theorem





$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

$$f(x) = x^3 - 3x + 2, \quad [-2, 2]$$

Solution

$f(x) = x^3 - 3x + 2, \quad [-2, 2]$

1. f is continuous on the closed interval $[a, b]$.

2. f is differentiable on the open interval (a, b).

f is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$ since polynomials are continuous and differentiable on $\mathbb{R}$.

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Solution

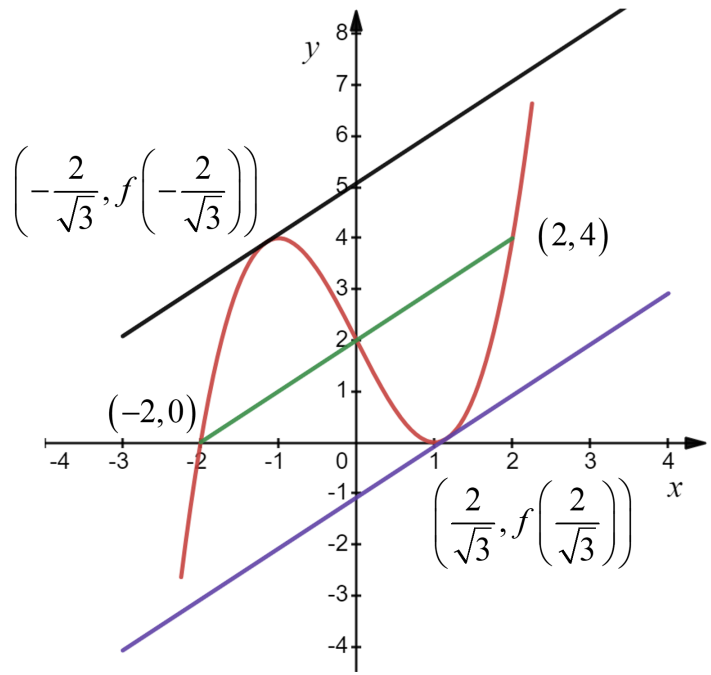
$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad f(x) = x^3 - 3x + 2, \quad [-2, 2] \qquad f'(x) = 3x^2 - 3$$

Then there is a number c in (a, b) such that

$$3c^2 - 3 = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{4 - 0}{4} = 1$$

$$3c^2 = 4$$

$$c^2 = \frac{4}{3}$$

$$c = \pm\frac{2}{\sqrt{3}} \in (-2, 2)$$

The slope of the secant line equals the slope of the tangent line.