

# 4

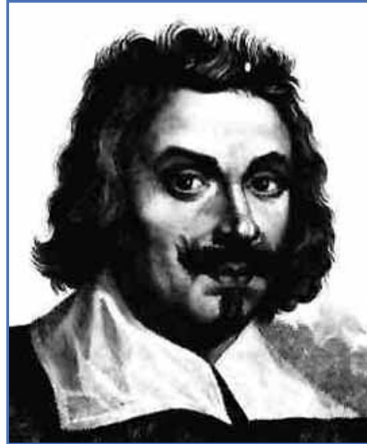
## Applications of Differentiation

**WE HAVE ALREADY INVESTIGATED SOME** of the applications of derivatives, but now that we know the differentiation rules, we are in a better position to pursue the applications of differentiation in greater depth. Here we learn what derivatives tell us about the shape of a graph of a function and, in particular, how they help us locate maximum and minimum values of functions. Many practical problems require us to minimize a cost or maximize an area or somehow find the best possible outcome of a situation. In particular, we will be able to investigate the optimal shape of a can and to explain the location of rainbows in the sky.



## 4.1

## Maximum and Minimum Values



**Evangelista Torricelli**  
1608 – 1647

**Evangelista Torricelli** was an Italian scientist who was the first man to create a sustained vacuum and to discover the principle of a barometer. He also achieved some important results in the development of the calculus.

## Dow Jones Industrial Average



How do we describe high and low points on a graph?

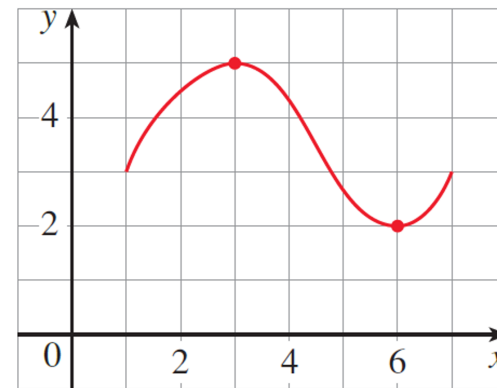
## ■ Absolute and Local Extreme Values

Even though the graphing calculator and the computer have eliminated the need to routinely use calculus to graph by hand and to find maximum and minimum values of functions, we still study the methods to increase our understanding of functions and the mathematics involved.

Absolute extreme values occur either at maximum or minimum points on a curve.

An absolute maximum or minimum is sometimes called a **global** maximum or minimum. The maximum and minimum values of  $f$  are called **extreme values** of  $f$ .

They are also sometimes called absolute extrema.  
(*Extrema* is the plural of the Latin *extremum*.)



Definition:

A function  $f$  has an absolute maximum at  $c$  if  $f(c) \geq f(x)$  for all  $x$  in the domain of  $f$ .

Terminology:

- $c$  is called an absolute maximizer
- $f(c)$  is called an absolute maximum value of  $f$ .
- $(c, f(c))$  is called an absolute maximum point.

Definition:

A function  $f$  has an absolute minimum at  $c$  if  $f(c) \leq f(x)$  for all  $x$  in the domain of  $f$ .

Terminology:

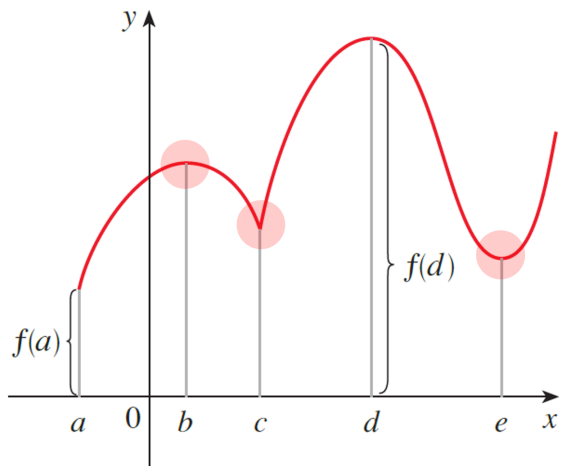
- $c$  is called an absolute minimizer
- $f(c)$  is called an absolute minimum value of  $f$ .
- $(c, f(c))$  is called an absolute minimum point.

Definition:

A function  $f$  has a local maximum at  $c$  if  $f(c) \geq f(x)$  when  $x$  is near  $c$ .

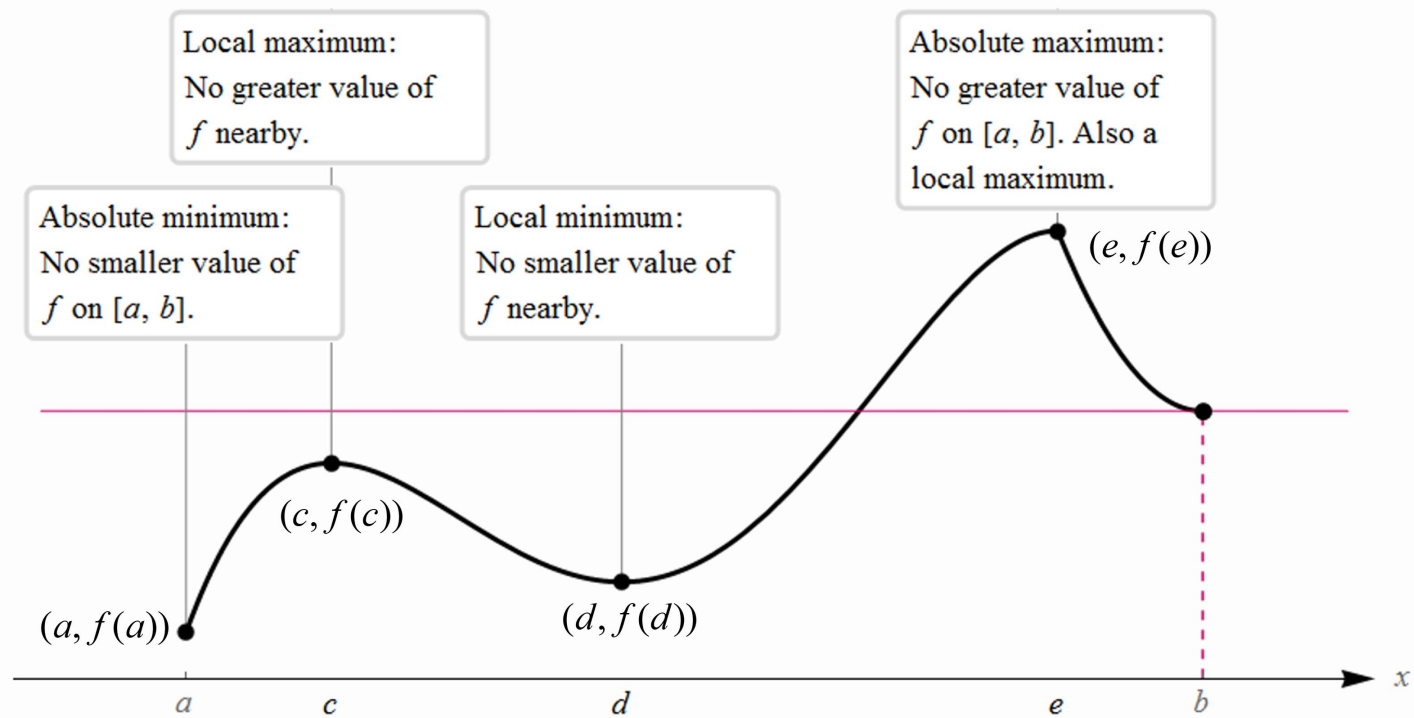
Definition:

A function  $f$  has a local minimum at  $c$  if  $f(c) \leq f(x)$  when  $x$  is near  $c$ .



If we say that something is **near**  $c$ , we mean that it is true on some open interval containing  $c$ . (Thus, a local maximum or minimum can't occur at an endpoint.)

## Local Maxima and Minima



## Extrema Terminology and Definitions

**Maxima**: absolute or local (relative)

The ordered pair  $(a, b)$  is an absolute maximum point if and only if  $y = b$  is an absolute maximum value and  $x = a$  is a maximizer.

The ordered pair  $(a, b)$  is a local maximum point if and only if  $y = b$  is a local maximum value and  $x = a$  is a maximizer.

**Minima**: absolute or local (relative)

The ordered pair  $(a, b)$  is an absolute minimum point if and only if  $y = b$  is an absolute minimum value,  $x = a$  is a minimizer.

The ordered pair  $(a, b)$  is a local minimum point if and only if  $y = b$  is a local minimum value,  $x = a$  is a minimizer.

**Example** The graph of the function

$$f(x) = 3x^4 - 16x^3 + 18x^2 \quad -1 \leq x \leq 4$$

is shown in the figure below.

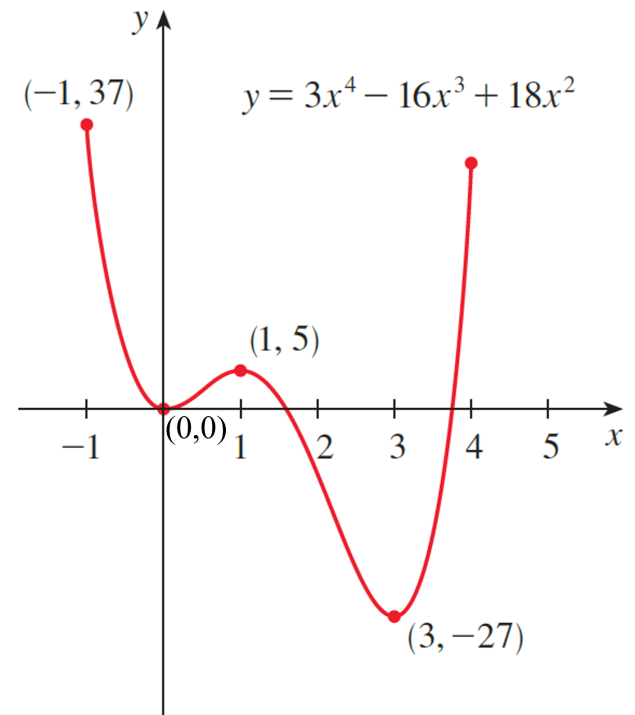
**Solution**

You can see that  $f(1) = 5$  is a local maximum value, whereas the absolute maximum value is  $f(-1) = 37$ .

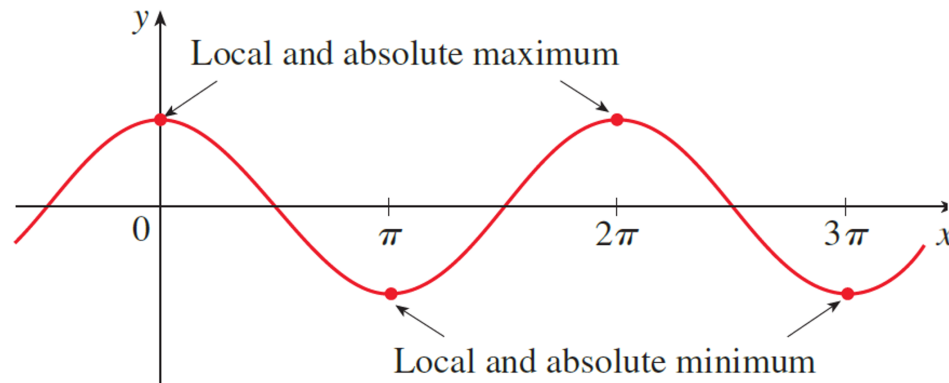
(This absolute maximum is not a local maximum because it occurs at an endpoint.)

Also,  $f(0) = 0$  is a local minimum value and  $f(3) = -27$  is both a local and an absolute minimum.

Note that  $f$  has neither a local nor an absolute maximum value at  $x = 4$ .



**Example** Consider  $f(x) = \cos x$



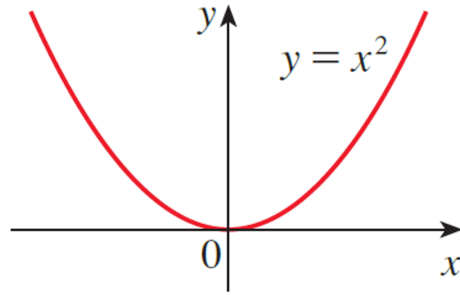
**Solution**

Cosine takes on its (local and absolute) maximum and minimum values of 1 and  $-1$ , respectively, infinitely many times. So, extreme values are not unique.

$\cos 2n\pi = 1$  for any integer  $n$  and  $-1 \leq \cos x \leq 1$  for all  $x$ .

Likewise,  $\cos(2n + 1)\pi = -1$  is its minimum value, where  $n$  is any integer.

**Example** Consider  $f(x) = x^2$



**Solution**

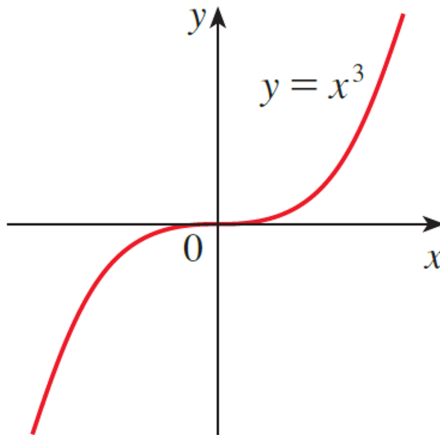
If  $f(x) = x^2$ , then  $f(x) \geq f(0)$  because  $x^2 \geq 0$  for all  $x$ .

Therefore,  $f(0) = 0$  is the absolute (and local) minimum value of  $f$ .

This corresponds to the fact that the origin is the lowest point on the parabola  $y = x^2$ .

However, there is no highest point on the parabola and so this function has no maximum value.

**Example** Consider  $f(x) = x^3$



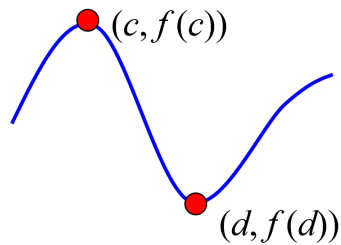
**Solution**

From the graph of the function  $f(x) = x^3$ , shown in the figure, we see that this function has neither an absolute maximum value nor an absolute minimum value. In fact, it has no local extreme values either.

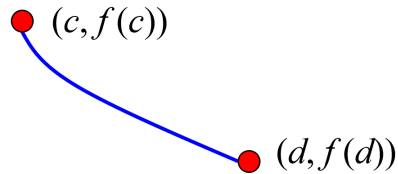
We have seen that some functions have extreme values, whereas others do not. The following theorem gives conditions under which a function is guaranteed to possess extreme values.

**EVT**

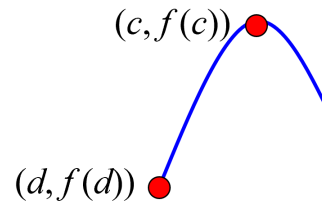
**3 The Extreme Value Theorem** If  $f$  is continuous on a closed interval  $[a, b]$ , then  $f$  attains an absolute maximum value  $f(c)$  and an absolute minimum value  $f(d)$  at some numbers  $c$  and  $d$  in  $[a, b]$ . Note: Check the endpoints for extreme values.



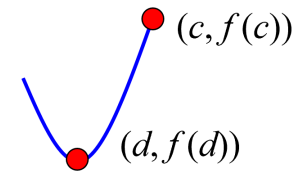
Absolute Maximum and Minimum points are interior points



Absolute Maximum and Minimum points are endpoints

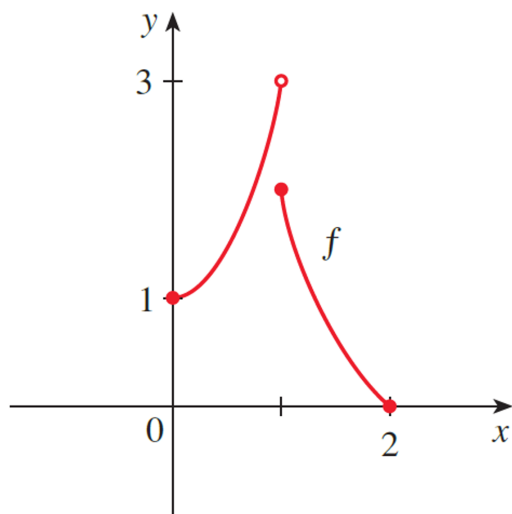


Absolute Maximum point is an interior point and Absolute Minimum point is an endpoint

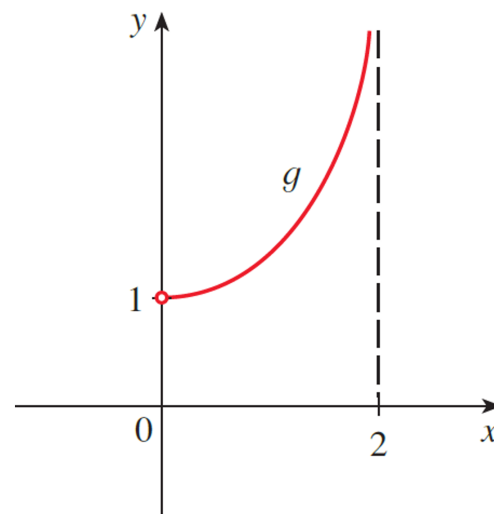


Absolute Minimum point is an interior point and Absolute Maximum point is an endpoint

The figures below show that a function need not possess extreme values if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem.



This function has minimum value  $f(2) = 0$ , but no maximum value.

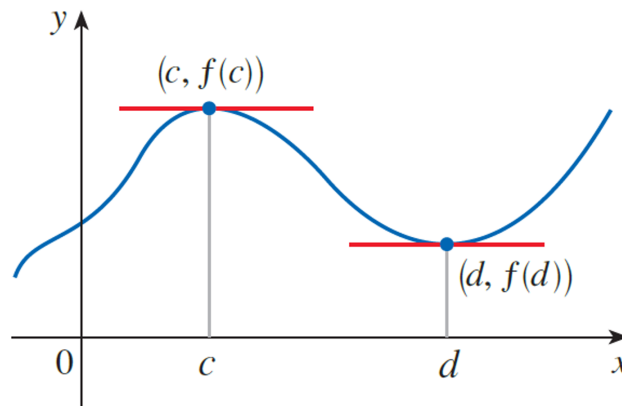


This continuous function  $g$  has no maximum or minimum.

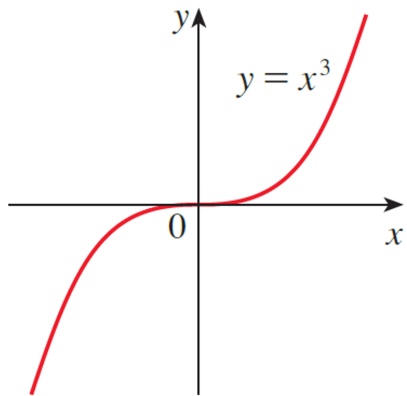
## ■ Critical Numbers and the Closed Interval Method

The Extreme Value Theorem says that a continuous function on a closed interval has a maximum value and a minimum value, but it does not tell us how to find these extreme values.

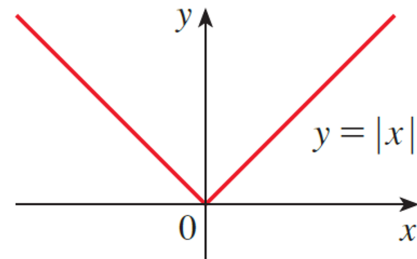
**4 Fermat's Theorem** If  $f$  has a local maximum or minimum at  $c$ , and if  $f'(c)$  exists, then  $f'(c) = 0$ .



**4 Fermat's Theorem** If  $f$  has a local maximum or minimum at  $c$ , and if  $f'(c)$  exists, then  $f'(c) = 0$ .



If  $f(x) = x^3$ , then  $f'(0) = 0$ , but  $f$  has no maximum or minimum.

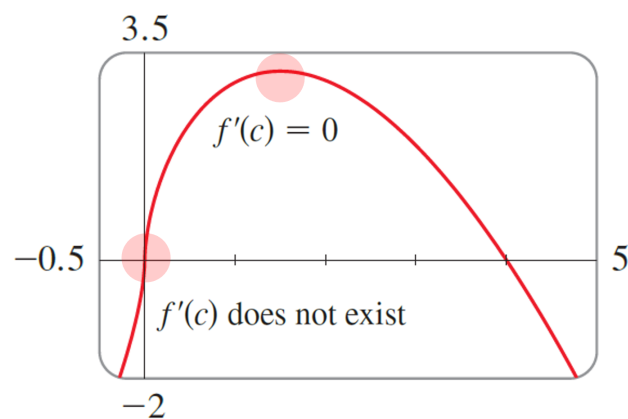


If  $f(x) = |x|$ , then  $f(0) = 0$  is a minimum value, but  $f'(0)$  does not exist.

**WARNING** These examples show that we must be careful when using Fermat's Theorem.  $y = x^3$  demonstrates that even when  $f'(c) = 0$  there need not be a maximum or minimum at  $c$ . (In other words, the converse of Fermat's Theorem is false in general.) Furthermore, there may be an extreme value even when  $f'(c)$  does not exist as in  $y = |x|$ .

Fermat's Theorem does suggest that we should at least *start* looking for extreme values of  $f$  at the numbers  $c$  where  $f'(c) = 0$  or where  $f'(c)$  does not exist. Such numbers are given a special name.

**6** **Definition** A **critical number** of a function  $f$  is a number  $c$  in the domain of  $f$  such that either  $f'(c) = 0$  or  $f'(c)$  does not exist.



**6 Definition** A **critical number** of a function  $f$  is a number  $c$  in the domain of  $f$  such that either  $f'(c) = 0$  or  $f'(c)$  does not exist.

**Example** Find the critical numbers of the function.

$$f(x) = x^{1/3}(4 - x)^{2/3}$$

**Solution**

$$f(x) = x^{1/3}(4 - x)^{2/3}$$

$$f'(x) = x^{1/3} \cdot \frac{2}{3}(4 - x)^{-1/3} \cdot (-1) + (4 - x)^{2/3} \cdot \frac{1}{3}x^{-2/3}$$

$$= \frac{1}{3}x^{-2/3}(4 - x)^{-1/3}[-2x + (4 - x)]$$

$$= \frac{4 - 3x}{3x^{2/3}(4 - x)^{1/3}}$$

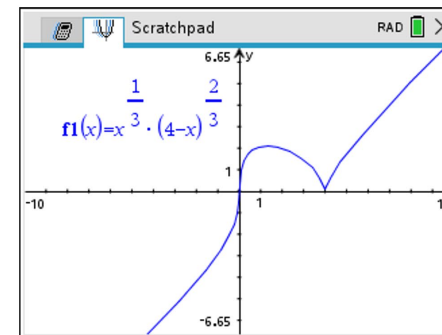
Let  $f'(x) = 0$

$$4 - 3x = 0 \quad x = \frac{4}{3}$$

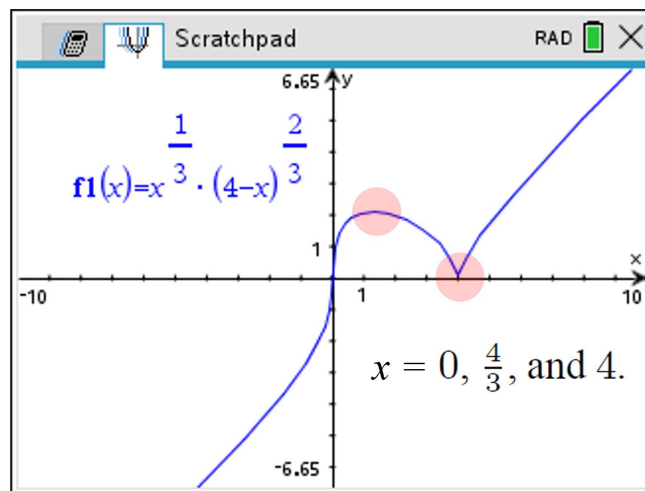
$f'(0)$  and  $f'(4)$  are undefined.

Thus, the three critical numbers are

$x = 0, \frac{4}{3},$  and  $4.$



7 If  $f$  has a local maximum or minimum at  $c$ , then  $c$  is a critical number of  $f$ .



We see that at  $x = 4/3$  and  $x = 4$  we have extreme points.

To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is local [in which case it occurs at a critical number] or it occurs at an endpoint of the interval. Thus, the following three-step procedure always works.

- The Closed Interval Method** To find the *absolute* maximum and minimum values of a continuous function  $f$  on a closed interval  $[a, b]$ :
1. Find the values of  $f$  at the critical numbers of  $f$  in  $(a, b)$ .
  2. Find the values of  $f$  at the endpoints of the interval.
  3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

**Example** Find the absolute maximum and absolute minimum values of  $f$  on the given interval.

(a)  $f(x) = x^{2/3}$ ,  $x \in [-2, 3]$

(b)  $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ ,  $[-2, 3]$

(c)  $f(t) = 2 \cos t + \sin 2t$ ,  $[0, \pi/2]$

**Solution**

(a)  $f(x) = x^{2/3}$ ,  $x \in [-2, 3]$  **The Closed Interval Method**

1. Find the values of  $f$  at the critical numbers of  $f$  in  $(a, b)$ .

$f'(x) = \frac{2}{3} x^{-1/3}$  There are no values of  $x$  that will make the first derivative equal to zero.

$f'(x) = \frac{2}{3\sqrt[3]{x}}$  The first derivative is undefined at  $x = 0$ , 0 is in the closed interval, so  $(0, 0)$  is a critical point.

2. Find the values of  $f$  at the endpoints of the interval.

3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Endpoint

At  $x = -2$   $f(-2) = (-2)^{2/3} = \sqrt[3]{4}$

Interior point (Critical Point)

At  $x = 0$   $f(0) = 0$

Endpoint

At  $x = 3$   $f(3) = (3)^{2/3} = \sqrt[3]{9}$

$x$	$f(x)$
-2	$\sqrt[3]{4}$
0	0 Absolute Minimum Value
3	$\sqrt[3]{9}$ Absolute Maximum Value

**By The Extreme Value Theorem**

Absolute Minimum Point:  $(0, 0)$

Absolute Maximum Point:  $(3, \sqrt[3]{9})$

(b)  $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ ,  $[-2, 3]$

**Solution**

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 1$$

$$f'(x) = 12x^3 - 12x^2 - 24x$$

$$= 12x(x^2 - x - 2)$$

$$= 12x(x + 1)(x - 2)$$

$$12x(x + 1)(x - 2) = 0$$

Critical numbers are

$$x = -1, 0, 2 \in [-2, 3]$$

Endpoint

$$f(-2) = 33 \quad \text{Absolute Maximum Value}$$

Interior points (Critical Points)

$$f(-1) = -4$$

$$f(0) = 1$$

$$f(2) = -31 \quad \text{Absolute Minimum Value}$$

Endpoint

$$f(3) = 28$$

**By The Extreme Value Theorem**

Absolute  
Minimum Point:  $(2, -31)$

Absolute  
Maximum Point:  $(-2, 33)$

(c)  $f(t) = 2 \cos t + \sin 2t$ ,  $[0, \pi/2]$

**Solution**

$$f(t) = 2 \cos t + \sin 2t$$

$$f'(t) = -2 \sin t + \cos 2t \cdot 2$$

$$= -2 \sin t + 2(1 - 2 \sin^2 t)$$

$$= -2 \sin t + 2(1 - 2 \sin^2 t)$$

$$= -2(2 \sin t - 1)(\sin t + 1)$$

$$f'(t) = 0$$

$$\begin{array}{ll} \sin t = \frac{1}{2} & \sin t = -1 \\ t = \frac{\pi}{6} & t = \frac{3\pi}{2} \notin [0, \pi/2] \end{array}$$

Critical number is

$$t = \frac{\pi}{6}$$

Endpoint

$$f(0) = 2$$

Interior point (Critical Point)

$$f\left(\frac{\pi}{6}\right) = \sqrt{3} + \frac{1}{2} \sqrt{3} = \frac{3}{2} \sqrt{3} \approx 2.598 \text{ Absolute Maximum Value}$$

Endpoint

$$f\left(\frac{\pi}{2}\right) = 0 \text{ Absolute Minimum Value}$$

**By The Extreme Value Theorem**

Absolute  
Minimum Point:  $\left(\frac{\pi}{2}, 0\right)$

Absolute  
Maximum Point:  $\left(\frac{\pi}{6}, \frac{3}{2}\sqrt{3}\right)$

**Example**

The water level, measured in feet above mean sea level, of Lake Lanier in Georgia, USA, during 2012 can be modeled by the function



$$L(t) = 0.01441t^3 - 0.4177t^2 + 2.703t + 1060.1 \quad [1,12]$$

where  $t$  is measured in months since January 1, 2012. Estimate when the water level was highest during 2012.

**Solution**

$$L(t) = 0.01441t^3 - 0.4177t^2 + 2.703t + 1060.1$$

$$L'(t) = 0.04323t^2 - 0.8354t + 2.703$$

Let  $L'(t) = 0$

$$t \stackrel{CAS}{\approx} 4.109 \quad \text{or} \quad t \stackrel{CAS}{\approx} 15.215 \notin [1,12]$$

Endpoint

$$L(0) \approx 1060.1$$

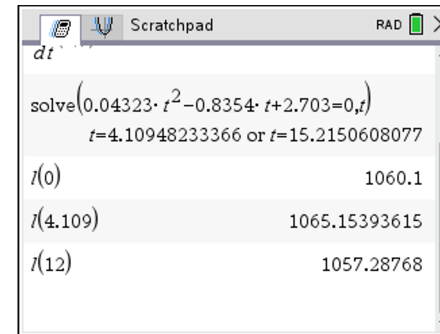
Interior point (Critical Point)

$$L(4.109) \approx 1065.154$$

Endpoint

$$L(12) \approx 1057.288$$

Absolute Maximum Value



Thus, by the **Extreme Value Theorem**, the water level was highest during 2012 at about 4.109 months after January 1.