3.9 Related Rates

Bonaventura Francesco Cavalieri
1598 – 1647

Bonaventura Cavalieri was an Italian mathematician who developed a method of indivisibles which became a factor in the development of the integral calculus.

In a related rates problem, the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time. A related rate problem is analyzing a problem over time.

Example Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Solution

I **Introduction**

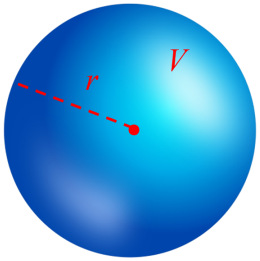
Assign symbols to all quantities that are functions of time.

Let

r = radius of the sphere (cm)

V = volume of the sphere (cm^3)

r and V are functions of time.



Derivatives

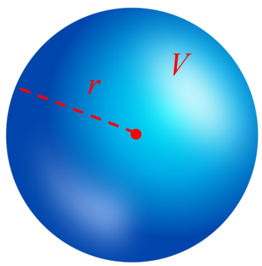
$$\frac{dr}{dt} \text{ (cm/s) and } \frac{dV}{dt} \text{ (cm}^3\text{/s)}$$

Solution

II **Known Derivative**

Express the given rate(s) in terms of derivatives.

The volume of air is increase at a rate of 100 cm^3/s



$$\frac{dV}{dt} = 100 \text{ cm}^3/\text{s}$$

III **Unknown Derivative**

Express the unknown rate(s) in terms of derivatives.

What rate is the radius increase when the diameter is 50 cm?

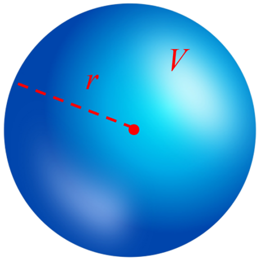
$$\left.\frac{dr}{dt}\right|_{r=25 \text{ cm}}$$

Solution

IV **Equation**

Write an equation that relates the various quantities of the problem that are functions of time. If necessary, use geometry of the situation to eliminate one of the variables by substitution.

$$V = \tfrac{4}{3}\pi r^3$$



V **Differential Equation**

Use the Chain Rule to differentiate both sides of the equation with respect to *t,* also known as implicit differentiation, to find a differential equation.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

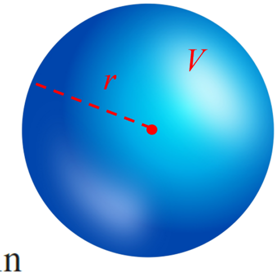
This differential equation relates r and V over time, thus a related rate!

Solution

VI **Snapshot**

Substitute the given information into the resulting equation and solve for the unknown derivative.

Now we solve for the unknown quantity:



$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

If we put $r = 25$ and $dV/dt = 100$ in this equation, we obtain

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2}100 = \frac{1}{25\pi} \approx 0.0127 \text{ cm/s}$$

VII **Conclusion**

State your conclusion to the problem.

The radius of the balloon is increasing at the rate of $1/(25\pi) \approx 0.0127$ cm/s when the diameter is 50 cm.

Steps for Related Rates Problems:

I **Introduction**

-Read the problem carefully.

-Draw a picture (sketch), if possible.

-Introduce notation. Assign symbols to all quantities that are functions of time.

II **Known Derivative**

Express the given rates in terms of derivatives.

III **Unknown Derivative**

Express the unknown rates in terms of derivatives.

IV **Equation**

Write an equation that relates the various quantities of the problem that are functions of time. If necessary, use geometry of the situation to eliminate one of the variables by substitution.

Steps for Related Rates Problems:

V **Differential Equation**

Use the Chain Rule to differentiate both sides of the equation with respect to *t,* also known as implicit differentiation, to find a differential equation.

VI **Snap Shot**

Substitute the given information into the resulting equation and solve for the unknown derivative.

VII **Conclusion**

State your conclusion to the problem.

Example Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

Solution

I **Introduction**

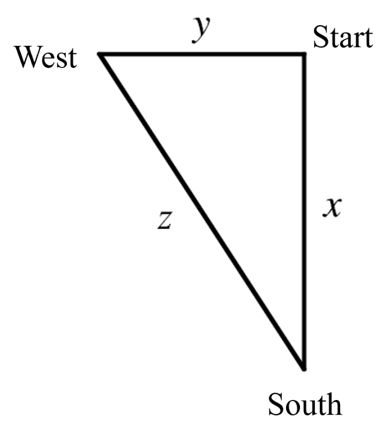
Let

x = car traveling south (mi)

y = car traveling west (mi)

z = distance between cars (mi)

x, y and z are functions of time.



Derivatives

$\frac{dx}{dt}$ (mi/h) $\frac{dy}{dt}$ (mi/h) $\frac{dz}{dt}$ (mi/h)

Example Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

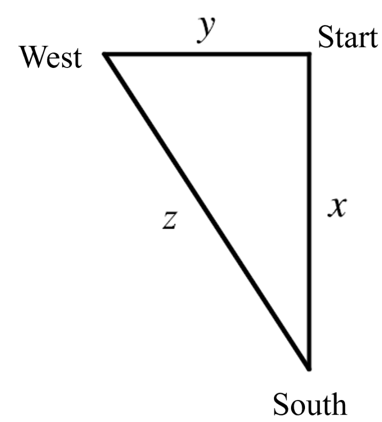
Solution

II **Known Derivative**

One travels south at 60 mi/h and the other travels west at 25 mi/h.

$$\frac{dx}{dt} = 60 \text{ mi/h}$$

$$\frac{dy}{dt} = 25 \text{ mi/h}$$



III **Unknown Derivative**

At what rate is the distance between the cars increasing two hours later?

$$\left.\frac{dz}{dt}\right|_{t=2 \text{ hours}}$$

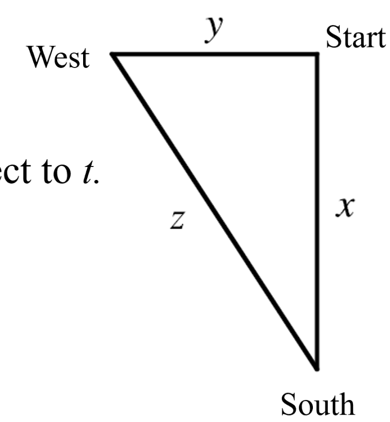
Example

Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

Solution

IV **Equation**

$$z^2 = x^2 + y^2$$



V **Differential Equation**

Use the Chain Rule to differentiate both sides of the equation with respect to *t*.

$$2z\,\frac{dz}{dt} = 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt}$$

$$z\,\frac{dz}{dt} = x\,\frac{dx}{dt} + y\,\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{1}{z}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right)$$

Differential Equation

Solution

VI **Snapshot**

Substitute the given information into the resulting equation.

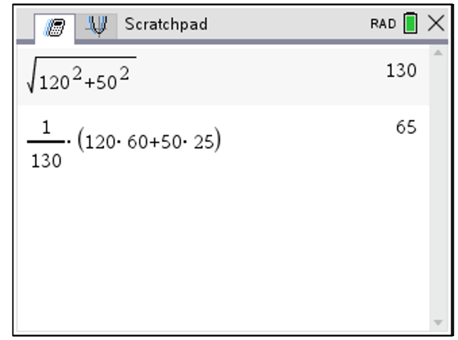
$$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

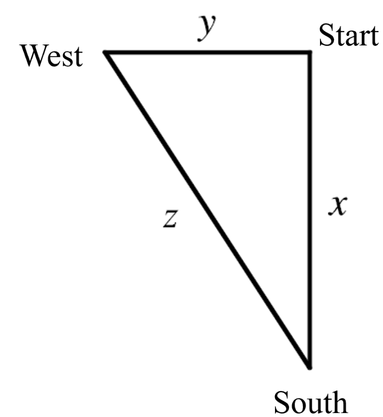
After 2 hours, $x = 2(60) = 120$ and $y = 2(25) = 50$

$$z^2 = x^2 + y^2$$

$$z = \sqrt{120^2 + 50^2} = 130 \qquad \frac{dx}{dt} = 60 \text{ mi/h} \qquad \frac{dy}{dt} = 25 \text{ mi/h}$$

$$\frac{dz}{dt} = \frac{1}{130}\left(120(60) + 50(25)\right) = 65 \text{ mi/h}$$





VII **Conclusion**

The distance between the cars after two hours is increasing at a rate of 65 mi/h.

Example The weight w of an astronaut (in pounds) is related to her height h above the surface of the earth (in miles) by

$$w = w_0\left(\frac{3960}{3960 + h}\right)^2$$

where w_0 is the weight of the astronaut on the surface of the earth. If the astronaut weighs 130 pounds on earth and is in a rocket, being propelled upward at a speed of 12 mi/s, find the rate at which her weight is changing (in lb/s) when she is 40 miles above the earth's surface.

Solution

I **Introduction**

Let

w = weight of astronaut (lb)

h = height above the surface of the earth (mi)

w, and h are functions of time.

Derivatives

$\frac{dw}{dt}$ (lb/s) $\frac{dh}{dt}$ (mi/s)

Solution

II **Known Derivative**

Being propelled upward at a speed of 12 mi/s.

$$\frac{dh}{dt} = 12 \text{ mi/s}$$

III **Unknown Derivative**

Find the rate at which her weight is changing she is 40 miles above the earth's surface.

$$\left.\frac{dw}{dt}\right|_{h=40 \text{ miles}}$$

IV **Equation**

$$w = w_0 \left(\frac{3960}{3960 + h} \right)^2 = w_0 \cdot 3960^2 (3960 + h)^{-2}$$

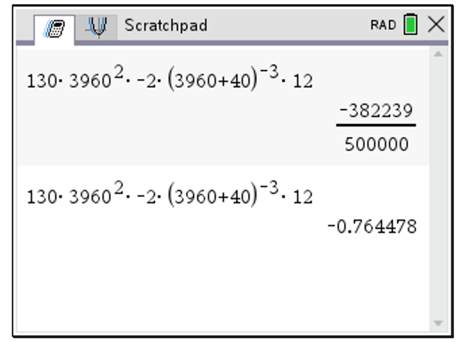
Solution

V **Differential Equation**

Use the Chain Rule to differentiate both sides of the equation with respect to *t*.

$$w = w_0\left(\frac{3960}{3960+h}\right)^2 = w_0 \cdot 3960^2(3960+h)^{-2}$$

$$\frac{dw}{dt} = w_0 \cdot 3960^2(-2)(3960+h)^{-3} \cdot \frac{dh}{dt}$$ Differential Equation



VI **Snapshot**

Substitute the given information into the differential equation above.

$w_0 = 130$ lb, $h = 40$ mi, and $\frac{dh}{dt} = 12$ mi/s

$$\frac{dw}{dt} = 130 \cdot 3960^2(-2)(3960+40)^{-3}(12) = -0.764478 \approx -0.7645 \text{ lb/s}$$

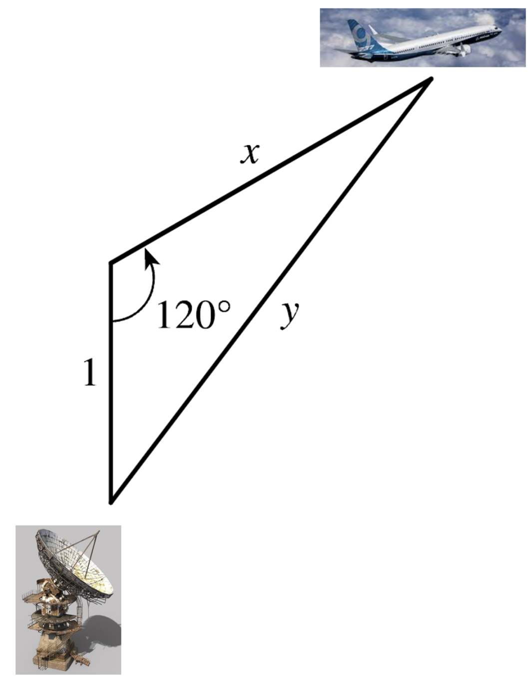
VII **Conclusion**

Her weight is decreasing at a rate of –0.7645 lb/s when she is 40 miles above the earth's surface.

Example

A plane flying with a constant speed of 300 km/h passes over a ground radar station at an altitude of 1 km and climbs at an angle of 30°. At what rate is the distance from the plane to the radar station increasing a minute later?

Solution



I **Introduction**

Let

x = distance plane traveled (km)

y = distance plane is from radar station (km)

x, and y are functions of time.

Derivatives

$\frac{dx}{dt}$ (km/h) $\quad$ $\frac{dh}{dt}$ (km/h)

Solution

II **Known Derivative**

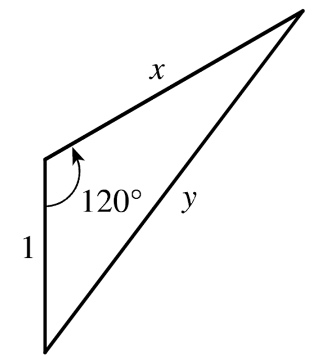
Plane flying with a constant speed of 300 km/h

$$\frac{dx}{dt} = 300 \text{ km/h}$$

III **Unknown Derivative**

Find the rate that the distance from the plane to the radar station increasing a minute later.

$$\left.\frac{dy}{dt}\right|_{t=1 \text{ minute}}$$



IV **Equation**

By the Law of Cosines, $y^2 = x^2 + z^2 - 2xz\cos(\theta)$

$$y^2 = x^2 + 1^2 - 2(1)(x)\cos 120^\circ$$

$$= x^2 + 1 - 2x\left(-\tfrac{1}{2}\right) = x^2 + x + 1$$

Solution

V **Differential Equation**

Use the Chain Rule to differentiate both sides of the equation with respect to t.

$$y^2 = x^2 + x + 1$$

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt} + \frac{dx}{dt}$$

$$2y\frac{dy}{dt} = (2x+1)\frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{2x+1}{2y}\frac{dx}{dt}$$ Differential Equation

VI **Snapshot**

Substitute the given information into the resulting equation.

After 1 minute, $x = \frac{300}{60} = 5$ km $\quad \frac{dx}{dt} = 300$ km/h $\quad y = \sqrt{x^2 + x + 1} = \sqrt{5^2 + 5 + 1} = \sqrt{31}$ km

$$\frac{dy}{dt} = \frac{2(5)+1}{2\sqrt{31}}(300) = \frac{1650}{\sqrt{31}} \approx 296 \text{ km/h}.$$

VII **Conclusion**

The distance from the plane to the radar station is increasing at a rate of 296 km/h after 1 minute.

Example

Water is draining from the bottom of a cone-shaped funnel at the rate 2 in^3/min. The height of the funnel is 16 inches and the radius at the top of the funnel is 4 in. At what rate is the height of the water in the funnel changing when the height of the water is 8 in?

Solution

I **Introduction**

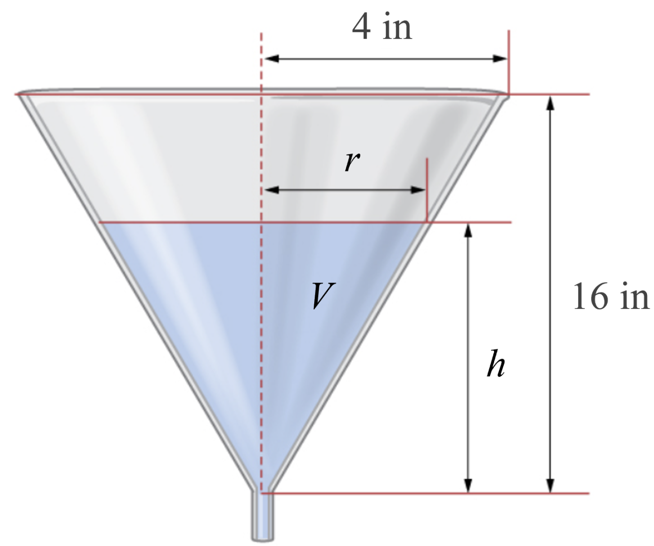
Let

V = volume of water (in^3)

r = radius of water level (in)

h = height of water level (in)

V, r, and h are functions of time.



Derivatives

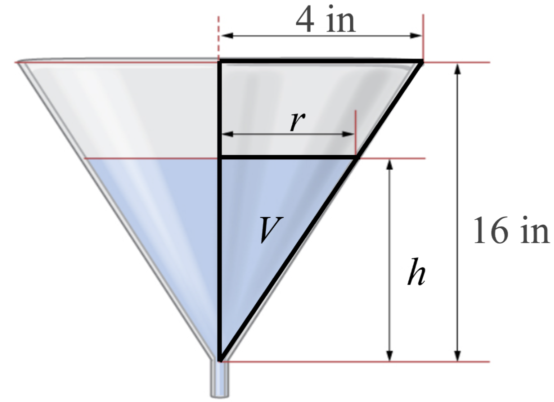
$\frac{dV}{dt}$ (in^3/min) $\quad$ $\frac{dr}{dt}$ (in/min) $\quad$ $\frac{dh}{dt}$ (in/min)

Solution

II **Known Derivative**

Water is draining out at the rate of 2 in³/min

$$\frac{dV}{dt} = -2 \text{ in}^3/\text{min}$$



III **Unknown Derivative**

Find the rate that the height of the water is changing when the water level is 8 in.

$$\left.\frac{dh}{dt}\right|_{h=8\text{ in}}$$

IV **Equation**

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h = \frac{1}{48}\pi h^3$$

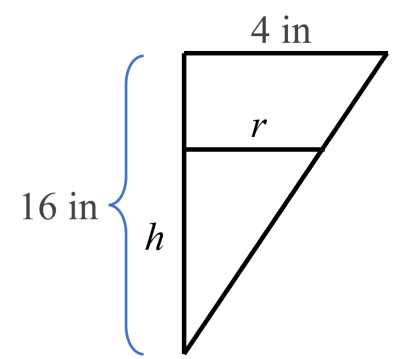
We see that we have similar triangles.

Therefore, the ratio of the sides in the two triangles is the same.

$$\frac{r}{4} = \frac{h}{16}$$

$$r = \frac{4h}{16}$$

$$r = \frac{h}{4}$$



Solution

V **Differential Equation**

Use the Chain Rule to differentiate both sides of the equation with respect to *t*.

$$V = \frac{1}{48}\pi h^3$$

$$\frac{dV}{dt} = \frac{1}{48}\cdot 3\cdot \pi h^2 \cdot \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{\pi h^2}{16}\cdot\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{16}{\pi h^2}\cdot\frac{dV}{dt}$$

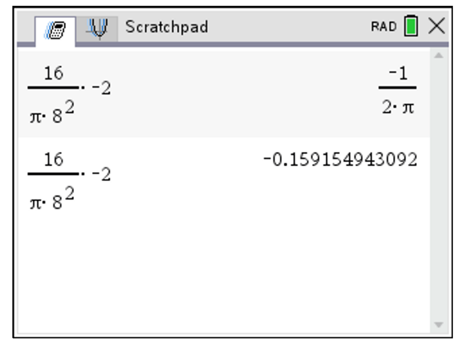
Differential Equation

VI **Snapshot**

Substitute the given information into the resulting equation.

$$h = 8 \text{ and } \frac{dV}{dt} = -2$$

$$\frac{dh}{dt} = \frac{16}{\pi h^2}\cdot\frac{dV}{dt} = \frac{16}{\pi (8)^2}\cdot(-2) = -\frac{1}{2\pi} \approx -0.159 \text{ in/min}$$



VII **Conclusion**

The height of the water is decreasing at a rate of –0.159 in/min when the water level is 8 in.