3.7 Rates of Change in the Natural and Social Sciences

Marin Mersenne
1588 – 1648

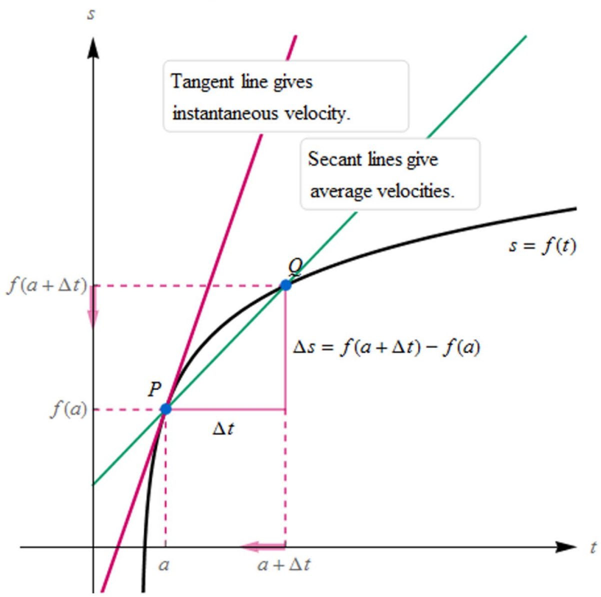
Marin Mersenne was a French monk who is best known for his role as a clearing house for correspondence between eminent philosophers and scientists and for his work in number theory.

Physics

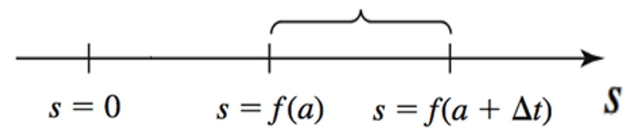
One-Dimensional Motion

Describing the motion of objects such as projectiles and planets was one of the challenges that led to the development of calculus in the 17th century. We begin by considering the motion of an object confined to one dimension; that is, the object moves along a line. This motion could be horizontal (for example, a car moving along a straight highway) or it could be vertical (such as a projectile launched vertically into the air).

If $s = f(t)$ is the position function of a particle that is moving in a straight line, then $\Delta s/\Delta t$ represents the average velocity over a time period Δt, and $v = ds/dt$ represents the instantaneous **velocity** (the rate of change of displacement with respect to time). The instantaneous rate of change of velocity with respect to time is **acceleration**: $a(t) = v'(t) = s''(t)$. This was discussed in Sections 2.7 and 2.8, but now that we know the differentiation formulas, we are able to more easily solve problems involving the motion of objects.



Displacement $\Delta s = f(a + \Delta t) - f(a)$



$$v_{avg} = \frac{\Delta s}{\Delta t} = \frac{f(a+\Delta t) - f(a)}{\Delta t}$$

$$v(a) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(a+\Delta t) - f(a)}{\Delta t} = f'(a) = s'$$

Velocity is the first derivative of position.

Free Fall Equation

$$s(t) = -\frac{1}{2} g \cdot t^2$$

$$s(t) = -\frac{1}{2} \cdot 32t^2$$

$$s(t) = -16t^2$$

$$v(t) = \frac{ds}{dt} = s'(t) = -32t$$

Speed is the absolute value of velocity.

Velocity is speed with direction.

Gravitational Constants on Earth

$$g = 32 \ \frac{\text{ft}}{\text{sec}^2}$$

$$g = 9.8 \ \frac{\text{m}}{\text{sec}^2}$$

$$g = 980 \ \frac{\text{cm}}{\text{sec}^2}$$

Acceleration is the derivative of velocity.

$$a(t) = v'(t) = \frac{dv}{dt} = s''(t) = \frac{d^2 s}{ds^2}$$

$$v(t) = -32t$$

$$a(t) = -32$$

If displacement, $s(t)$, is in: feet or meters

Velocity would be in: $\frac{\text{feet}}{\text{sec}}$ or $\frac{\text{meters}}{\text{sec}}$

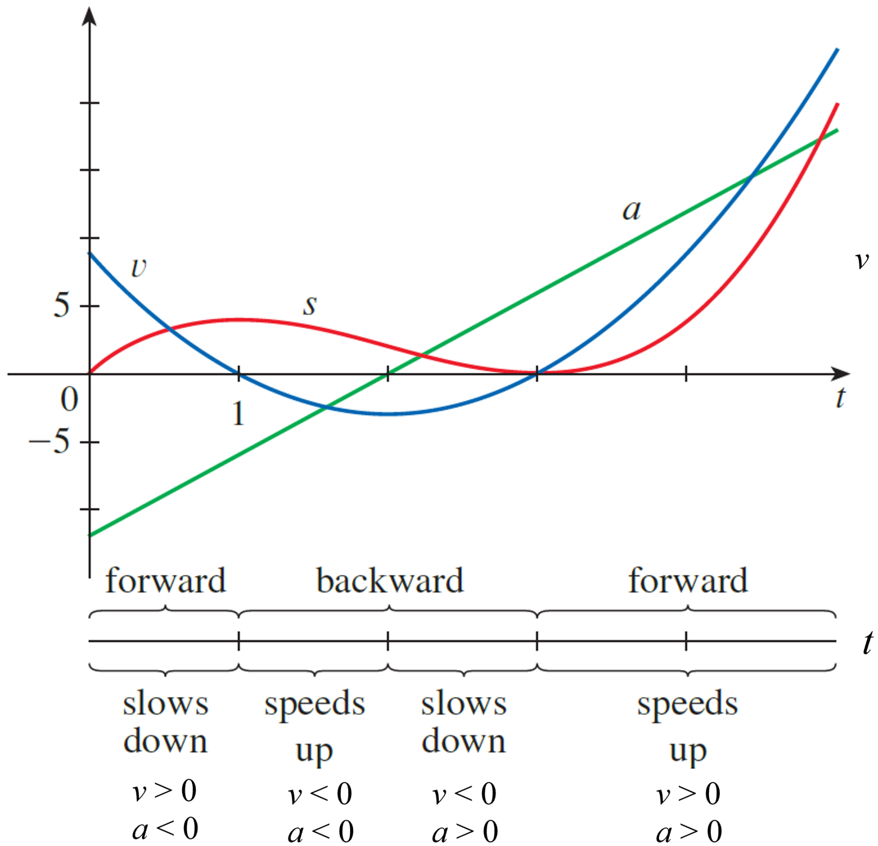
Acceleration would be in: $\dfrac{\frac{\text{ft}}{\text{sec}}}{\text{sec}} = \frac{\text{ft}}{\text{sec}^2}$ or $\dfrac{\frac{\text{m}}{\text{sec}}}{\text{sec}} = \frac{\text{m}}{\text{sec}^2}$

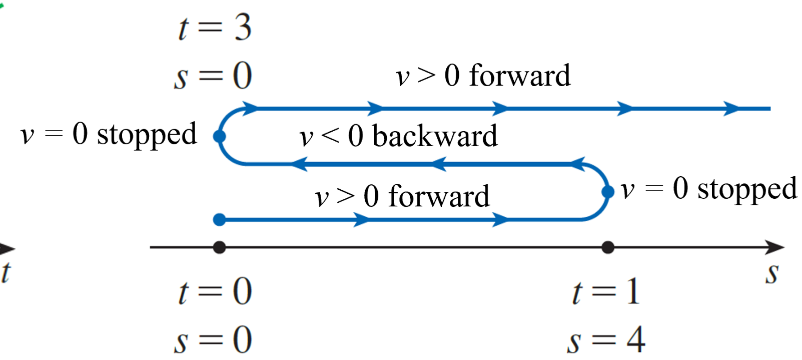
It is important to understand the relationship between velocity, acceleration and the behavior of the particle.

Velocity	Acceleration	Behavior of Particle
Positive	Positive	Speeding Up
Positive	Negative	Slowing Down
Negative	Positive	Slowing Down
Negative	Negative	Speeding Up
Zero	Positive or Negative	Stopped
Positive or Negative	Zero	Constant Speed

The particle speeds up when the velocity is positive and increasing (v and a are both positive) and also when the velocity is negative and decreasing (v and a are both negative). In other words, the particle speeds up when the velocity and acceleration have the same sign. The particle is slowing down when velocity and acceleration have opposite sign.

The figure below summarizes the motion of a particle moving horizontally.





How about acceleration?

Example A particle moves according to a law of motion $s(t) = t^2e^{-t}$, $t \geq 0$, where t is measured in seconds and s in feet.

(a) Find the velocity at time t.

(b) What is the velocity after 1 second?

(c) When is the particle at rest?

(d) When is the particle moving in the positive direction?

(e) Find the total distance traveled during the first 6 seconds.

(f) Draw a diagram like Figure 2 to illustrate the motion of the particle.

(g) Find the acceleration at time t and after 1 second.

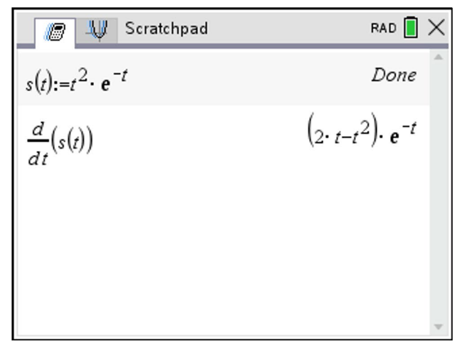
(h) Graph the position, velocity, and acceleration functions for $0 \leqslant t \leqslant 6$.

(i) When is the particle speeding up? When is it slowing down?

Solution

(a) Find the velocity at time t.

$$s(t) = t^2e^{-t} \qquad v(t) \overset{CAS}{=} (2t - t^2)\cdot e^{-t}$$

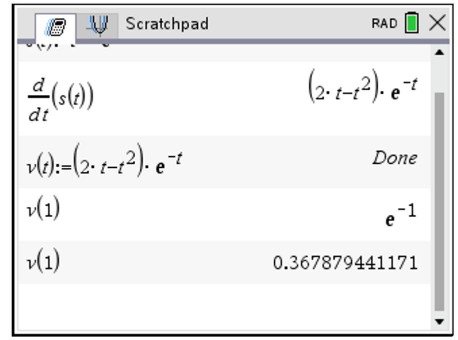


Solution

(b) What is the velocity after 1 second?

$$v(t) \overset{CAS}{=} (2t - t^2) \cdot e^{-t}$$

$$v(1) \overset{CAS}{=} e^{-1} \approx 0.369 \text{ ft/sec}$$



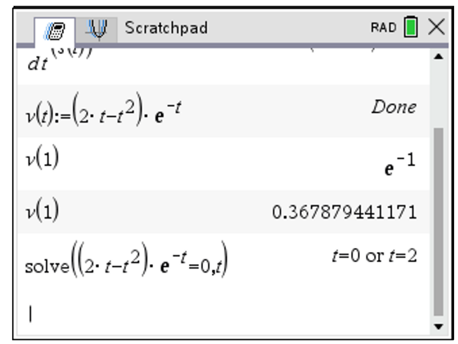
(c) When is the particle at rest, stopped?

Let $v(t) = 0$

$$(2t - t^2) \cdot e^{-t} = 0$$

$$t(2 - t) \cdot e^{-t} = 0$$

Particle is stopped when $t = 0$ seconds and $t = 2$ seconds.



Solution

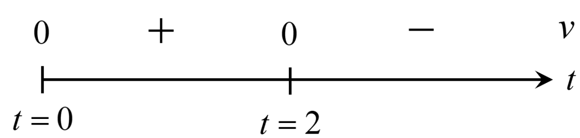
(d) When is the particle moving in the positive direction, forward?

The particle is moving in the positive direction when $v(t) > 0$

$v(t) = (2t - t^2) \cdot e^{-t}$

$t(2-t) \cdot e^{-t} > 0$

$t(2-t) > 0$



The particle is moving in the positive direction (forward) when

$0 < t < 2$ seconds

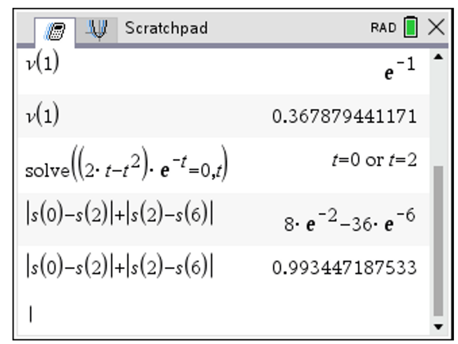
(e) Find the total distance traveled during the first 6 seconds.

v changes sign at $t = 2$ in the interval $[0, 6]$.

The total distance traveled during the first 6 seconds is

$s(t) = t^2 e^{-t}$

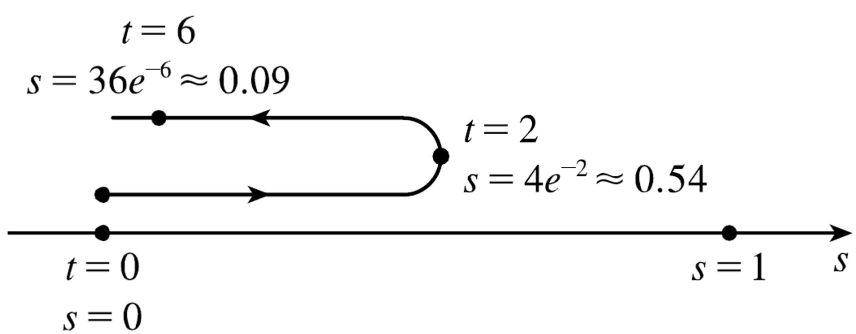
$$|s(0) - s(2)| + |s(2) - s(6)| \overset{CAS}{=} 8e^{-2} - 36e^{-6} \overset{CAS}{\approx} 0.993 \text{ feet}$$

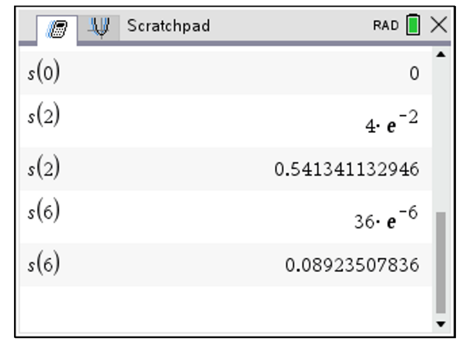


Solution

(f) Draw a diagram to illustrate the motion of the particle.

$s(t) = t^2 e^{-t}$





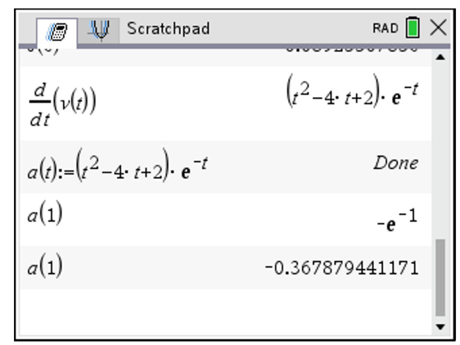
(g) Find the acceleration at time *t* and after 1 second.

$$v(t) = (2t - t^2) \cdot e^{-t}$$

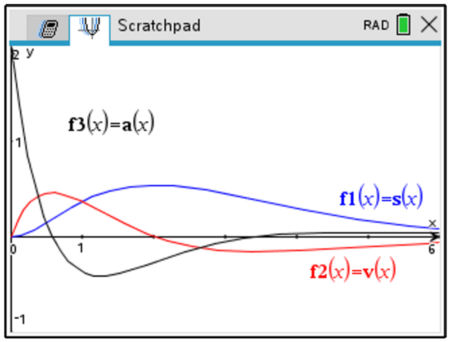
$$a(t) = v'(t)$$

$$a(t) = \left(t^2 - 4t + 2\right) \cdot e^{-t}$$

$$a(1) \overset{CAS}{=} -e^{-1} \overset{CAS}{\approx} -0.368 \text{ ft/sec}^2$$



Solution (h) Graph the position, velocity, and acceleration functions for $0 < t < 6$.



(i) When is the particle speeding up? When is it slowing down?

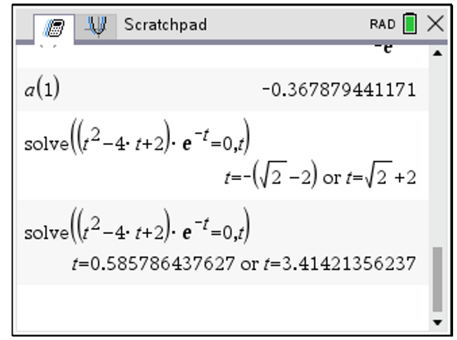
Let $a(t) = 0$

$$a(t) = (t^2 - 4t + 2) \cdot e^{-t}$$

$$(t^2 - 4t + 2) \cdot e^{-t} = 0$$

$$t = 2 - \sqrt{2} \approx 0.586 \text{ sec and } t = 2 + \sqrt{2} \approx 3.414 \text{ sec}$$

The particle is speeding up when v and a have the same sign.



Using the previous information and the figure in part (h), we see that v and a are both positive when

$$0 < t < 2 - \sqrt{2}$$

both negative when

$$2 < t < 2 + \sqrt{2}$$

The particle is slowing down when v and a have the same sign. This occurs when

$$2 - \sqrt{2} < t < 2 \text{ and } t > 2 + \sqrt{2}$$

Example If a ball is thrown vertically upward with a velocity of 80 ft/s, then its height after t seconds is $s = 80t - 16t^2$.

(a) What is the maximum height reached by the ball?

(b) What is the velocity of the ball when it is 96 ft above the ground on its way up? On its way down?

Solution

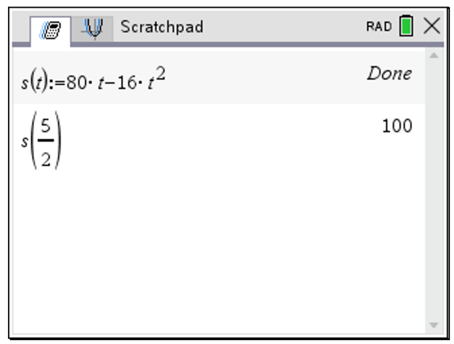
(a) What is the maximum height reached by the ball?

At maximum height the velocity of the ball is $v = 0$ ft/sec.

$$v(t) = s'(t) = 80 - 32t = 0$$

$$32t = 80$$

$$t = \frac{5}{2} = 2.5 \text{ sec}$$



So the maximum height is

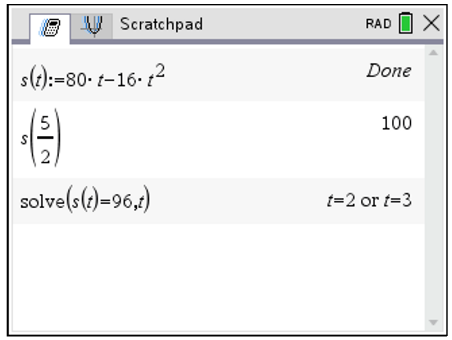
$$s\left(\tfrac{5}{2}\right) = 80\left(\tfrac{5}{2}\right) - 16\left(\tfrac{5}{2}\right)^2$$

$= 100$ ft is the maximum height reached by the ball.

Solution

(b) What is the velocity of the ball when it is 96 ft above the ground on its way up? On its way down?

We let



$$s(t) = 80t - 16t^2 = 96$$

$$16t^2 - 80t + 96 = 0$$

$$16(t^2 - 5t + 6) = 0$$

$$16(t-3)(t-2) = 0$$

So the ball has a height of 96 ft on the way up at $t = 2$ and on the way down at $t = 3$.

$$v(t) = 80 - 32t$$

$$v(2) = 80 - 32(2) = 16 \text{ ft/s}$$

$$v(3) = 80 - 32(3) = -16 \text{ ft/s}$$