3.6 Derivatives of Logarithmic and Inverse Trigonometric Functions

John Napier
1550 to 1617

John Napier was a Scottish scholar who is best known for his invention of logarithms, but other mathematical contributions include a mnemonic for formulas used in solving spherical triangles and two formulas known as Napier's analogies.

Now we want to find the derivative of logarithmic functions.

$$y = \ln x$$

$$e^y = x$$

$$\frac{d}{dx}\left(e^y\right) = \frac{d}{dx}(x)$$

$$e^y \frac{dy}{dx} = 1$$

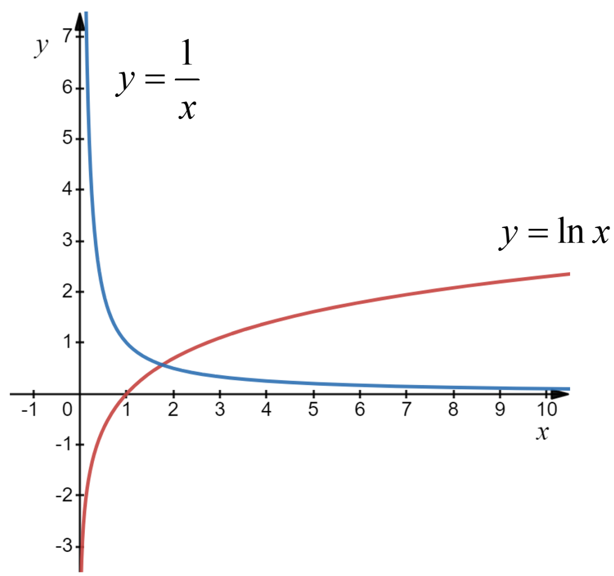
$$\frac{dy}{dx} = \frac{1}{e^y}$$

But, $e^y = x$

So, $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$

The generalized derivative is:

$$\frac{d}{dx}\ln u = \frac{1}{u}\cdot\frac{du}{dx}$$



Example Find $\dfrac{d}{dx} \ln \dfrac{x+1}{\sqrt{x-2}}$.

Solution

$$\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}} = \frac{1}{\dfrac{x+1}{\sqrt{x-2}}} \frac{d}{dx} \frac{x+1}{\sqrt{x-2}}$$

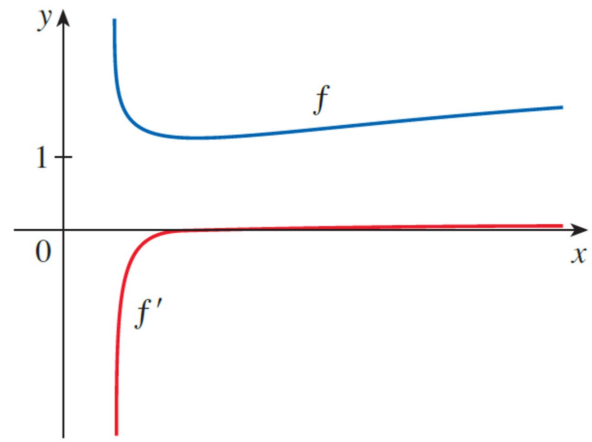
$$\frac{d}{dx} \ln u = \frac{1}{u} \cdot \frac{du}{dx}$$

$$= \frac{\sqrt{x-2}}{x+1} \frac{\sqrt{x-2} \cdot 1 - (x+1)\left(\frac{1}{2}\right)(x-2)^{-1/2}}{x-2}$$

$$= \frac{x - 2 - \frac{1}{2}(x+1)}{(x+1)(x-2)} \cdot \frac{2}{2}$$

$$= \frac{2x-4-x-1}{2(x+1)(x-2)}$$

$$= \frac{x-5}{2(x+1)(x-2)}$$



Properties of Logarithms

$\log_a(xy) = \log_a x + \log_a y$

$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

$\log_a x^p = p \cdot \log_a x$

$x = y$ if and only if $\log_a x = \log_a y$

Change-of-Base Formula

Changing from base b to base e we get $\log_b x = \frac{\ln x}{\ln b}$

Example Find $\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}}$.

Solution

$$\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}} = \frac{d}{dx}\left[\ln(x+1) - \ln(x-2)^{1/2}\right]$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$= \frac{d}{dx}\ln(x+1) - \frac{d}{dx}\left[\frac{1}{2}\cdot \ln(x-2)\right]$$

$$= \frac{1}{x+1} - \frac{1}{2}\cdot\frac{1}{x-2}$$

$$= \frac{2x-4-(x+1)}{2(x+1)(x-2)}$$

$$= \frac{x-5}{2(x+1)(x-2)}$$

A lot easier!

To find the derivative of any logarithmic function of any base, you can use the change of base rule for logs:

$$\frac{d}{dx}\log_b x = \frac{d}{dx}\frac{\ln x}{\ln b} = \frac{1}{\ln b}\cdot\frac{d}{dx}\ln x = \frac{1}{\ln b}\cdot\frac{1}{x}$$

Change of Base to e

The generalized derivative formula of the log of any base is:

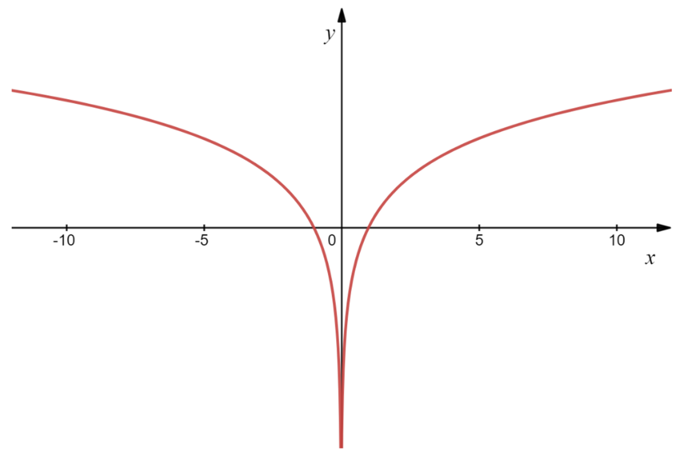
$$\frac{d}{dx}\log_b u = \frac{1}{u}\cdot\frac{1}{\ln b}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\cdot\frac{du}{dx}$$

To expand the domain of logarithmic functions we take the absolute value of the argument. So, let's examine

$$y = \log_b |x|, \quad b>1$$

$$x \in \mathbb{R}, \; x \neq 0$$



It can be shown that,

$$\frac{d}{dx}\log_b |u| = \frac{1}{u}\cdot\frac{1}{\ln b}\cdot\frac{du}{dx}$$

The derivative is the same!!

Example Differentiate the function.

(a) $y = \log_8(x^2 + 3x)$

(b) $g(t) = \ln \dfrac{t(t^2+1)^4}{\sqrt[3]{2t-1}}$

Solution

(a) $y = \log_8(x^2 + 3x)$

$$y' = \frac{1 \cdot 1}{(x^2+3x)\ln 8} \cdot \frac{d}{dx}(x^2+3x)$$

$$= \frac{1}{(x^2+3x)\ln 8} \cdot (2x+3)$$

$$= \frac{2x+3}{(x^2+3x)\ln 8}$$

(b) $g(t) = \ln \dfrac{t(t^2+1)^4}{\sqrt[3]{2t-1}}$

$$= \ln t + \ln(t^2+1)^4 - \ln \sqrt[3]{2t-1}$$

$$= \ln t + 4\ln(t^2+1) - \tfrac{1}{3}\ln(2t-1)$$

$$g'(t) = \frac{1}{t} + 4 \cdot \frac{1}{t^2+1} \cdot 2t - \frac{1}{3} \cdot \frac{1}{2t-1} \cdot 2$$

$$= \frac{1}{t} + \frac{8t}{t^2+1} - \frac{2}{3(2t-1)}$$

Example Find an equation of the tangent line to the curve at the given point.

$$y = \ln(x^2 - 3x + 1), \quad (3, 0)$$

Solution

$$y = \ln(x^2 - 3x + 1)$$

$$y' = \frac{1}{x^2 - 3x + 1} \cdot (2x - 3)$$

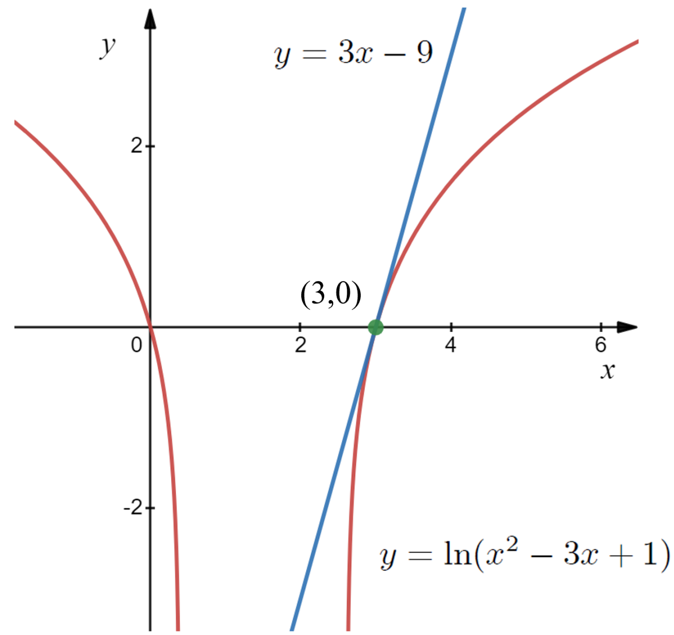
So, the slope of tangent line to the curve at (3,0) is

$$m = y'(3) = \frac{1}{1} \cdot 3 = 3$$

An equation of the tangent line is

$$y - 0 = 3(x - 3)$$

$$y = 3x - 9$$



Logarithmic Differentiation

Example Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$. Quotient Rule?

Solution We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Because we have an explicit expression for y, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to expand the expression.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y' and replace y by $f(x)$.

You should distinguish carefully between the Power Rule $[(x^n)' = nx^{n-1}]$, where the base is variable and the exponent is constant, and the rule for differentiating exponential functions $[(a^x)' = a^x \ln a]$, where the base is constant and the exponent is variable.

In general there are four cases for exponents and bases:

1. $\dfrac{d}{dx}(a^b) = 0 \qquad (a \text{ and } b \text{ are constants})$

2. $\dfrac{d}{dx}[f(x)]^b = b[f(x)]^{b-1}f'(x)$

3. $\dfrac{d}{dx}[a^{g(x)}] = a^{g(x)}(\ln a)g'(x)$

4. To find $(d/dx)[f(x)]^{g(x)}$, logarithmic differentiation can be used, as in the next example.

Example Use logarithmic differentiation to find the derivative of the function.

(a) $y = \sqrt{x}\, e^{x^2 - x}(x + 1)^{2/3}$ (b) $y = (\ln x)^{\cos x}$

Solution

(a) $y = \sqrt{x}\, e^{x^2 - x}(x + 1)^{2/3}$

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to expand the expression.

$$\ln y = \ln\left[x^{1/2} e^{x^2 - x}(x+1)^{2/3}\right]$$

$$\ln y = \tfrac{1}{2}\ln x + (x^2 - x) + \tfrac{2}{3}\ln(x+1)$$

2. Differentiate implicitly with respect to x.

$$\frac{1}{y}\, y' = \frac{1}{2}\cdot\frac{1}{x} + 2x - 1 + \frac{2}{3}\cdot\frac{1}{x+1}$$

3. Solve the resulting equation for y' and replace y by $f(x)$.

$$y' = y\left(\frac{1}{2x} + 2x - 1 + \frac{2}{3x+3}\right)$$

$$y' = \sqrt{x}\, e^{x^2 - x}(x+1)^{2/3}\left(\frac{1}{2x} + 2x - 1 + \frac{2}{3x+3}\right)$$

Example Use logarithmic differentiation to find the derivative of the function.

(a) $y = \sqrt{x}\, e^{x^2 - x}(x+1)^{2/3}$ (b) $y = (\ln x)^{\cos x}$

Solution

(b) $y = (\ln x)^{\cos x}$ $y' = \cos x (\ln x)^{\cos x - 1}$? NO!!! Logarithmic Differentiation

$$\ln y = \cos x \cdot \ln(\ln x)$$

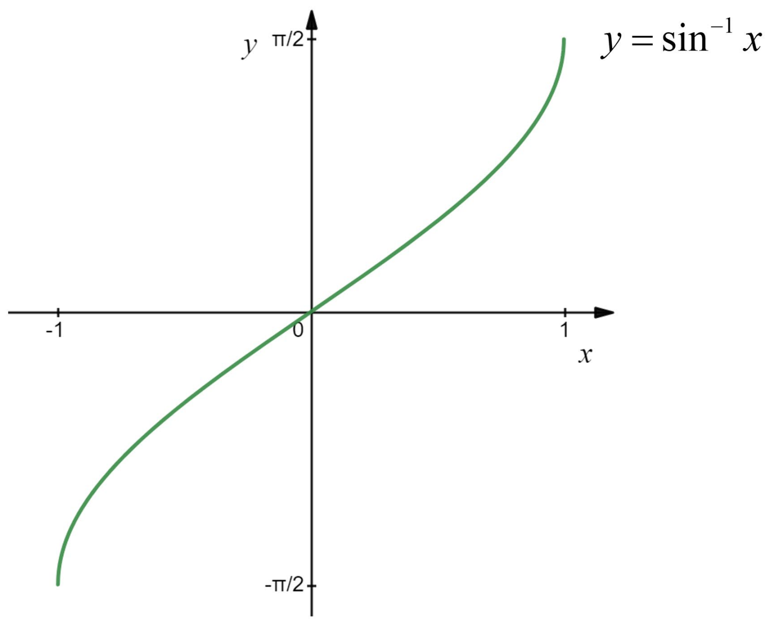
$$\frac{1}{y} y' = \cos x \cdot \frac{1}{\ln x} \cdot \frac{1}{x} + (\ln \ln x)(-\sin x)$$

$$y' = y \cdot \left(\frac{\cos x}{x \ln x} - \sin x \cdot \ln \ln x \right)$$

$$y' = (\ln x)^{\cos x} \left(\frac{\cos x}{x \ln x} - \sin x \ln \ln x \right)$$

Derivatives of Inverse Trigonometric Functions

Recall $y = \sin^{-1} x$.



I can read your minds, what is $\frac{dy}{dx}$.

We can use implicit differentiation to find $\dfrac{d}{dx}\sin^{-1} x$

$$y = \sin^{-1} x$$

$$\sin y = x$$

$$\frac{d}{dx}\sin y = \frac{d}{dx}x$$

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\sin^2 y + \cos^2 y = 1$$

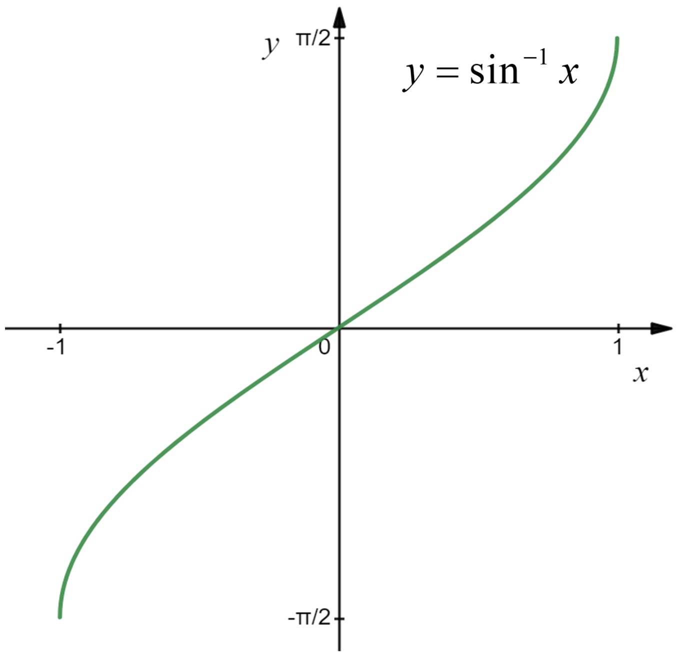
$$\cos^2 y = 1 - \sin^2 y$$

$$\cos y = \pm\sqrt{1-\sin^2 y}$$

But $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

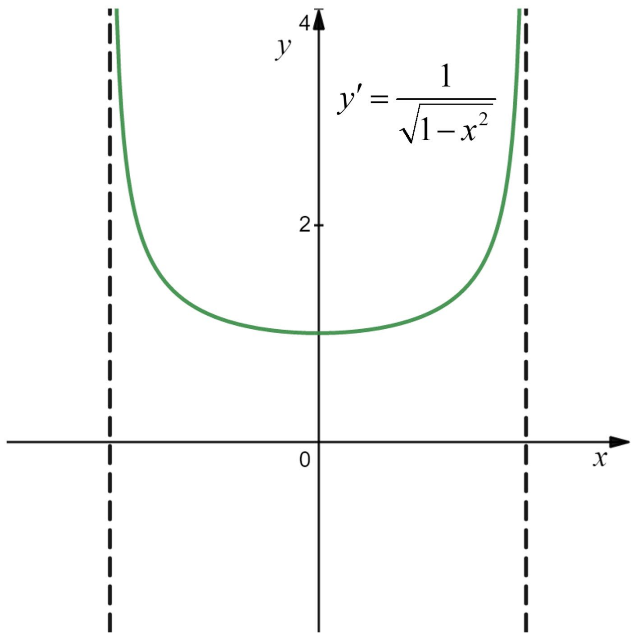
so $\cos y$ is positive.

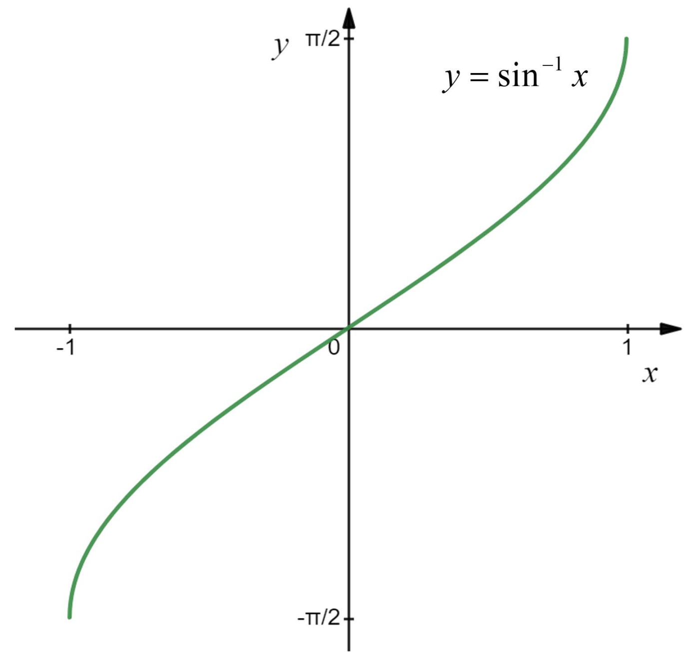
$$\therefore \cos y = \sqrt{1-\sin^2 y}$$



$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}}$$

This result is consistent with the graph of $f(x) = \sin^{-1} x$





We could use the same technique to find $\frac{d}{dx}\tan^{-1}x$ and $\frac{d}{dx}\sec^{-1}x$.

$$\csc^{-1}x = \frac{\pi}{2} - \sec^{-1}x$$

$$\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\cot^{-1}u = -\frac{1}{1+u^2} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\csc^{-1}u = -\frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

Example   Find the derivative of the function. Simplify where possible.

(a) $y = x \cdot \sin^{-1} x + \sqrt{1 - x^2}$ (b) $y = \tan^{-1}\left(\dfrac{x}{a}\right) + \ln\sqrt{\dfrac{x - a}{x + a}}$

Solution

(a) $y = x \cdot \sin^{-1} x + \sqrt{1 - x^2}$

$$y' = x \cdot \frac{1}{\sqrt{1-x^2}} + (\sin^{-1} x)(1) + \frac{1}{2}(1 - x^2)^{-1/2}(-2x)$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$

$$= \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x - \frac{x}{\sqrt{1-x^2}}$$

$$= \sin^{-1} x$$

Example Find the derivative of the function. Simplify where possible.

(a) $y = x \cdot \sin^{-1} x + \sqrt{1 - x^2}$ (b) $y = \tan^{-1}\left(\frac{x}{a}\right) + \ln\sqrt{\frac{x - a}{x + a}}$

Solution

(b) $y = \tan^{-1}\left(\frac{x}{a}\right) + \ln\sqrt{\frac{x - a}{x + a}}$

$$= \tan^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\ln\left(\frac{x - a}{x + a}\right)$$

$$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

$$y' = \frac{1}{1 + \left(\frac{x}{a}\right)^2} \cdot \frac{1}{a} + \frac{1}{2} \cdot \frac{1}{\frac{x - a}{x + a}} \cdot \frac{(x + a) \cdot 1 - (x - a) \cdot 1}{(x + a)^2}$$

$$= \frac{1}{a + \frac{x^2}{a}} + \frac{1}{2} \cdot \frac{x + a}{x - a} \cdot \frac{2a}{(x + a)^2}$$

Example Find the derivative of the function. Simplify where possible.

(a) $y = x \cdot \sin^{-1} x + \sqrt{1 - x^2}$ (b) $y = \tan^{-1}\left(\frac{x}{a}\right) + \ln\sqrt{\frac{x-a}{x+a}}$

Solution

$$= \frac{1}{a + \frac{x^2}{a}} + \frac{1}{\cancel{2}} \cdot \frac{\cancel{x+a}}{x-a} \cdot \frac{\cancel{2}a}{(x+a)^{\cancel{2}}}$$

$$= \frac{1}{a + \frac{x^2}{a}} \cdot \frac{a}{a} + \frac{a}{(x-a)(x+a)}$$

$$= \frac{a}{x^2 + a^2} + \frac{a}{x^2 - a^2}$$

That was fun!! I can't wait to do my homework!!