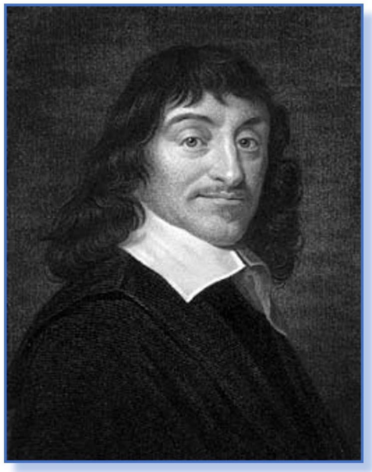
3.5 Implicit Differentiation

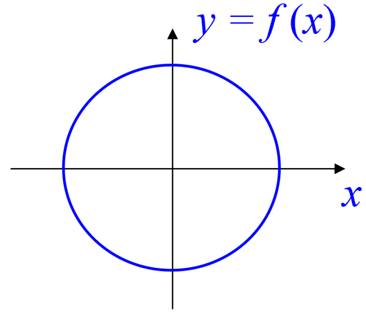


René Descartes
1596 – 1650

René Descartes was a French philosopher whose work, *La géométrie*, includes his application of algebra to geometry from which we now have Cartesian geometry. His work had a great influence on both mathematicians and philosophers.

Implicit Equation

$$x^2 + y^2 = 1$$



This is not a function, but it would still be nice to be able to find the slope of tangents line to the circle.

$$m = \frac{dy}{dx}$$

$$x^2 + (f(x))^2 = 1$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}(f(x))^2 = \frac{d}{dx}1$$

← Do the same thing to both sides.

Note use of chain rule, and $f(x)$ is a function of x.

$$2x + 2f(x) \cdot \frac{d}{dx} f(x) = 0$$

Remember $y = f(x)$.

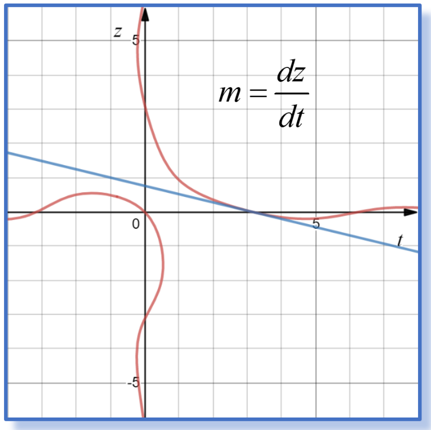
$$2x + 2y \cdot \frac{dy}{dx} = 0$$

Solving for dy/dx.

$$2y \cdot \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$



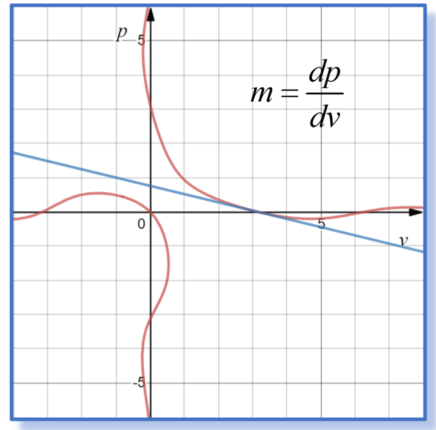
z is a function of *t*.

$$m = \frac{dz}{dt}$$

Examples

$$\frac{d}{dt}\left[t^3\right] = 3t^2 \cdot \frac{dt}{dt} = 3t^2$$

$$\frac{d}{dt}\left[z^{1/2}\right] = \frac{1}{2} z^{-1/2} \cdot \frac{dz}{dt}$$



p is a function of *v*.

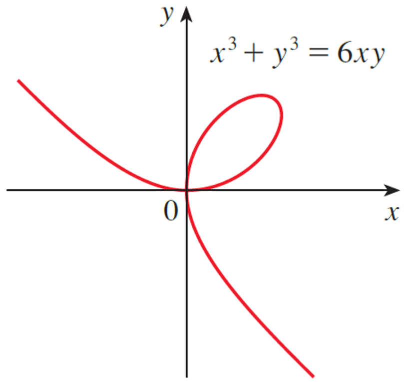
$$m = \frac{dp}{dv}$$

Examples

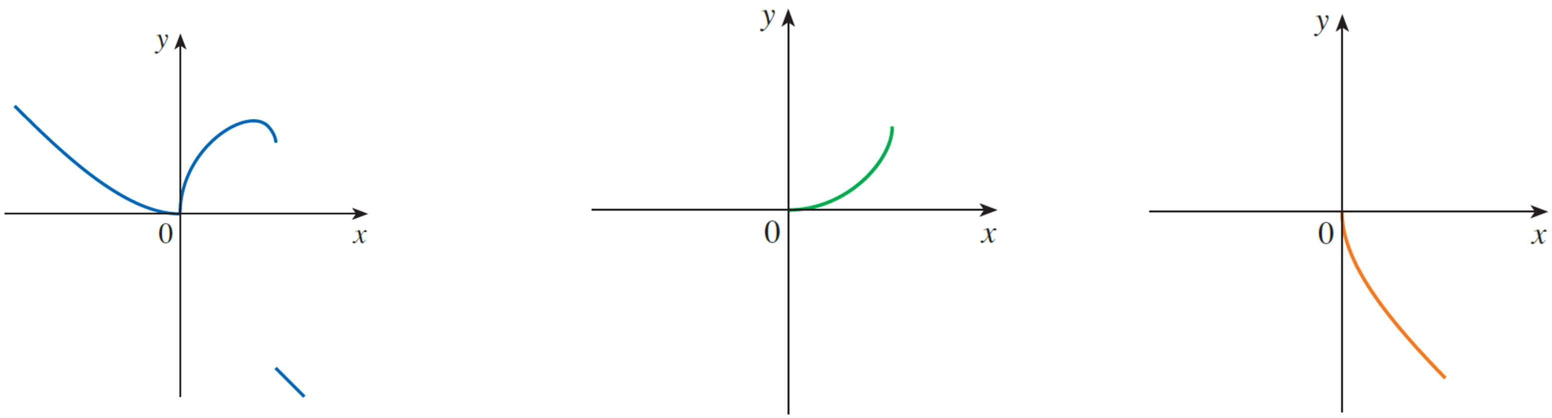
$$\frac{d}{dv}\left[3^v\right] = 3^v \cdot \ln 3 \cdot \frac{dv}{dv} = 3^v \cdot \ln 3$$

$$\frac{d}{dv}\left[\sin(p)\right] = \cos(p) \cdot \frac{dp}{dv}$$

The folium of Descartes



Graphs of three functions defined implicitly by the folium of Descartes



Implicit Differentiation

Fortunately, we don't need to solve an equation for y in terms of x in order to find the derivative of y. Instead, we can use the method of **implicit differentiation**. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for dy/dx. In the examples and exercises of this section it is always assumed that the given equation determines y implicitly as a differentiable function of x so that the method of implicit differentiation can be applied.

Example Differentiate $x^3 + y^3 = 6xy$.

Solution Differentiate both sides with respect to x.

$$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = \frac{d}{dx}6xy$$

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 6x \cdot \frac{dy}{dx} + 6y$$

Solve for $\frac{dy}{dx}$.

$$3y^2 \cdot \frac{dy}{dx} - 6x \cdot \frac{dy}{dx} = 6y - 3x^2$$

Factor out $\frac{dy}{dx}$.

$$\frac{dy}{dx} \cdot \left(3y^2 - 6x\right) = 6y - 3x^2$$

Divide both sides by $3y^2 - 6x$.

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$$

Simplify.

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

Prime Notation

$x^3 + y^3 = 6xy$

Differentiate both sides with respect to x.

$3x^2 + 3y^2 \cdot y' = 6x \cdot y' + 6y$

Solve for y'.

$3y^2 \cdot y' - 6x \cdot y' = 6y - 3x^2$

Factor out y'.

$y'\left(3y^2 - 6x\right) = 6y - 3x^2$

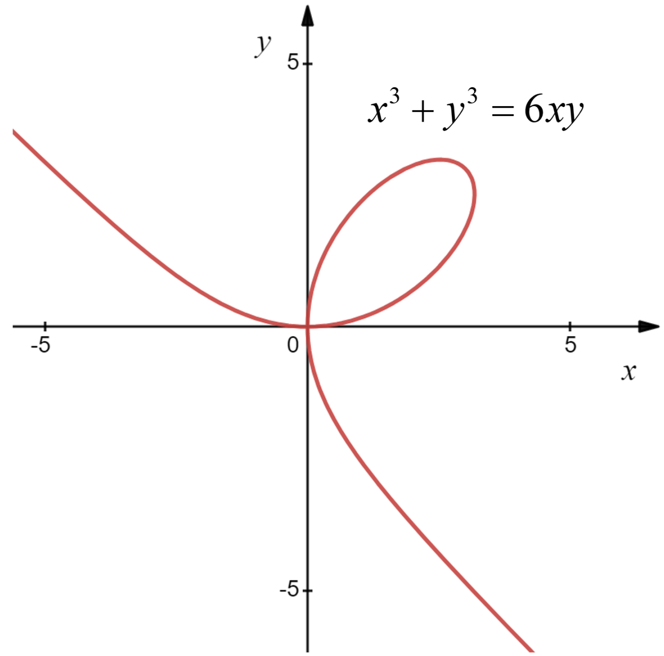
$$y' = \frac{6y - 3x^2}{3y^2 - 6x}$$

Prime Notation

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

Leibniz Notation

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$



Example Consider the implicit equation $2y = x^2 + \sin y$, find $\dfrac{dy}{dx}$.

Solution This can't be solved for y.

$$2y = x^2 + \sin y$$

$$\frac{d}{dx}2y = \frac{d}{dx}x^2 + \frac{d}{dx}\sin y$$

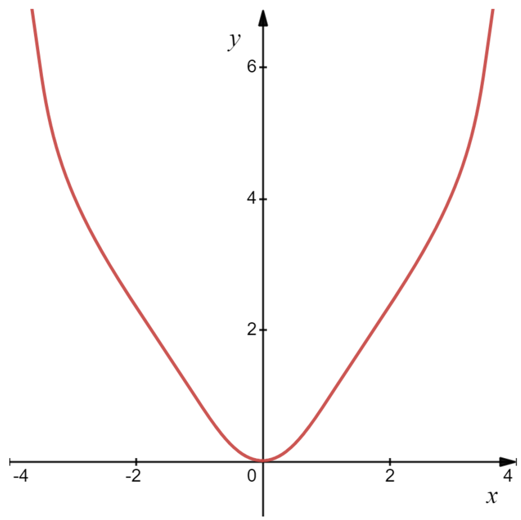
$$2\frac{dy}{dx} = 2x + \cos y\frac{dy}{dx}$$

$$2\frac{dy}{dx} - \cos y\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx}(2 - \cos y) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{2 - \cos y}$$

The graph of the curve is:



Again, this technique is called <u>Implicit Differentiation</u>.

1. Differentiate both sides of the equation with respect to x.
2. Solve for $\dfrac{dy}{dx}$.

Example Find dy/dx by implicit differentiation.

$$\sqrt{x+y} = x^4 + y^4$$

Solution

$$\frac{d}{dx}\sqrt{x+y} = \frac{d}{dx}\left(x^4+y^4\right)$$

$$\tfrac{1}{2}(x+y)^{-1/2}(1+y') = 4x^3 + 4y^3y'$$

$$\frac{1}{2\sqrt{x+y}} + \frac{1}{2\sqrt{x+y}}\,y' = 4x^3 + 4y^3y'$$

$$\frac{1}{2\sqrt{x+y}} - 4x^3 = 4y^3y' - \frac{1}{2\sqrt{x+y}}\,y'$$

$$\frac{1-8x^3\sqrt{x+y}}{2\sqrt{x+y}} = \frac{8y^3\sqrt{x+y}-1}{2\sqrt{x+y}}\,y'$$

$$\frac{2\sqrt{x+y}}{8y^3\sqrt{x+y}-1}\cdot\frac{1-8x^3\sqrt{x+y}}{2\sqrt{x+y}} = \frac{2\sqrt{x+y}}{8y^3\sqrt{x+y}-1}\cdot\frac{8y^3\sqrt{x+y}-1}{2\sqrt{x+y}}\,y'$$

$$y' = \frac{1-8x^3\sqrt{x+y}}{8y^3\sqrt{x+y}-1}$$

Example Find the equations of the lines tangent and normal to the curve $x^2 - xy + y^2 = 7$ at $(-1, 2)$.

Solution We need the slope. Since it's not easy to solve for y, we'll use implicit differentiation to solve for $\frac{dy}{dx}$.

$$x^2 - xy + y^2 = 7$$

Note product rule.

$$2x - \left[x\frac{dy}{dx} + y\right] + 2y\frac{dy}{dx} = 0$$

$$2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0$$

$$(2y - x)\frac{dy}{dx} = y - 2x$$

$$\boxed{\frac{dy}{dx} = \frac{y - 2x}{2y - x}}$$

The point of tangency is $(-1, 2)$, so the slope is

$$m = \left.\frac{dy}{dx}\right|_{(-1,2)} = \frac{2 - 2(-1)}{2 \cdot 2 - (-1)} = \frac{2+2}{4+1} = \frac{4}{5}$$

Tangent Line:

$$y - 2 = \frac{4}{5}(x+1)$$

$$y - 2 = \frac{4}{5}x + \frac{4}{5}$$

$$\boxed{y = \frac{4}{5}x + \frac{14}{5}}$$

Normal Line:

$$y - 2 = -\frac{5}{4}(x+1)$$

$$y - 2 = -\frac{5}{4}x - \frac{5}{4}$$

$$\boxed{y = -\frac{5}{4}x + \frac{3}{4}}$$

Here is the graph!