3.4 The Chain Rule

Johannes Kepler
1571 – 1630

Johannes Kepler was a German mathematician and astronomer who discovered that the Earth and planets travel about the sun in elliptical orbits. He gave three fundamental laws of planetary motion. He also did important work in optics and geometry.

The Chain Rule

We now have a pretty good list of "shortcuts" to find derivatives of simple functions.

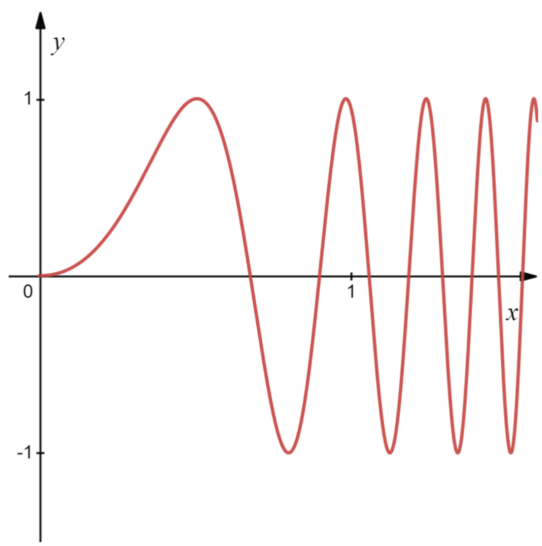
$\frac{d}{dx}(c) = 0$	$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(e^x) = e^x$
$(cf)' = cf'$	$(f+g)' = f' + g'$	$(f-g)' = f' - g'$
$(fg)' = fg' + gf'$	$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$	

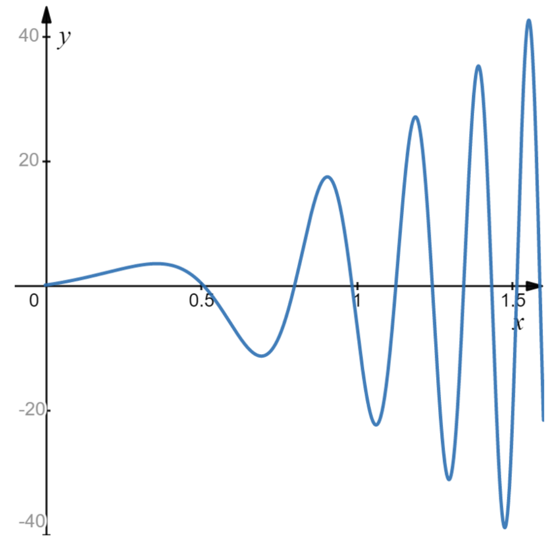
Of course, many of the functions that we will encounter are not so simple. What is needed is a way to combine derivative rules to evaluate more complicated combinations of functions. Functions that include addition, subtraction, multiplication, division, and composition.

For example, find y' if $y = \sin(3x^2 \cdot e^{\sqrt{x}})$.

$$y = \sin(3x^2 \cdot e^{\sqrt{x}})$$

$$y' = f'(x) = ?$$





We'll do this derivative later!

Consider a simple composite function:

$$y = 6x - 10$$

$$y = 2(3x - 5)$$

Let $u = 3x - 5$

then $y = 2u$

$$y = 6x - 10 \qquad y = 2u \qquad u = 3x - 5$$

$$\frac{dy}{dx} = 6 \qquad \frac{dy}{du} = 2 \qquad \frac{du}{dx} = 3$$

$$6 = 2 \cdot 3$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Consider:

$$y = 9x^2 + 6x + 1$$

$$y = (3x+1)^2$$

Let $u = 3x + 1$

then $y = u^2$

$$y = 9x^2 + 6x + 1 \qquad y = u^2 \qquad u = 3x + 1$$

$$\frac{dy}{dx} = 18x + 6 \qquad \frac{dy}{du} = 2u \qquad \frac{du}{dx} = 3$$

$$\frac{dy}{du} = 2(3x+1)$$

$$\frac{dy}{du} = 6x + 2$$

$$18x + 6 = (6x+2) \cdot 3$$

This pattern is called the **Chain Rule!**

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Chain Rule

Leibniz Notation

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

If $y = f(g(x))$, and we let $u = g(x)$, then $y = f(u)$.

So we get

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \cdot g'(x) = \underbrace{f'(g(x))} \cdot \underbrace{g'(x)}$$

Differentiate the outside function evaluate at the inside function…

Times the derivative of the inside function.

Hence,

If $f \circ g$ is the composite of $y = f(u)$ and $u = g(x)$, then

$$(f \circ g)'(x) = [f(g(x))]' = f'(g(x)) \cdot g'(x)$$

Chain Rule

Prime Notation

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x) \tag{1}$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \tag{2}$$

$$\frac{d}{dx}\underbrace{f}_{\text{outer function}}\ \underbrace{(g(x))}_{\text{evaluated at inner function}} = \underbrace{f'}_{\text{derivative of outer function}}\ \underbrace{(g(x))}_{\text{evaluated at inner function}} \cdot \underbrace{g'(x)}_{\text{derivative of inner function}}$$

Example

For the function below, write the composite function in the form $f(g(x))$. [Identify the inner function $u = g(x)$ and the outer function $y = f(u)$.] Then find the derivative dy/dx.

$$y = \sin\left(x^2 - 4\right)$$

Solution

$$y = f\left(g\left(x\right)\right) = \sin\left(x^2 - 4\right)$$

Let $u = x^2 - 4$ Inner Function

$y = \sin u$ Outer Function

$$\frac{dy}{du} = \cos u \qquad \frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$ **Chain Rule**

$$\frac{dy}{dx} = \cos u \cdot 2x$$

$$\frac{dy}{dx} = \cos\left(x^2 - 4\right) \cdot 2x$$

Example Find the derivative of the function.

$$y = \sin\left(x^2 - 4\right)$$

Solution Here is a faster way to find the derivative:

$$y = \sin\left(x^2 - 4\right)$$

Differentiate the outside function evaluate at the inside function…

$$y' = \cos\left(x^2 - 4\right) \cdot \frac{d}{dx}\left(x^2 - 4\right)$$

Times the derivative of the inside function.

Simplifying

$$y' = \cos\left(x^2 - 4\right) \cdot 2x$$

Example Find the derivative of the function.

$$y = \cos^2(3x)$$

Solution

$$y' = \frac{d}{dx}\cos^2(3x)$$

$$= \frac{d}{dx}\left[\cos(3x)\right]^2$$

$$= 2\left[\cos(3x)\right]\cdot\frac{d}{dx}\cos(3x)$$

derivative of the outside function evaluate at the inside function

derivative of the inside function

$$= 2\cos(3x)\cdot\left(-\sin(3x)\right)\cdot\frac{d}{dx}(3x)$$

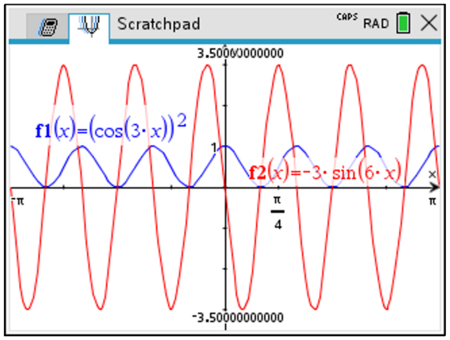
The chain rule can be used more than once.

(That's what makes the "chain" in the "chain rule"!)

$$= -2\cos(3x)\cdot\sin(3x)\cdot 3$$

$$= -6\cos(3x)\sin(3x)$$

$$= -3\sin(6x)$$ WHAT!!



Every derivative problem could be thought of as a chain-rule problem:

$$\frac{d}{dx}x^2 = \underbrace{2x^1}\cdot\underbrace{\frac{d}{dx}x} = 2x\cdot 1 = 2x$$

derivative of outside function evaluated at the inside function

derivative of inside function

The derivative of x is one.

Generalized Derivative formulas include the chain rule!

$$\frac{d}{dx}u^n = nu^{n-1}\cdot\frac{du}{dx} \qquad \frac{d}{dx}\sin u = \cos u\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\cos u = -\sin u\cdot\frac{du}{dx} \qquad \frac{d}{dx}\tan u = \sec^2 u\cdot\frac{du}{dx}$$

$$\frac{d}{dx}e^u = e^u\cdot\frac{du}{dx}$$

and the list continues…

Remember that u is a function of x!

Example Find the derivative of the function.

(a) $f(x) = (x^5 + 3x^2 - x)^{50}$ (b) $F(t) = \left(\dfrac{1}{2t + 1}\right)^4$ (c) $A(r) = \sqrt{r} \cdot e^{r^2+1}$

Solution

(a) $f(x) = (x^5 + 3x^2 - x)^{50}$

$$f'(x) = \underbrace{50(x^5 + 3x^2 - x)^{49}}_{\text{derivative of outside function evaluated at the inside function}} \cdot \underbrace{\frac{d}{dx}(x^5 + 3x^2 - x)}_{\text{derivative of the inside function}}$$

derivative of outside function evaluated at the inside function

derivative of the inside function

$$= 50(x^5 + 3x^2 - x)^{49}(5x^4 + 6x - 1)$$

Example Find the derivative of the function.

(a) $f(x) = (x^5 + 3x^2 - x)^{50}$ (b) $F(t) = \left(\dfrac{1}{2t + 1}\right)^4$ (c) $A(r) = \sqrt{r} \cdot e^{r^2+1}$

Solution

(b) $$F(t) = \left(\frac{1}{2t + 1}\right)^4 = [(2t+1)^{-1}]^4 = (2t+1)^{-4}$$

$$F'(t) = -4(2t+1)^{-5} \cdot \frac{d}{dt}(2t+1) = -4(2t+1)^{-5}(2) = -\frac{8}{(2t+1)^5}$$

Example Find the derivative of the function.

(a) $f(x) = (x^5 + 3x^2 - x)^{50}$ (b) $F(t) = \left(\dfrac{1}{2t + 1}\right)^4$ (c) $A(r) = \sqrt{r} \cdot e^{r^2+1}$

Solution

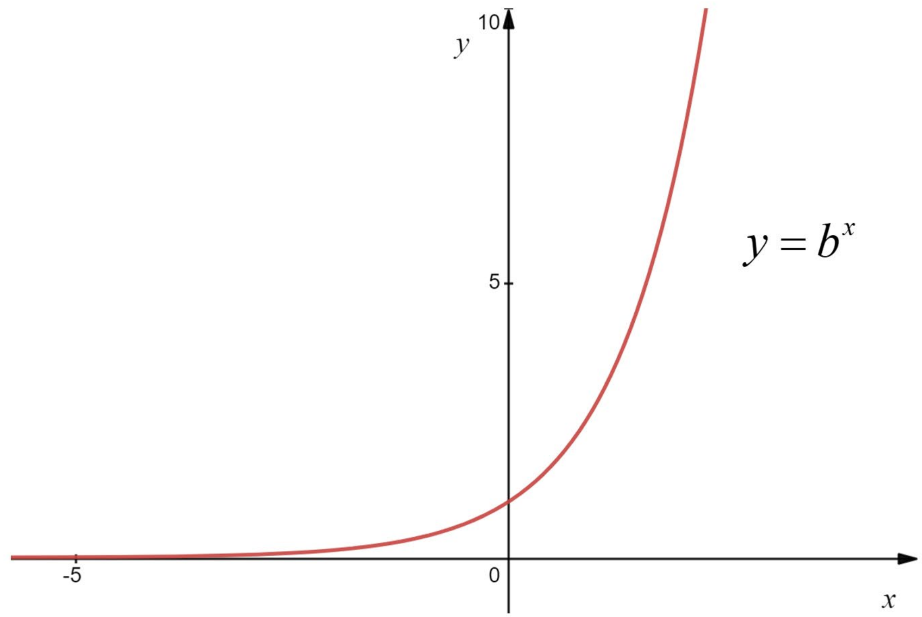
(c) $A(r) = \sqrt{r} \cdot e^{r^2+1}$

$$A'(r) = \sqrt{r} \cdot e^{r^2+1} \cdot \frac{d}{dr}\left(r^2 + 1\right) + e^{r^2+1} \cdot \frac{d}{dr}\left(\sqrt{r}\right)$$

$$= \sqrt{r} \cdot e^{r^2+1} \cdot 2r + e^{r^2+1} \cdot \frac{1}{2\sqrt{r}}$$

$$= e^{r^2+1}\left(2r\sqrt{r} + \frac{1}{2\sqrt{r}}\right) \text{ or } e^{r^2+1}\left(\frac{4r^2+1}{2\sqrt{r}}\right)$$

Derivatives of General Exponential Functions



Consider the exponential function $y = b^x$, $b > 0$ and $b \neq 1$.

We would like to find its derivative.

Recall, if $f(x) = \ln x$ and $g(x) = e^x$ then

$$f(g(x)) = \ln(e^x) = x$$

and

$$g(f(x)) = g(\ln x) = e^{\ln x} = x.$$

f and g are inverse functions of each other.

Inverse Properties for e^x and $\ln x$

1. $e^{\ln x} = x$, for $x > 0$, and $\ln(e^x) = x$, for all x.

2. $y = \ln x$ if and only if $x = e^y$.

3. For real numbers x and $b > 0$, $b^x = e^{(\ln b^x)} = e^{x \ln b}$.

So $y = b^x$

$$= e^{\ln b^x}$$

$$= e^{x \cdot (\ln b)}$$

Using the chain rule we get

$$y' = e^{x \cdot (\ln b)} \cdot \ln b$$

$$= e^{\ln b^x} \cdot \ln b$$

$$= b^x \cdot \ln b$$

$$y' = b^x \cdot \ln b$$

$$\frac{d}{dx}\left(b^u\right) = b^u \cdot \ln b \cdot \frac{du}{dx}$$

Example Find the derivative of the function.

(a) $r(t) = 10^{2\sqrt{t}}$ (b) $y = e^{\sin 2x} + \sin(e^{2x})$ (c) $y = (3^{\cos(x^2)} - 1)^4$

Solution

(a) $r(t) = 10^{2\sqrt{t}}$

$$\frac{d}{dx}\left(b^u\right) = b^u \cdot \ln b \cdot \frac{du}{dx}$$

$$r'(t) = 10^{2\sqrt{t}} \ln 10 \frac{d}{dt}\left(2\sqrt{t}\right)$$

$$= 10^{2\sqrt{t}} \ln 10 \left(2 \cdot \tfrac{1}{2} t^{-1/2}\right)$$

$$= \frac{(\ln 10)\, 10^{2\sqrt{t}}}{\sqrt{t}}$$

Example Find the derivative of the function.

(a) $r(t) = 10^{2\sqrt{t}}$ (b) $y = e^{\sin 2x} + \sin(e^{2x})$ (c) $y = (3^{\cos(x^2)} - 1)^4$

Solution

(b) $y = e^{\sin 2x} + \sin(e^{2x})$

$$y' = e^{\sin 2x} \frac{d}{dx} \sin 2x + \cos(e^{2x}) \frac{d}{dx} e^{2x}$$

$$= e^{\sin 2x} (\cos 2x) \cdot 2 + \cos(e^{2x})\, e^{2x} \cdot 2$$

$$= 2 \cos 2x\, e^{\sin 2x} + 2e^{2x} \cos(e^{2x})$$

Example Find the derivative of the function.

(a) $r(t) = 10^{2\sqrt{t}}$ (b) $y = e^{\sin 2x} + \sin(e^{2x})$ (c) $y = (3^{\cos(x^2)} - 1)^4$

Solution

(c) $y = (3^{\cos(x^2)} - 1)^4$

$$= 4\left(3^{\cos\left(x^2\right)} - 1\right)^3 \cdot \frac{d}{dx}\left(3^{\cos\left(x^2\right)} - 1\right)$$

$$= 4\left(3^{\cos\left(x^2\right)} - 1\right)^3 \cdot 3^{\cos\left(x^2\right)} \cdot \ln 3 \cdot \frac{d}{dx}\left(\cos\left(x^2\right)\right)$$

$$= 4\left(3^{\cos\left(x^2\right)} - 1\right)^3 \cdot 3^{\cos\left(x^2\right)} \cdot \ln 3 \cdot \left(-\sin\left(x^2\right)\right) \cdot \frac{d}{dx}\left(x^2\right)$$

$$= 4\left(3^{\cos\left(x^2\right)} - 1\right)^3 \cdot 3^{\cos\left(x^2\right)} \cdot \ln 3 \cdot \left(-\sin\left(x^2\right)\right) \cdot 2x$$

$$= -8x \cdot \ln 3 \cdot \sin\left(x^2\right) \cdot 3^{\cos\left(x^2\right)} \cdot \left(3^{\cos\left(x^2\right)} - 1\right)^3$$

Example Find an equation of the tangent line to the curve at the given point.

$$y = \sqrt{1 + x^3}, \quad (2, 3)$$

Solution

$$y = \sqrt{1+x^3} = (1+x^3)^{1/2}$$

$$y' = \frac{1}{2}(1+x^3)^{-1/2} \cdot 3x^2$$

$$= \frac{3x^2}{2\sqrt{1+x^3}}$$

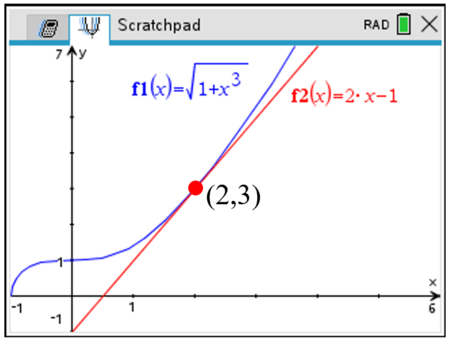
At the point of tangency (2,3) the slope is

$$m = y' = \frac{3 \cdot 4}{2\sqrt{9}} = 2$$

and an equation of the tangent line is

$$y - 3 = 2(x - 2)$$

$$y = 2x - 1$$



Example If $h(x) = \sqrt{4 + 3f(x)}$, where $f(1) = 7$ and $f'(1) = 4$, find $h'(1)$.

Solution

$$h(x) = \sqrt{4 + 3f(x)}$$

$$h'(x) = \tfrac{1}{2}(4 + 3f(x))^{-1/2} \cdot 3f'(x)$$

$$h'(1) = \tfrac{1}{2}(4 + 3f(1))^{-1/2} \cdot 3f'(1)$$

$$= \tfrac{1}{2}(4 + 3 \cdot 7)^{-1/2} \cdot 3 \cdot 4$$

$$= \frac{6}{\sqrt{25}}$$

$$= \frac{6}{5}$$

Example A table of values for f, g, f', and g' is given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

(a) If $F(x) = f(f(x))$, find $F'(2)$.

(b) If $G(x) = g(g(x))$, find $G'(3)$.

Solution

(a) If $F(x) = f(f(x))$, find $F'(2)$.

$F'(x) = f'(f(x)) \cdot f'(x)$

so $F'(2) = f'(f(2)) \cdot f'(2)$

$= f'(1) \cdot 5$

$= 4 \cdot 5$

$= 20$

Example A table of values for f, g, f', and g' is given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

(a) If $F(x) = f(f(x))$, find $F'(2)$.

(b) If $G(x) = g(g(x))$, find $G'(3)$.

Solution

(b) If $G(x) = g(g(x))$, find $G'(3)$.

$$G'(x) = g'(g(x)) \cdot g'(x)$$

so $G'(3) = g'(g(3)) \cdot g'(3)$

$$= g'(2) \cdot 9$$

$$= 7 \cdot 9$$

$$= 63$$

Example If g is a twice differentiable function and $f(x) = xg(x^2)$, find f'' in terms of g, g', and g''.

Solution

$$f(x) = xg(x^2)$$

$$f'(x) = xg'(x^2)\,2x + g(x^2)\cdot 1$$
$$= 2x^2g'(x^2) + g(x^2)$$

$$f''(x) = 2x^2g''(x^2)\,2x + g'(x^2)\,4x + g'(x^2)\,2x$$
$$= 4x^3g''(x^2) + 4xg'(x^2) + 2xg'(x^2)$$
$$= 6xg'(x^2) + 4x^3g''(x^2)$$