



### 3.3

## Derivatives of Trigonometric Functions

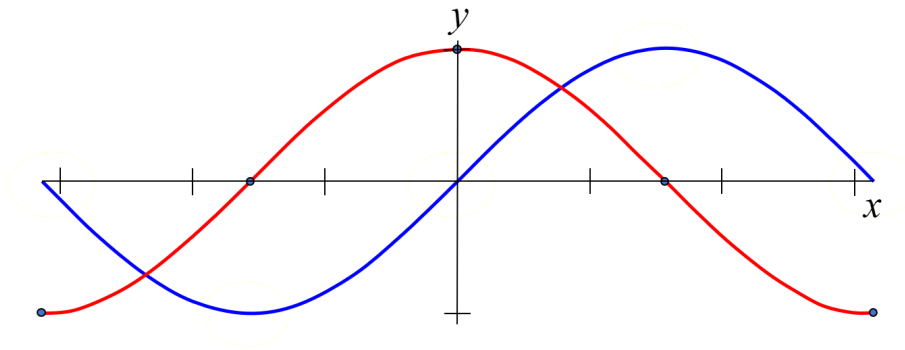


**Hypatia of Alexandria**  
370 – 415

**Hypatia** was an Alexandrine Greek Neoplatonist philosopher in Egypt who was one of the earliest mothers of mathematics. As head of the Platonist school at Alexandria, she also taught philosophy and astronomy.

Consider the function  $y = \sin x$

We could make a graph of the slope:



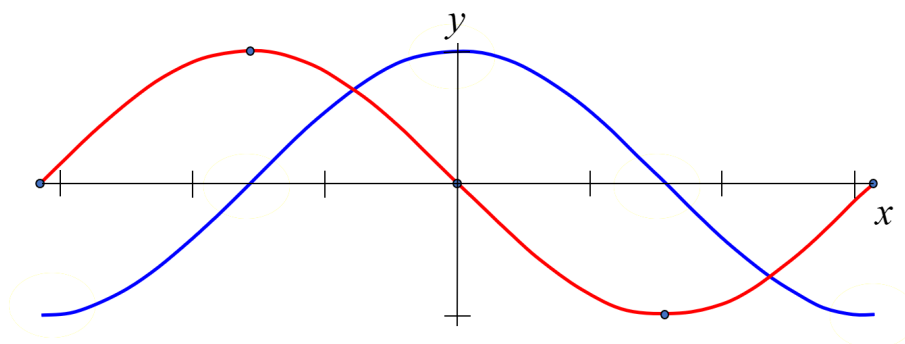
Now we connect the dots!

The resulting curve is a cosine curve.

$x$	slope
$-\pi$	$-1$
$-\frac{\pi}{2}$	$0$
$0$	$1$
$\frac{\pi}{2}$	$0$
$\pi$	$-1$

$$\frac{d}{dx} \sin x = \cos x$$

We can do the same thing for  $y = \cos x$



The resulting curve is a sine curve that has been reflected about the  $x$ -axis.

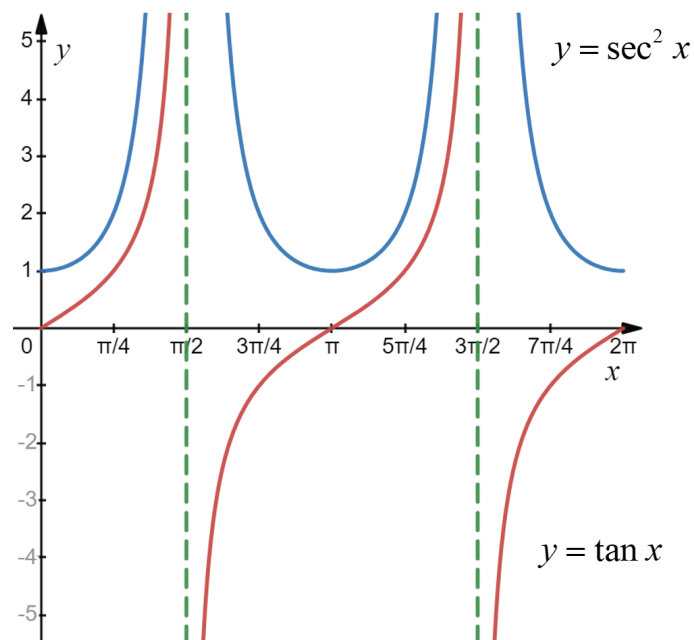
$x$	slope
$-\pi$	0
$-\frac{\pi}{2}$	1
0	0
$\frac{\pi}{2}$	-1
$\pi$	0

$$\frac{d}{dx} \cos x = -\sin x$$

We can find the derivative of tangent  $x$  by using the quotient rule.

$$\begin{aligned} & \frac{d}{dx} \tan x \\ &= \frac{d \sin x}{dx \cos x} \\ &= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x \end{aligned}$$

$$\frac{d}{dx} \tan x = \sec^2 x$$



Derivatives of the remaining trigonometric functions can be determined the same way.

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \sec x = \sec x \cdot \tan x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \csc x = -\csc x \cdot \cot x$$

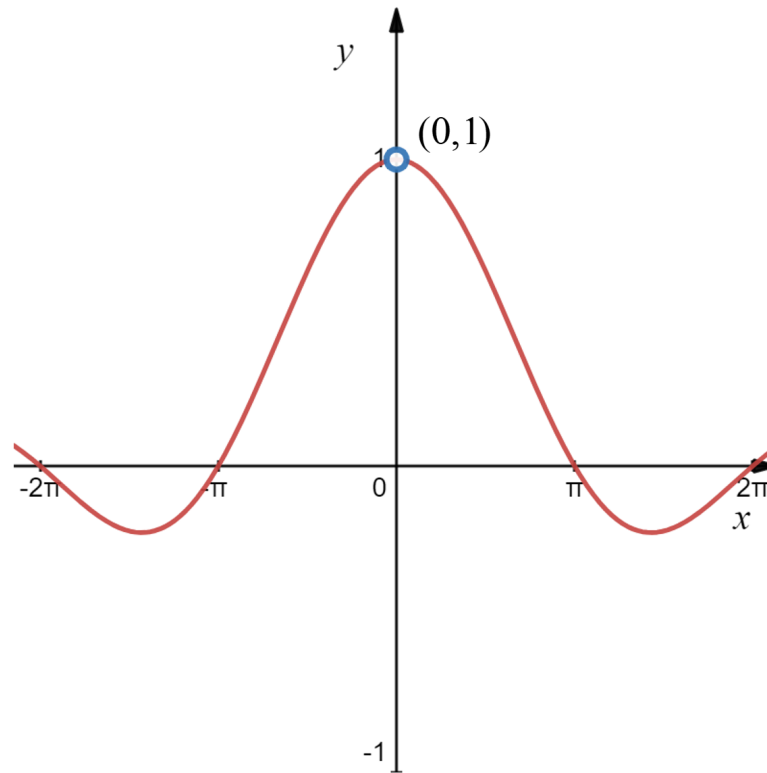
## Fundamental Limits with Sine and Cosine

Consider:

$$f(x) = \frac{\sin x}{x}$$

Hence:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$



## Fundamental Limits with Sine and Cosine

Consider:

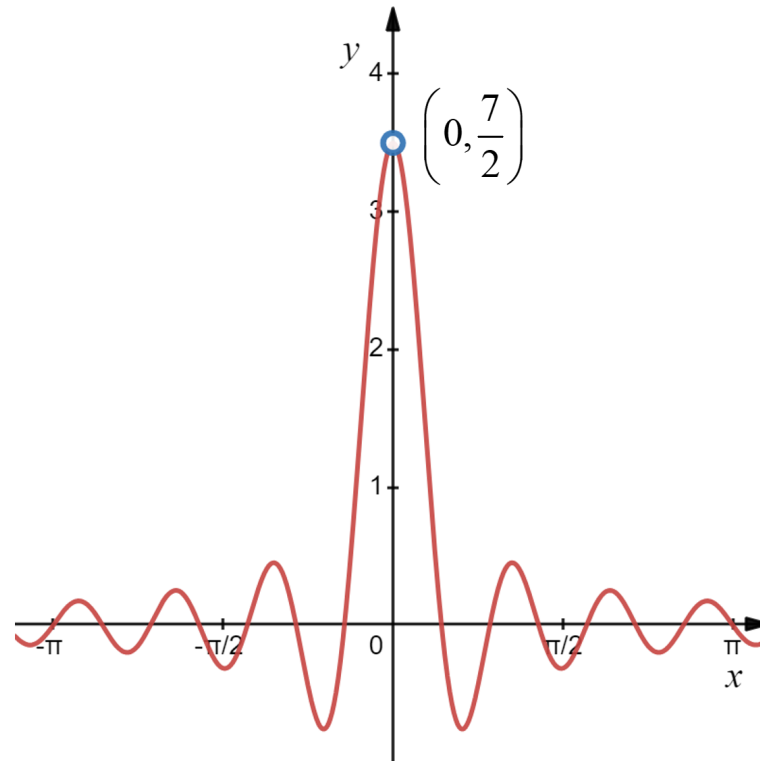
$$f(x) = \frac{\sin(7x)}{2x}$$

So

$$\lim_{x \rightarrow 0} \frac{\sin(7x)}{2x} = \frac{7}{2}$$

Hence:

$$\lim_{x \rightarrow 0} \frac{\sin(mx)}{nx} = \frac{m}{n}$$



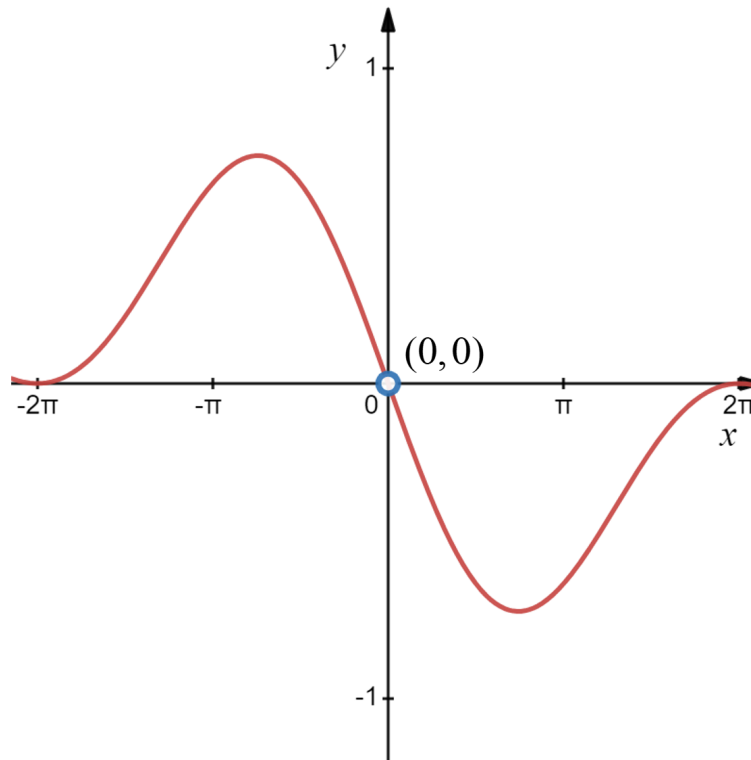
## Fundamental Limits with Sine and Cosine

Consider:

$$f(x) = \frac{\cos x - 1}{x}$$

Hence:

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$$



## Fundamental Limits with Sine and Cosine

Consider:

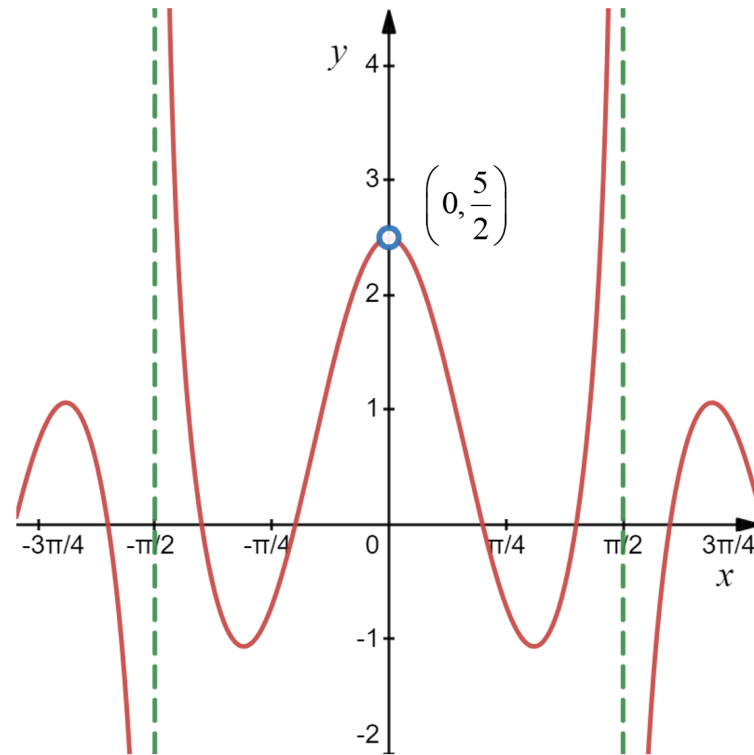
$$f(x) = \frac{\sin(5x)}{\sin(2x)}$$

So

$$\lim_{x \rightarrow 0} \frac{\sin(5x)}{\sin(2x)} = \frac{5}{2}$$

Hence:

$$\lim_{x \rightarrow 0} \frac{\sin(mx)}{\sin(nx)} = \frac{m}{n}$$



**Example** Differentiate.

$$(a) \quad g(x) = 3x + x^2 \cos x$$

$$(b) \quad g(z) = \frac{z}{\sec z + \tan z}$$

$$(c) \quad y = \frac{\cos x}{1 - \sin x}$$

**Solution**

$$(a) \quad g(x) = 3x + x^2 \cos x$$

$$\begin{aligned} g'(x) &= 3 + x^2(-\sin x) + (\cos x)(2x) \\ &= 3 - x^2 \sin x + 2x \cos x \end{aligned}$$

**Example** Differentiate.

(a)  $g(x) = 3x + x^2 \cos x$

(b)  $g(z) = \frac{z}{\sec z + \tan z}$

(c)  $y = \frac{\cos x}{1 - \sin x}$

**Solution**

(a)  $g(x) = 3x + x^2 \cos x$

$$\begin{aligned} g'(x) &= 3 + x^2(-\sin x) + (\cos x)(2x) \\ &= 3 - x^2 \sin x + 2x \cos x \end{aligned}$$

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(b)  $g(z) = \frac{z}{\sec z + \tan z}$

$$\begin{aligned} g'(z) &= \frac{(\sec z + \tan z)(1) - z(\sec z \tan z + \sec^2 z)}{(\sec z + \tan z)^2} \\ &= \frac{(\sec z + \tan z)(1) - z \sec z (\tan z + \sec z)}{(\sec z + \tan z)^2} \end{aligned}$$

$$\begin{aligned} &= \frac{(1 - z \sec z)(\sec z + \tan z)}{(\sec z + \tan z)^2} \\ &= \frac{1 - z \sec z}{\sec z + \tan z} \end{aligned}$$

**Example** Differentiate.

(a)  $g(x) = 3x + x^2 \cos x$

(b)  $g(z) = \frac{z}{\sec z + \tan z}$

(c)  $y = \frac{\cos x}{1 - \sin x}$

**Solution**

(c)  $y = \frac{\cos x}{1 - \sin x}$

$$\frac{dy}{dx} = \frac{(1 - \sin x)(-\sin x) - \cos x(-\cos x)}{(1 - \sin x)^2}$$

$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2}$$

Pythagorean Identity  $\sin^2 x + \cos^2 x = 1$

$$= \frac{-\sin x + 1}{(1 - \sin x)^2}$$

$$= \frac{1}{1 - \sin x}$$

**Example**

(a) Find an equation of the tangent line to the curve  $y = 2x \sin x$  at the point  $(\pi/2, \pi)$ .



(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

**Solution**

(a)  $y = 2x \sin x$

$$y' = 2(x \cos x + \sin x \cdot 1)$$

At  $(\frac{\pi}{2}, \pi)$ ,

$$y' = 2\left(\frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2}\right)$$

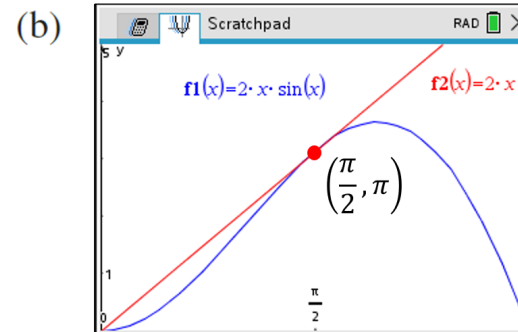
$$= 2(0 + 1)$$

$$= 2$$

An equation of the tangent line with  $m = 2$  and the point of tangency  $(\frac{\pi}{2}, \pi)$  is

$$y - \pi = 2\left(x - \frac{\pi}{2}\right)$$

$$y = 2x$$



**Example** Find  $\lim_{x \rightarrow 0} \frac{\sin 4x}{6x^3 - 7x}$

**Solution**

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin 4x}{6x^3 - 7x} &= \lim_{x \rightarrow 0} \frac{\sin 4x}{x(6x^2 - 7)} \\ &= \lim_{x \rightarrow 0} \left[ \frac{\sin 4x}{x} \cdot \frac{1}{6x^2 - 7} \right] \\ &= \lim_{x \rightarrow 0} \left[ \frac{\sin 4x}{x} \right] \cdot \lim_{x \rightarrow 0} \left[ \frac{1}{6x^2 - 7} \right] \\ &= 4 \cdot \left[ -\frac{1}{7} \right] \\ &= -\frac{4}{7}\end{aligned}$$

