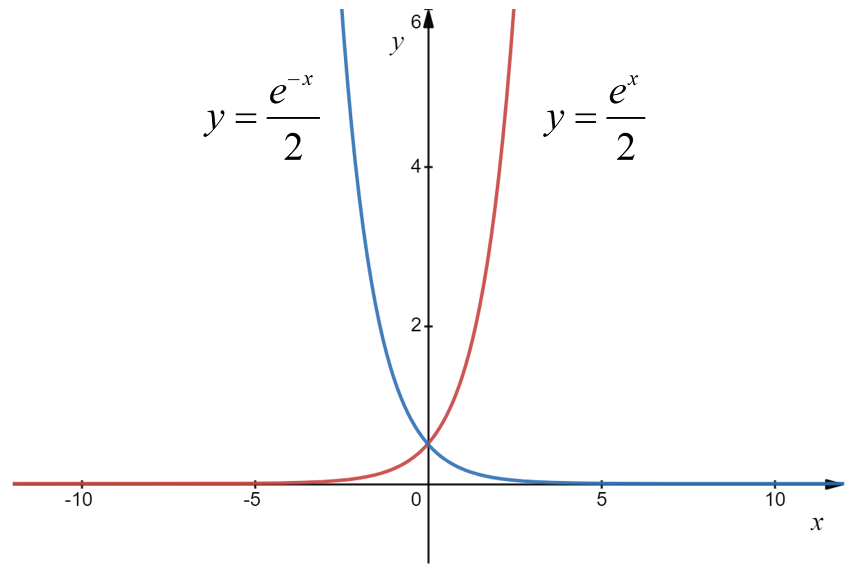
3.11 Hyperbolic Functions

Georg Friedrich Bernhard Riemann
1826 – 1866

Riemann's ideas concerning geometry of space had a profound effect on the development of modern theoretical physics. He clarified the notion of integral by defining what we now call the Riemann integral.

Let's examine the exponential functions:

$$y = \frac{e^x}{2} \text{ and } y = \frac{e^{-x}}{2}.$$

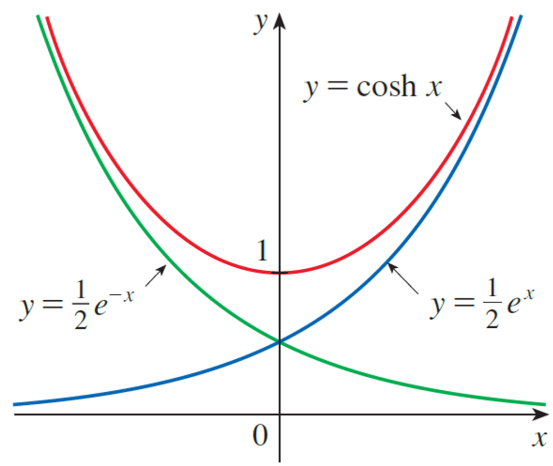


Now, lets look at the combination of these two functions with addition.

$$y = \frac{e^x}{2} + \frac{e^{-x}}{2} = \frac{e^x + e^{-x}}{2}.$$

We call this function hyperbolic cosine. $y = \frac{e^x + e^{-x}}{2} = \cosh x$

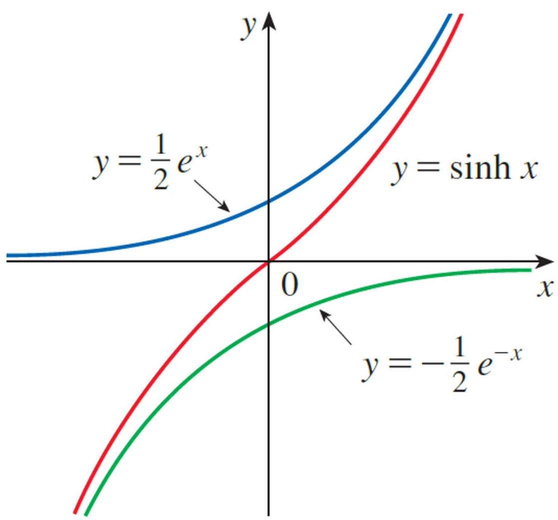
Any curve of the form $y = c + b\cosh\left(\frac{x}{a}\right)$ is called a catenary curve. (Note: x is not an angle!)

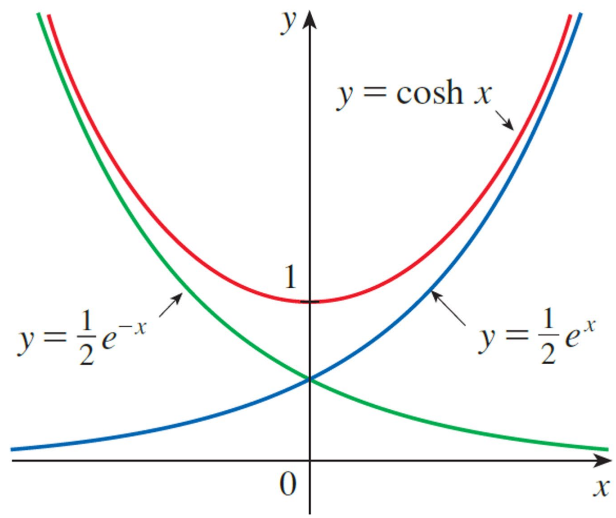


Next, lets look at the combination of these two functions with subtraction.

$$y = \frac{e^x}{2} - \frac{e^{-x}}{2} = \frac{e^x - e^{-x}}{2}.$$

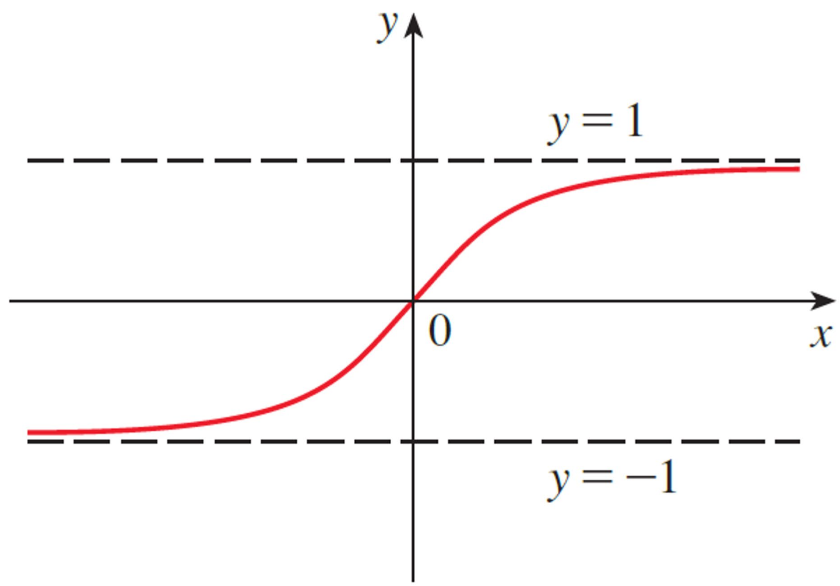
We call this function hyperbolic sine. $y = \dfrac{e^x - e^{-x}}{2} = \sinh x.$





Note that sinh has domain $\mathbb{R}$ and range $\mathbb{R}$, while cosh has domain $\mathbb{R}$ and range $[1, \infty)$.

$y = \tanh x$



The graph of tanh is shown above. Domain is $\mathbb{R}$, and it's range is $(-1,1)$.
It has the horizontal asymptotes $y = \pm 1$.

Why hyperbolic, you ask? Watch this…

Take the unit hyperbola

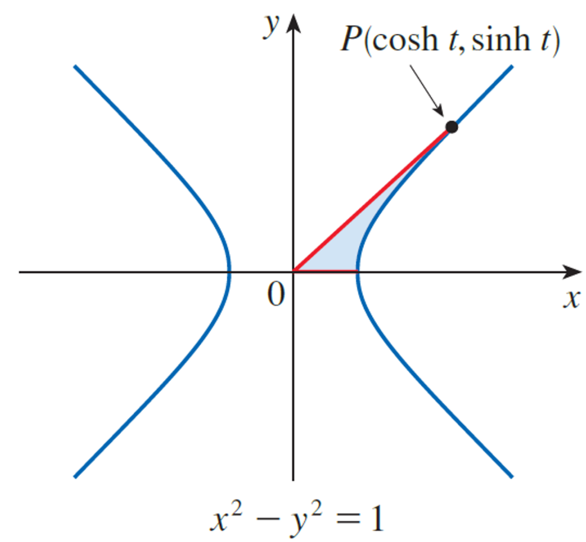
$$x^2 - y^2 = 1$$

Let

$$\cosh t = \frac{e^t + e^{-t}}{2}$$

and

$$\sinh t = \frac{e^t - e^{-t}}{2}.$$



Substituting into the unit hyperbola we get

$$x^2 - y^2 = 1$$

$$\cosh^2 t - \sinh^2 t = \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2$$

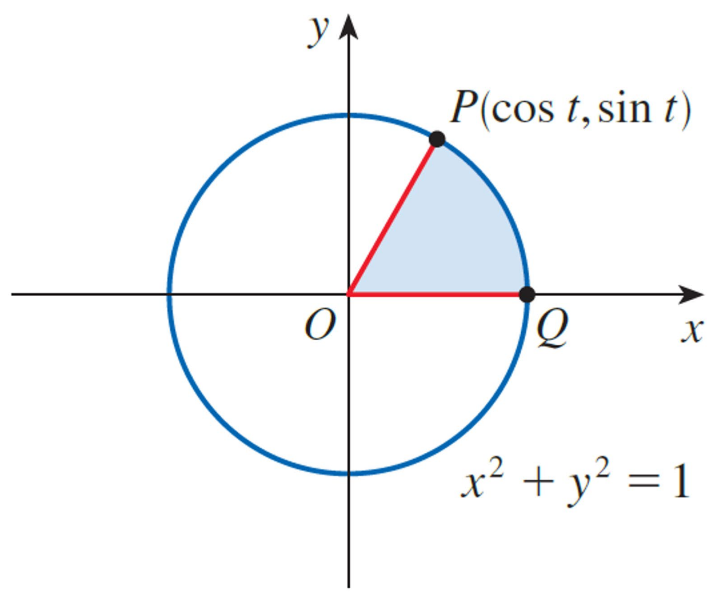
$$= \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4}$$

$$= \frac{4}{4}$$

$$= 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

If t is any real number, then the point $P(\cos t, \sin t)$ lies on the unit circle $x^2 + y^2 = 1$ because $\cos^2 t + \sin^2 t = 1$. In fact, t can be interpreted as the radian measure of $\angle POQ$ in Figure 6. For this reason the trigonometric functions are sometimes called *circular* functions.



Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

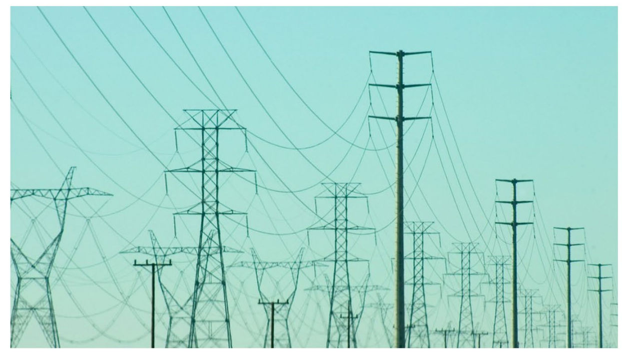
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

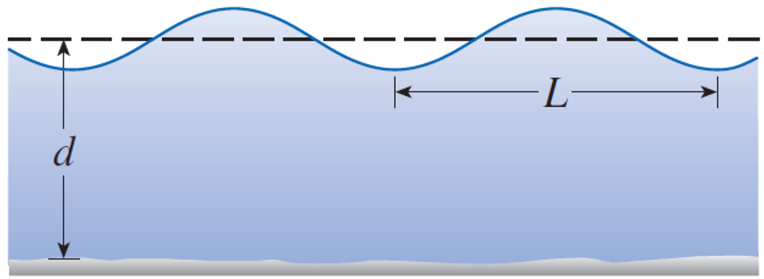
Some of the mathematical uses of hyperbolic functions will be seen in Chapter 7. Applications to science and engineering occur whenever an entity such as light, velocity, electricity, or radioactivity is gradually absorbed or extinguished because the decay can be represented by hyperbolic functions. The most famous application is the use of hyperbolic cosine to describe the shape of a hanging wire. It can be proved that if a heavy flexible cable (such as an overhead power line) is suspended between two points at the same height, then it takes the shape of a curve with equation

$$y = c + a\cosh(x/a)$$

called a *catenary* (The Latin word *catena* means "chain.")



Idealized Ocean Wave



Another application of hyperbolic functions occurs in the description of ocean waves: the velocity of a water wave with length L moving across a body of water with depth d is modeled by the function

$$v = \sqrt{\frac{gL}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)}$$

where g is the acceleration due to gravity.

The hyperbolic functions satisfy a number of identities that are similar to well-known trigonometric identities.

Hyperbolic Identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

I can read your minds, what are their derivatives?

1 Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

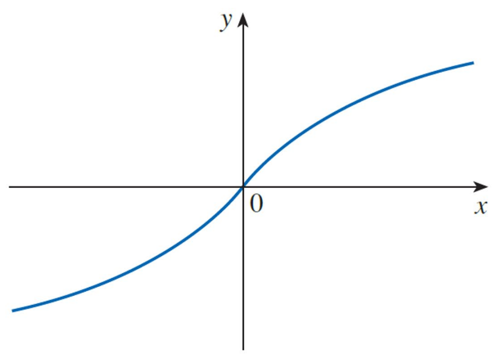
$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

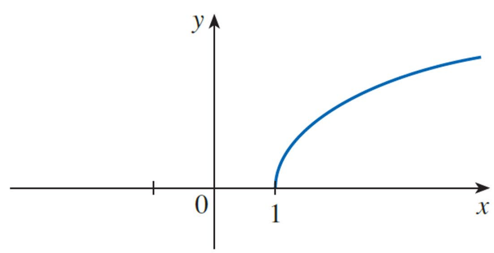
Note the analogy with the differentiation formulas for trigonometric functions, but note that the signs are different in some cases.

Inverse Hyperbolic Functions and Their Derivatives

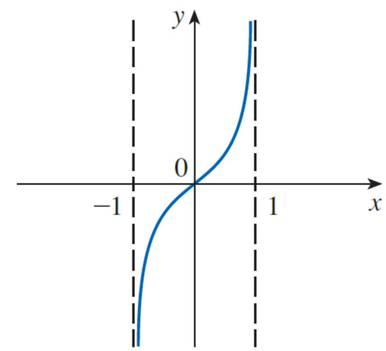
$$y = \sinh^{-1}x \iff \sinh y = x$$
$$y = \cosh^{-1}x \iff \cosh y = x \quad \text{and} \quad y \geq 0$$
$$y = \tanh^{-1}x \iff \tanh y = x$$



$y = \sinh^{-1}x$
domain $= \mathbb{R}$ range $= \mathbb{R}$



$y = \cosh^{-1}x$
domain $= [1, \infty)$ range $= [0, \infty)$



$y = \tanh^{-1}x$
domain $= (-1, 1)$ range $= \mathbb{R}$

Since the hyperbolic functions are defined in terms of exponential functions, it's not surprising to learn that the inverse hyperbolic functions can be expressed in terms of logarithms. In particular, we have:

3 $$\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad x \in \mathbb{R}$$

4 $$\cosh^{-1}x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad x \geqslant 1$$

5 $$\tanh^{-1}x = \tfrac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) \qquad -1 < x < 1$$

6 Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1 + x^2}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2 + 1}}$$

$$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2 - 1}} \qquad \frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1 - x^2} \qquad \frac{d}{dx}(\coth^{-1}x) = \frac{1}{1 - x^2}$$

Example Find the derivative. Simplify where possible.

(a) $F(t) = \ln(\sinh t)$ (b) $f(t) = \dfrac{1 + \sinh t}{1 - \sinh t}$ (c) $y = \cosh^{-1}(\sec \theta) \quad 0 \leqslant \theta < \pi/2$

Solution

(a) $F(t) = \ln(\sinh t)$

$$F'(t) = \frac{1}{\sinh t} \cdot \frac{d}{dt} \sinh t$$

$$= \frac{1}{\sinh t} \cosh t$$

$$= \coth t$$

(b) $f(t) = \dfrac{1 + \sinh t}{1 - \sinh t}$

$$f'(t) = \frac{(1 - \sinh t)\ \cosh t - (1 + \sinh t)(-\cosh t)}{(1 - \sinh t)^2}$$

$$= \frac{\cosh t - \cancel{\sinh t \cosh t} + \cosh t + \cancel{\sinh t \cosh t}}{(1 - \sinh t)^2}$$

$$= \frac{2 \cosh t}{(1 - \sinh t)^2}$$

Example Find the derivative. Simplify where possible.

(a) $F(t) = \ln(\sinh t)$ (b) $f(t) = \dfrac{1 + \sinh t}{1 - \sinh t}$ (c) $y = \cosh^{-1}(\sec\theta) \quad 0 \leqslant \theta < \pi/2$

Solution

(c) $y = \cosh^{-1}(\sec\theta)$

$$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2 - 1}}$$

$$y' = \frac{1}{\sqrt{\sec^2\theta - 1}} \cdot \frac{d}{d\theta}(\sec\theta)$$

$$= \frac{1}{\sqrt{\tan^2\theta}} \cdot \sec\theta\ \tan\theta$$

$$= \frac{1}{\tan\theta} \cdot \sec\theta\ \tan\theta$$

$$= \sec\theta$$

Example

The Gateway Arch The Gateway Arch in St. Louis was designed by Eero Saarinen and was constructed using the equation

$$y = 211.49 - 20.96 \cosh 0.03291765x$$

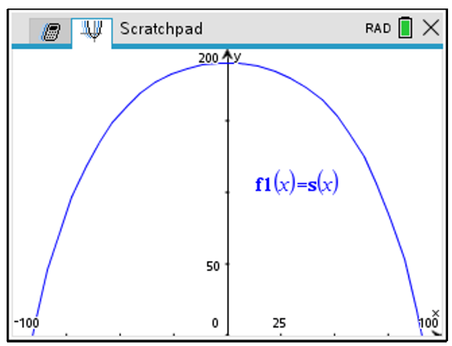
for the central curve of the arch, where x and y are measured in meters and $|x| \leq 91.20$.

(a) Graph the central curve.
(b) What is the height of the arch at its center?
(c) At what points is the height 100 m?
(d) What is the slope of the arch at the points in part (c)?

Solution

(a) Graph the central curve.

Let $y = s(x)$



Solution

(b) What is the height of the arch at its center?

Height of the arch at its center in when $x = 0$.

The height of the arch at its center is 190.53 m.

(c) At what points is the height 100 m?

Let $s(x) = 100$, and solve.

$$x \overset{CAS}{\approx} \pm 71.558 \text{ m}$$

The points are approximately $(\pm 71.558, 100)$.

(d) What is the slope of the arch at the points in part (c)?

So, the slope at (71.558, 100) is about −3.605 and the slope at (−71.558, 100) is about 3.605.