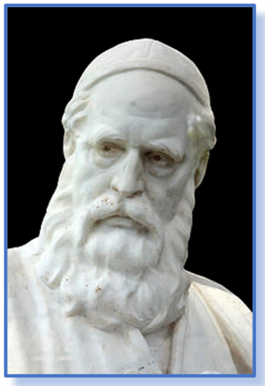
3 Differentiation Rules

WE HAVE SEEN HOW TO INTERPRET derivatives as slopes and rates of change. We have used the definition of a derivative to calculate the derivatives of functions defined by formulas. But it would be tedious if we always had to use the definition, so in this chapter we develop rules for finding derivatives without having to use the definition directly. These differentiation rules enable us to calculate with relative ease the derivatives of polynomials, rational functions, algebraic functions, exponential and logarithmic functions, and trigonometric and inverse trigonometric functions. We then use these rules to solve problems involving rates of change and the approximation of functions.

3.1 Derivatives of Polynomials and Exponential Functions

3.2 The Product and Quotient Rules



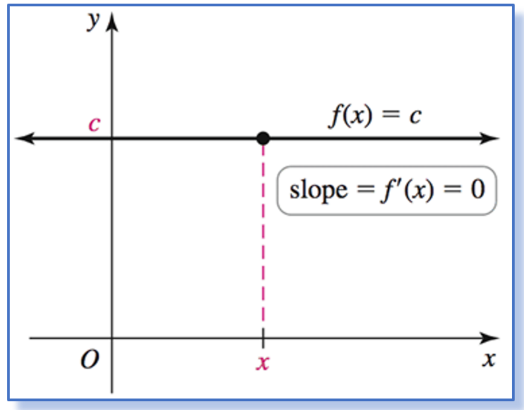
Omar Khayyam
1048 – 1131

Omar Khayyam was a Persian mathematician, astronomer, philosopher, and poet, who is widely considered to be one of the most influential scientists of the middle ages. He wrote numerous treatises on mechanics, geography, mineralogy, and astronomy.

If the derivative of a function is its slope, then for a constant function, the derivative must be zero.

$$\frac{d}{dx}(c) = 0$$

The derivative of a constant is zero.



Example Differentiate:

$$y = 3$$

$$y' = 0$$

We saw that if $y = x^2$, then $y' = 2x$.

This is part of a pattern.

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$ ← **Power Rule**

Example Differentiate:

(a) $f(x) = x^4$ (b) $y = x^{4/5}$

Solution

(a) $f'(x) = 4x^{4-1}$

$= 4x^3$

(b) $y = x^{4/5}$

$$\frac{dy}{dx} = \frac{4}{5}x^{-1/5}$$

$$\frac{dy}{dx} = \frac{4}{5x^{1/5}} = \frac{4}{5\sqrt[5]{x}}$$

Constant Multiple Rule

$$\frac{d}{dx}(c \cdot f) = c \cdot \frac{df}{dx}$$

The derivative of a constant times a function is the constant time the derivative of the function.

Example Differentiate:

(a) $\frac{d}{dx} cx^n = c \cdot \frac{d}{dx} x^n = cnx^{n-1}$

The Power Rule (General Version)

(b) $\frac{d}{dx} 9x^{2/3} = 9 \cdot \frac{2}{3} x^{-1/3} = 6x^{-1/3} = \frac{6}{x^{1/3}}$

(c) $f(x) = \frac{-15}{x^6} = -15x^{-6}$

$f'(x) = -15(-6)x^{-7} = 90x^{-7} = \frac{90}{x^7}$

Sum and Difference Rules

$$\frac{d}{dx}\left(f(x)\pm g(x)\right)=\frac{d}{dx}f(x)\pm\frac{d}{dx}g(x)$$

$$\left(f\pm g\right)'=f'\pm g'$$

Example Differentiate:

(a) $y=x^4+32x$ (b) $y=6x^4-3x^2+2^\pi$ (c) $f(x)=\frac{12}{7}x^{7/3}-24x^{4/3}+9$

Solution

(a) $y=x^4+32x$

$y'=4x^3+32$

$y'=4(x^3+8)$

$y'=4(x+2)(x^2-2x+4)$

(b) $y=6x^4-3x^2+2^\pi$

$y'=24x^3-6x$

$y'=6x(4x^2-1)$

$y'=6x(2x-1)(2x+1)$

(c) $f(x)=\frac{12}{7}x^{7/3}-24x^{4/3}+9$

$\frac{d}{dx}\left[f(x)\right]=4x^{4/3}-32x^{1/3}$

$\frac{d}{dx}\left[f(x)\right]=4x^{1/3}\cdot(x-8)$

Product Rule Notice that this is not just the product of two derivatives.

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right)=f(x)\cdot\frac{d}{dx}g(x)+g(x)\cdot\frac{d}{dx}f(x)$$

$$\left(fg\right)'=f\cdot g'+g\cdot f'$$

Example Differentiate:

$$\left(fg\right)'=f\cdot g'+g\cdot f'$$

$$\frac{d}{dx}\left[\left(x^2+3\right)\left(2x^3+5x\right)\right]=\left(x^2+3\right)\left(6x^2+5\right)+\left(2x^3+5x\right)\left(2x\right)$$

$$=\frac{d}{dx}\left(2x^5+5x^3+6x^3+15x\right)$$

$$=\frac{d}{dx}\left(2x^5+11x^3+15x\right)$$

$$=10x^4+33x^2+15$$

$$=6x^4+5x^2+18x^2+15+4x^4+10x^2$$

$$=10x^4+33x^2+15$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\cdot\frac{d}{dx}f(x)-f(x)\cdot\frac{d}{dx}g(x)}{\left[g(x)\right]^2}$$

$$\left(\frac{f}{g}\right)'=\frac{g\cdot f'-f\cdot g'}{g^2}$$

"Low dee high minus high dee low draw the line and denominator squared we go."

Example Differentiate:

$$\frac{d}{dx}\left(\frac{2x^3+5x}{x^2+3}\right)=\frac{\left(x^2+3\right)\left(6x^2+5\right)-\left(2x^3+5x\right)\left(2x\right)}{\left(x^2+3\right)^2}$$

Derivative of the Natural Exponential Function

$$\frac{d}{dx}\left(e^{x}\right)=e^{x}$$

TABLE OF DIFFERENTIATION FORMULAS

Shortcut Formulas

$\frac{d}{dx}(c) = 0$	$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(e^x) = e^x$
$(cf)' = cf'$	$(f+g)' = f' + g'$	$(f-g)' = f' - g'$
$(fg)' = fg' + gf'$	$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$	

Use $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ only when asked to.

Example Differentiate the function.

(a) $G(r) = \dfrac{3r^{3/2} + r^{5/2}}{r}$ (b) $F(z) = \dfrac{A + Bz + Cz^2}{z^2}$ (c) $g(x) = \left(x + 2\sqrt{x}\right) e^x$

(d) $h(r) = \dfrac{ae^r}{b + e^r}$ (e) $f(x) = \dfrac{x^2 e^x}{x^2 + e^x}$

Solution

(a)

$$G(r) = \frac{3r^{3/2} + r^{5/2}}{r}$$

$$= \frac{3r^{3/2}}{r} + \frac{r^{5/2}}{r}$$

$$= 3r^{3/2-2/2} + r^{5/2-2/2}$$

$$= 3r^{1/2} + r^{3/2}$$

Power Functions

$$G'(r) = 3\left(\tfrac{1}{2}r^{-1/2}\right) + \tfrac{3}{2}r^{1/2}$$

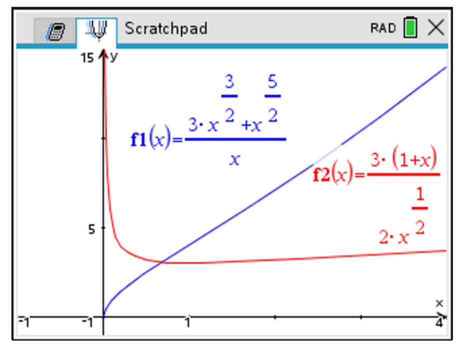
$$= \tfrac{3}{2}r^{-1/2} + \tfrac{3}{2}r^{1/2}$$

$$= \frac{3}{2}r^{-1/2}(1+r)$$

$$= \frac{3(1+r)}{2r^{1/2}}$$

Power Rule

$$\frac{d}{dx}cx^n = cnx^{n-1}$$



Example Differentiate the function.

(b) $F(z) = \dfrac{A + Bz + Cz^2}{z^2}$ (c) $g(x) = \left(x + 2\sqrt{x}\right) e^x$

Solution

(b) $F(z) = \dfrac{A + Bz + Cz^2}{z^2}$

$$= \frac{A}{z^2} + \frac{Bz}{z^2} + \frac{Cz^2}{z^2}$$

$$= Az^{-2} + Bz^{-1} + C$$

Power Rule

$$\frac{d}{dx} cx^n = cnx^{n-1}$$

$$F'(z) = A(-2z^{-3}) + B(-1z^{-2}) + 0$$

$$= -2Az^{-3} - Bz^{-2}$$

$$= -\frac{2A}{z^3} - \frac{B}{z^2}$$

$$= -\frac{2A + Bz}{z^3}$$

(c) $g(x) = \left(x + 2\sqrt{x}\right) e^x$

Product Rule

$$(fg)' = fg' + gf'$$

$$g'(x) = (x + 2\sqrt{x})(e^x)' + e^x(x + 2\sqrt{x})'$$

$$= (x + 2\sqrt{x})e^x + e^x\left(1 + 2 \cdot \tfrac{1}{2}x^{-1/2}\right)$$

$$= e^x\left[\left(x + 2\sqrt{x}\right) + \left(1 + \frac{1}{\sqrt{x}}\right)\right]$$

$$= e^x\left[x + 1 + 2\sqrt{x} + \frac{1}{\sqrt{x}}\right]$$

Example Differentiate the function.

Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

(d) $h(r) = \dfrac{ae^r}{b + e^r}$ (e) $f(x) = \dfrac{x^2e^x}{x^2 + e^x}$

Solution

(d) $h(r) = \dfrac{ae^r}{b + e^r}$

$$\frac{dh}{dr} = \frac{(b + e^r)(ae^r) - (ae^r)(e^r)}{(b + e^r)^2}$$

$$= \frac{abe^r + \cancel{ae^{2r}} - \cancel{ae^{2r}}}{(b + e^r)^2}$$

$$= \frac{abe^r}{(b + e^r)^2}$$

(e) $f(x) = \dfrac{x^2e^x}{x^2 + e^x}$

$$\frac{df}{dx} = \frac{(x^2 + e^x)\left[x^2e^x + e^x(2x)\right] - x^2e^x(2x + e^x)}{(x^2 + e^x)^2}$$

$$= \frac{x^4e^x + \cancel{2x^3e^x} + \cancel{x^2e^{2x}} + 2xe^{2x} - \cancel{2x^3e^x} - \cancel{x^2e^{2x}}}{(x^2 + e^x)^2}$$

$$= \frac{x^4e^x + 2xe^{2x}}{(x^2 + e^x)^2}$$

$$= \frac{xe^x(x^3 + 2e^x)}{(x^2 + e^x)^2}$$

Example

Find the horizontal tangent lines of $y = x^4 - 2x^2 + 2$.

Solution

$$\frac{dy}{dx} = 4x^3 - 4x$$

Horizontal tangent lines occur when slope is zero.

$$4x^3 - 4x = 0$$

$$4x\left(x^2 - 1\right) = 0$$

$$4x\left(x+1\right)\left(x-1\right) = 0$$

$$x = 0, -1,\ 1$$

Plugging the x-values into the original equation, we get the points of tangency to be:

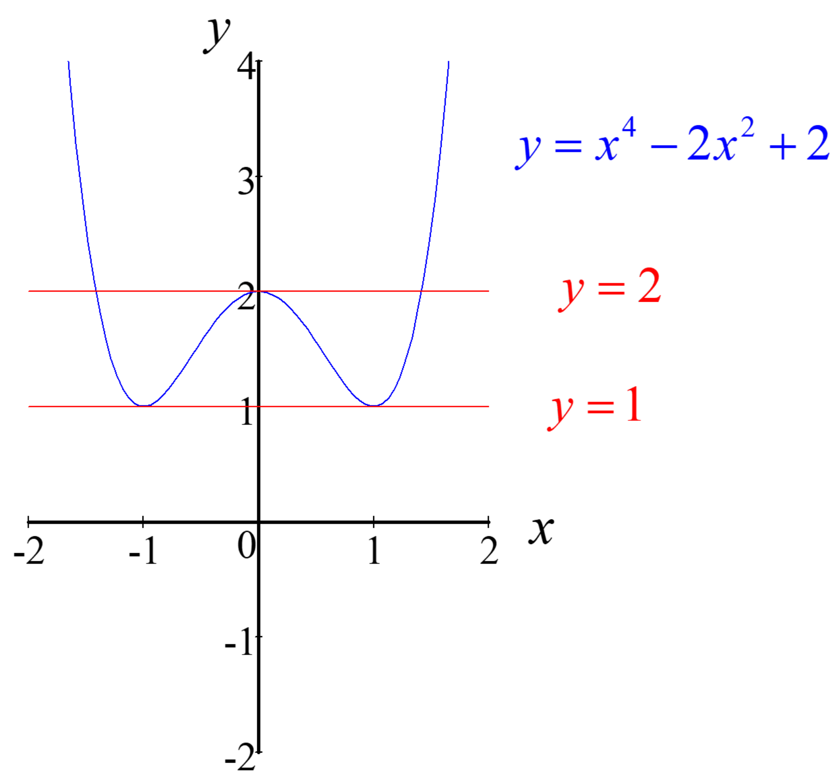
$$(0,2),\ \ (-1,1) \text{ and } (1,1)$$

Hence, the horizontal tangent lines are:

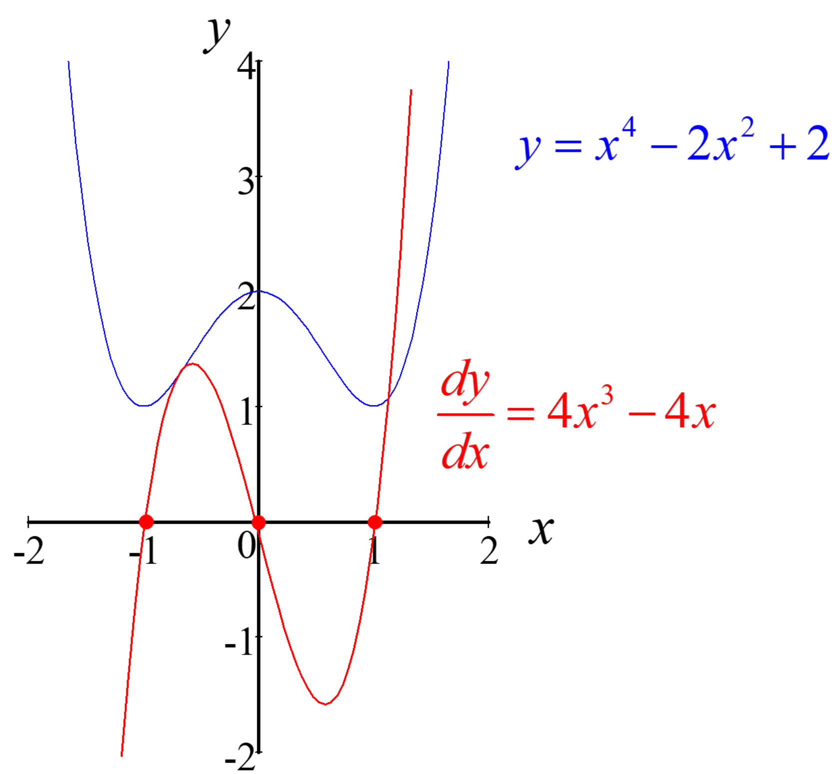
$$y = 2,\ \ y = 1 \text{ and } y = 1$$

Why do we get the same tangent line?

Let’s look at the graph.



The function is even, that is why we get the same horizontal tangent at $y = 1$.



First derivative (slope) is zero at $x = 0, -1, 1$. The x-intercepts of the graph of the derivative.

Example At what point on the curve $y = 1 + 2e^x - 3x$ is the tangent line parallel to the line $3x - y = 5$? Illustrate by graphing the curve and both lines.

Solution

The slope of $y = 1 + 2e^x - 3x$ is given by

$$m = y' = 2e^x - 3$$

The slope of $3x - y = 5$

$$y = 3x - 5 \text{ is } 3$$

$$m = 3$$

$$2e^x - 3 = 3$$

$$e^x = 3$$

$$x = \ln 3$$

The point of tangency occurs at

$$y(\ln 3) = 1 + 2 \cdot e^{\ln 3} - 3 \cdot \ln 3 = 1 + 2 \cdot 3 - 3 \cdot \ln 3$$

$$y(\ln 3) = 1 + 6 - 3 \cdot \ln 3$$

$$= 7 - 3 \cdot \ln 3$$

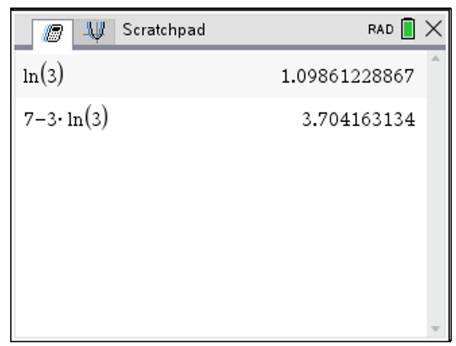
This occurs at the point $(\ln 3, 7 - 3\ln 3) \approx (1.098, 3.704)$

The equation of the tangent line is $y - y_1 = m(x - x_1)$

$$y - (7 - 3\ln 3) = 3(x - \ln 3)$$

$$y - 7 + 3\ln 3 = 3x - 3\ln 3$$

$$y = 3x + 7 - 6\ln 3$$



Example Find equations of the tangent line and normal line to the given curve at the specified point.

$$y = \frac{3x}{1 + 5x^2}, \quad \left(1, \tfrac{1}{2}\right)$$

Solution

$$y' = \frac{(1+5x^2)(3) - 3x(10x)}{(1+5x^2)^2}$$

$$= \frac{3 + 15x^2 - 30x^2}{(1+5x^2)^2}$$

$$= \frac{3 - 15x^2}{(1+5x^2)^2}$$

At $\left(1, \frac{1}{2}\right)$, $y' = \dfrac{3 - 15(1^2)}{(1 + 5 \cdot 1^2)^2} = \dfrac{-12}{6^2} = -\dfrac{1}{3}$

The slope at $\left(1, \frac{1}{2}\right)$ is $m = -\frac{1}{3}$

An equation of the tangent line at $\left(1, \frac{1}{2}\right)$ is

$$y - y_1 = m(x - x_1)$$

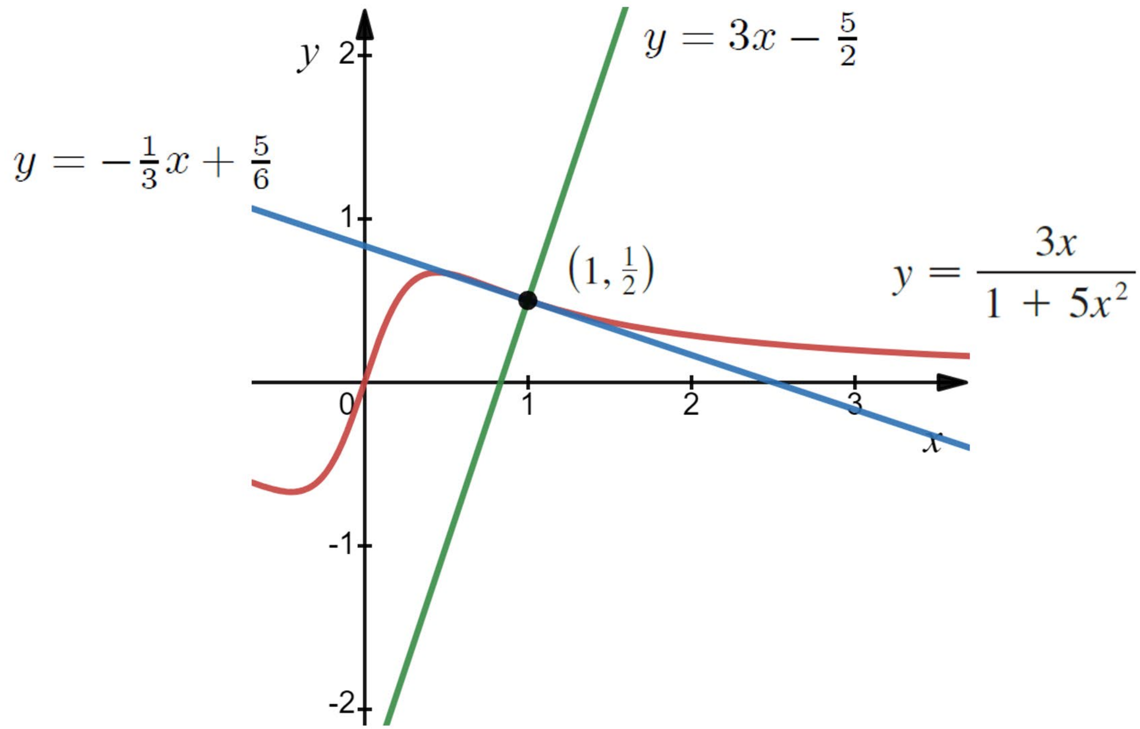
$$y - \tfrac{1}{2} = -\tfrac{1}{3}(x - 1)$$

$$y = -\tfrac{1}{3}x + \tfrac{5}{6}$$

The slope of the normal line is 3, so an equation of the normal line is

$$y - \tfrac{1}{2} = 3(x - 1)$$

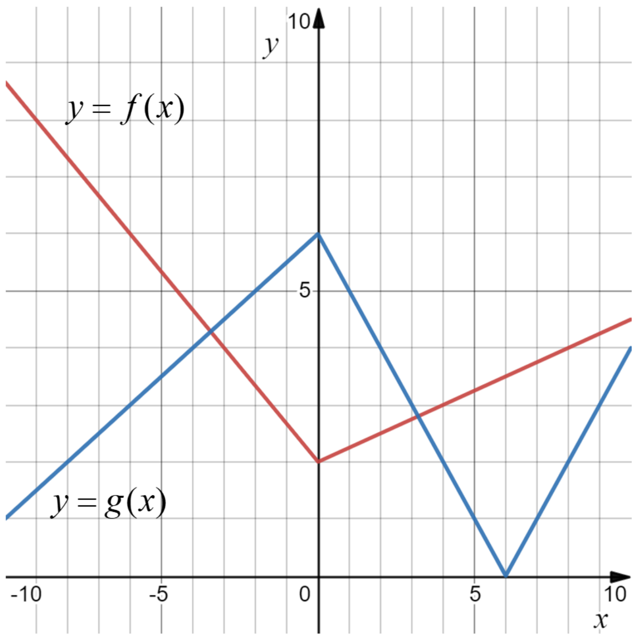
$$y = 3x - \tfrac{5}{2}$$



y
x
2
1
0
1
2
3
-1
-2
$y = 3x - \frac{5}{2}$
$y = -\frac{1}{3}x + \frac{5}{6}$
$(1, \frac{1}{2})$
$y = \dfrac{3x}{1 + 5x^2}$

Example

If f and g are the function whose graphs are shown below, let $u(x)=f(x)\cdot g(x)$, and $v(x)=\dfrac{f(x)}{g(x)}$.



Find the following: (a) $u'(a),\ a=8$ (b) $\left.\dfrac{d}{dx}v(x)\right|_{x=4}$

Solution

(a) Find $u'(a),\ a=8$

$$u(x)=f(x)\cdot g(x)$$

$$u'(x)=f(x)\cdot g'(x)+g(x)\cdot f'(x)$$

$$u'(8)=f(8)\cdot g'(8)+g(8)\cdot f'(8)$$

$$u'(8)=4\cdot 1+2\cdot\frac{1}{4}=\frac{9}{2}$$

(b)
$$\left.\frac{d}{dx}v(x)\right|_{x=4}=\left.\frac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{[g(x)]^2}\right|_{x=4}$$

$$=\frac{g(4)\cdot f'(4)-f(4)\cdot g'(4)}{[g(4)]^2}$$

$$=\frac{2\cdot\frac{1}{4}-3\cdot(-1)}{[2]^2}$$

$$=\frac{\frac{1}{2}+3}{4}=\frac{\frac{7}{2}}{4}=\frac{7}{8}$$

Example If $g(x)=\dfrac{f(x)}{3x^2+x-1}$ where $f(-2)=3$ and $f'(-2)=-4$, find $g'(-2)$.

Solution

$$g'(x)=\frac{(3x^2+x-1)\cdot f'(x)-f(x)\cdot(3x^2+x-1)'}{\left[3x^2+x-1\right]^2}$$

$$g'(x)=\frac{(3x^2+x-1)\cdot f'(x)-f(x)\cdot(6x+1)}{\left[3x^2+x-1\right]^2}$$

$$g'(-2)=\frac{(3(-2)^2+(-2)-1)\cdot f'(-2)-f(-2)\cdot(6(-2)+1)}{\left[3(-2)^2+(-2)-1\right]^2}$$

$$g'(-2)=\frac{(12-2-1)\cdot(-4)-3\cdot(-12+1)}{[12-2-1]^2}$$

$$g'(-2)=\frac{(9)\cdot(-4)-3\cdot(-11)}{[9]^2}$$

$$g'(-2)=\frac{-36+33}{81}$$

$$g'(-2)=\frac{-3}{81}$$

$$g'(-2)=-\frac{1}{27}$$

Example A coin is tossed up from the top of the Freedom Tower which is 1776 feet tall. The coin is tossed up at an initial velocity of 49 ft/sec. Determine the following if the position function of the coin is given by $s(t) = 1776 + 49t - 16t^2$.

a) Find the average velocity of the coin over the time interval 1.978 second to 2.645 seconds.

b) Find the velocity function $v(t)$ for the coin.

c) Find the velocity and direction of the coin when $t = 1.978$ seconds.

d) What is the instantaneous velocity of the coin when it hits the ground?

Solution

a) Find the average velocity of the coin over the time interval 1.978 second to 2.645 seconds.

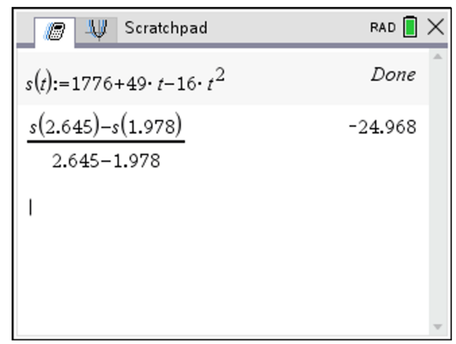
$$s(t) = 1776 + 49t - 16t^2$$

$$v_{avg} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

Let $t_1 = 1.978$ and $t_2 = 2.645$, we get

$$v_{avg} = \frac{s(2.645) - s(1.978)}{2.645 - 1.978}$$

$$v_{avg} \overset{CAS}{=} -24.968 \text{ ft/sec}$$



Solution

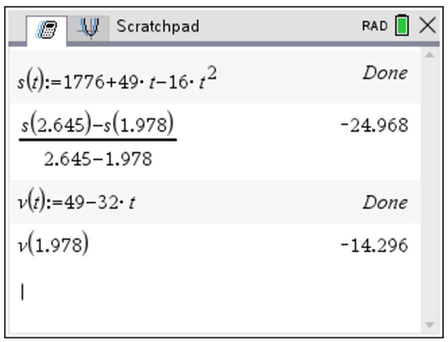
b) Find the velocity function $v(t)$ for the coin.

$$s(t) = 1776 + 49t - 16t^2$$

$$v(t) = s'(t)$$

$$v(t) = 49 - 32t$$

c) Find the velocity and direction of the coin when $t = 1.978$ seconds.



$$v(1.978) \overset{CAS}{=} -14.296 \text{ ft/sec}$$

The coin is falling at 14.296 ft/sec.

Solution

d) What is the instantaneous velocity of the coin when it hits the ground?

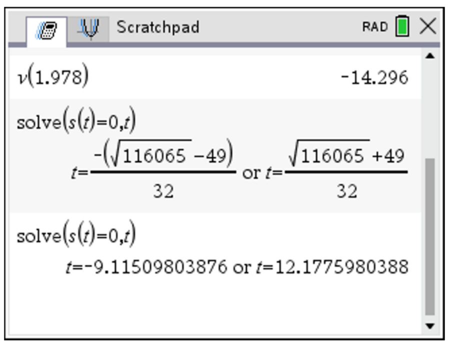
To determine the time for the coin to hit the ground we let $s(t) = 0$.

$$s(t) = 1776 + 49t - 16t^2$$

$$1776 + 49t - 16t^2 = 0$$

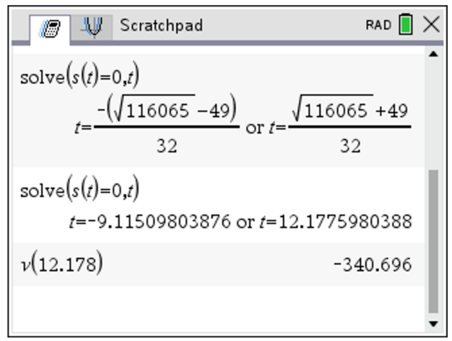
$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Or better yet, let's use CAS.



It takes 12.178 seconds for the coin to hit the ground. So we get

$$v(t) = 49 - 32t$$



$$v(12.178) \overset{CAS}{=} -340.696 \text{ ft/sec}$$

The coin hits the ground at a velocity of -340.696 ft/sec.

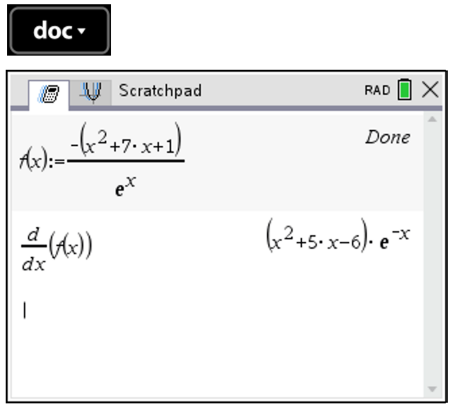
Example Consider the function $f(x) = -\dfrac{x^2+7x+1}{e^x}$.

a) Find $f'(x)$.

b) Find the x-coordinates of all the horizontal tangent lines to the function $f(x)$.

c) Find the equations of the tangent and normal lines at the x-values you found in part (b).

Solution



a) Find $f'(x)$.

Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Or better yet, let's use CAS.

$$f'(x) \overset{CAS}{=} (x^2+5x-6)\cdot e^{-x}$$

$$f'(x) = \frac{x^2+5x-6}{e^x}$$

Solution

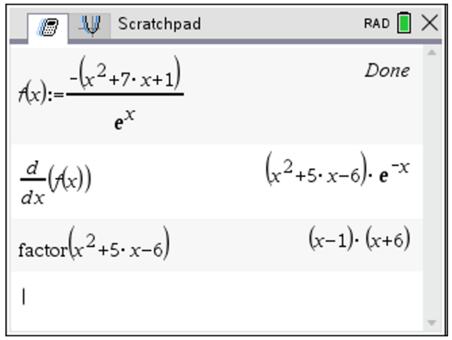
b) Find the x-coordinates of all the horizontal tangent lines to the function $f(x)$.

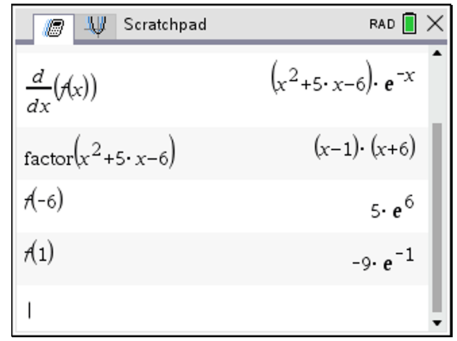
Let $f'(x)=0$.

$$\frac{x^2+5x-6}{e^x}=0$$

$$x^2+5x-6=0$$

$$(x+6)(x-1)=0$$





$x=-6$ and $x=1$ are the x-coordinates of all the horizontal tangent lines to the function $f(x)$

c) Find the equations of the tangent and normal lines at the x-values you found in part (b).

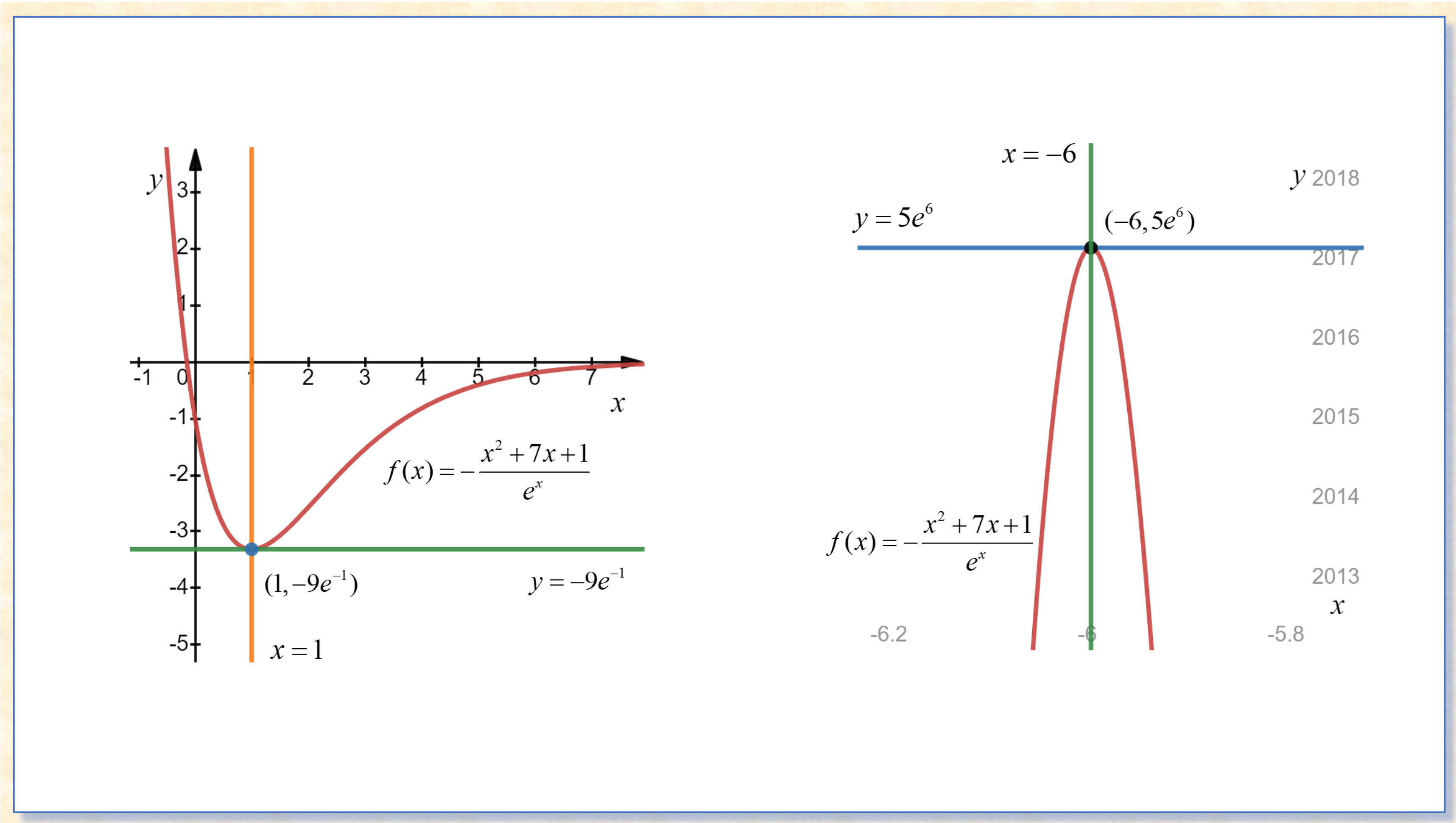
The points of tangency are

$(-6, f(-6))$ and $(1, f(1))$

$(-6, 5e^6)$ and $(1, -9e^{-1})$

The tangent lines are $y=5e^6$ and $y=-9e^{-1}$.

The normal lines are $x=-6$ and $x=1$.


y
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-3
-4
-5
$f(x) = -\frac{x^2+7x+1}{e^x}$
$(1, -9e^{-1})$
$y = -9e^{-1}$
$x = 1$
$x = -6$
$y = 5e^6$
$(-6, 5e^6)$
y 2018
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2016
2015
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x
$f(x) = -\frac{x^2+7x+1}{e^x}$
-6.2
-6
-5.8