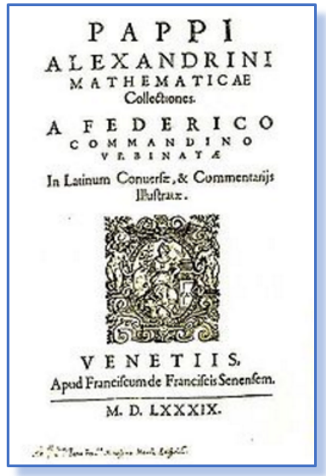
2.8 The Derivative as a Function



PAPPI
ALEXANDRINI
MATHEMATICAE
Collectiones.

A FEDERICO
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VRBINATE

In Latinum Conuersæ, & Commentarijs
Illustratæ.

VENETIIS.
Apud Franciscum de Franciscis Senensem.

M. D. LXXXIX.

Pappus of Alexandria
290 – 350 A.D.

Pappus of Alexandria is the last of the great Greek geometers and one of his theorems is cited as the basis of modern projective geometry.

Definition **The Derivative**

The **derivative** of f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists and x is in the domain of f. If $f'(x)$ exists, we say f is **differentiable** at x. If f is differentiable at every point of an open interval I, we say that f is differentiable on I.

"The derivative of f with respect to x is …" These words are going to become very familiar.

In summary, here are all the forms that represent the derivative of the function f.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x_1) = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

There are many ways to write the derivative of $y = f(x)$

$f'(x)$	"f prime of x"	or	"the derivative of f with respect to x"
y'	"y prime"	or	"the derivative of y with respect to x"
$\dfrac{dy}{dx}$	"dee why dee ecks"	or	"the derivative of y with respect to x"
$\dfrac{df}{dx}$	"dee eff dee ecks"	or	"the derivative of f with respect to x"
$\dfrac{d}{dx}f(x)$	"dee dee ecks uv eff uv ecks" (d dx of f of x)	or	"the derivative of f with respect to x"

Most Used Derivative Notation

$$f'(x)$$

$$y'$$

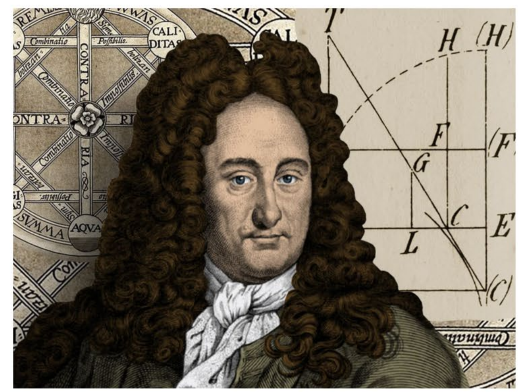
"Optimus" Prime Notation

$$\frac{dy}{dx}$$

$$\frac{df}{dx}$$

$$\frac{d}{dx}f(x)$$

Leibniz Notation



Other Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

The symbols D and d/dx are called **differentiation operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative.

$$\left.\frac{dy}{dx}\right|_{x=a} \quad \text{or} \quad \left.\frac{dy}{dx}\right]_{x=a}$$

which is a synonym for $f'(a)$. The vertical bar means "evaluate at."

Slope of Tangent Line!!

Note:

dx does not mean d times x

dy does not mean d times y

Note:

$$\frac{dy}{dx} \text{ does } \underline{\text{not}} \text{ mean } dy \div dx$$

(except when it is convenient to think of it as division.)

$$\frac{df}{dx} \text{ does } \underline{\text{not}} \text{ mean } df \div dx$$

(except when it is convenient to think of it as division.)

Note:

$$\frac{d}{dx}f(x) \text{ does } \underline{\text{not}} \text{ mean } \frac{d}{dx} \text{ times } f(x)$$

(except when it is convenient to treat it that way.)


In the future, all will become clear.

Example Consider the function $f(x) = 4 + 8x - 5x^2$.

a) Find $f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}[y]$. Recall: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, provide the limit exists.

b) Find the slope of the tangent line and normal line to the curve when $x = 1$.

c) Find the equation of the tangent and normal lines to the curve when $x = 1$.

Solution a) Find $f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}[y]$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{Provide the limit exists.}$$

$$= \lim_{h \to 0} \frac{\left[4 + 8(x+h) - 5(x+h)^2\right] - (4 + 8x - 5x^2)}{h}$$

$$= \lim_{h \to 0} \frac{4 + 8x + 8h - 5(x^2 + 2xh + h^2) - 4 - 8x + 5x^2}{h}$$

$$= \lim_{h \to 0} \frac{8h - \cancel{5x^2} - 10xh - 5h^2 + \cancel{5x^2}}{h}$$

$$= \lim_{h \to 0} \frac{8h - 10xh - 5h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(8 - 10x - 5h)}{\cancel{h}}$$

$$= \lim_{h \to 0} (8 - 10x - \cancelto{0}{5h})$$

$$= 8 - 10x$$

Solution b) Find the slope of the tangent line and normal line to the curve when $x=1$.

Slope of tangent line $m = f'(x) = 8 - 10x$

Tangent line:

1) Slope

$$m|_{x=1} = f'(1) = 8 - 10(1) = 8 - 10 = -2$$

Slope of normal line $m = -\dfrac{1}{f'(x)}$, negative reciprocal.

Normal line:

1) Slope

$$m|_{x=1} = -\frac{1}{f'(1)} = \frac{1}{2}$$

c) Find the equation of the tangent and normal lines to the curve when $x=1$.

2) Point of Tangency

$$(1, f(1)) = (1, 7)$$

$$f(x) = 4 + 8x - 5x^2$$

$$f(1) = 4 + 8(1) - 5(1)^2 = 4 + 8 - 5 = 7$$

3) Formula (Point Slope)

$$y - y_1 = m(x - x_1)$$

$$y - 7 = -2(x - 1)$$

$y = -2x + 9$, tangent line

2) Point of Tangency

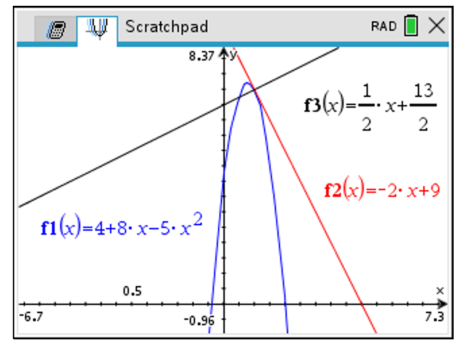
$$(1, f(1)) = (1, 7)$$

3) Formula (Point Slope)

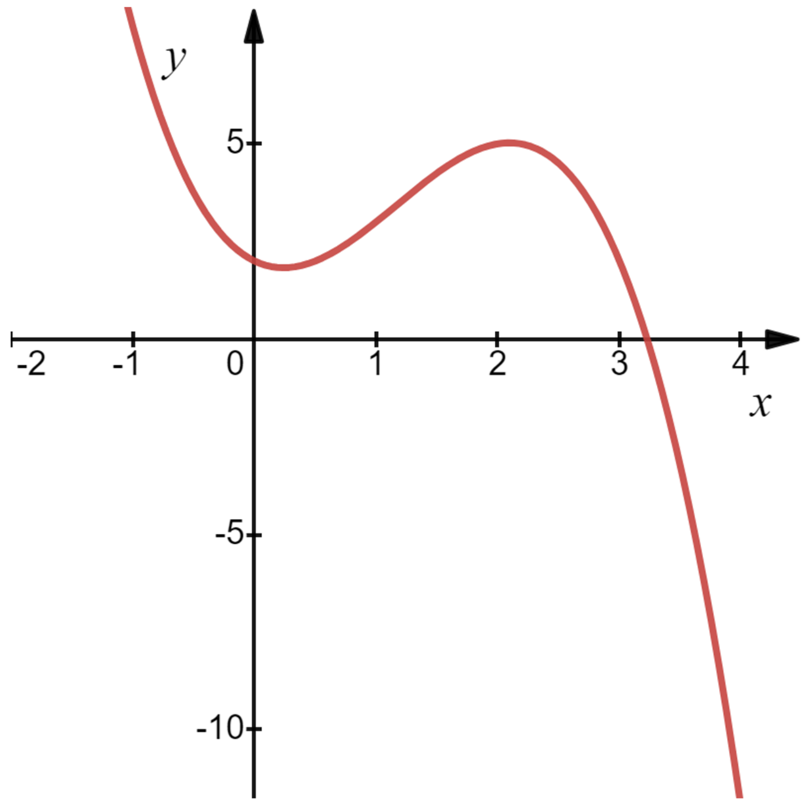
$$y - y_1 = m(x - x_1)$$

$$y - 7 = \frac{1}{2}(x - 1)$$

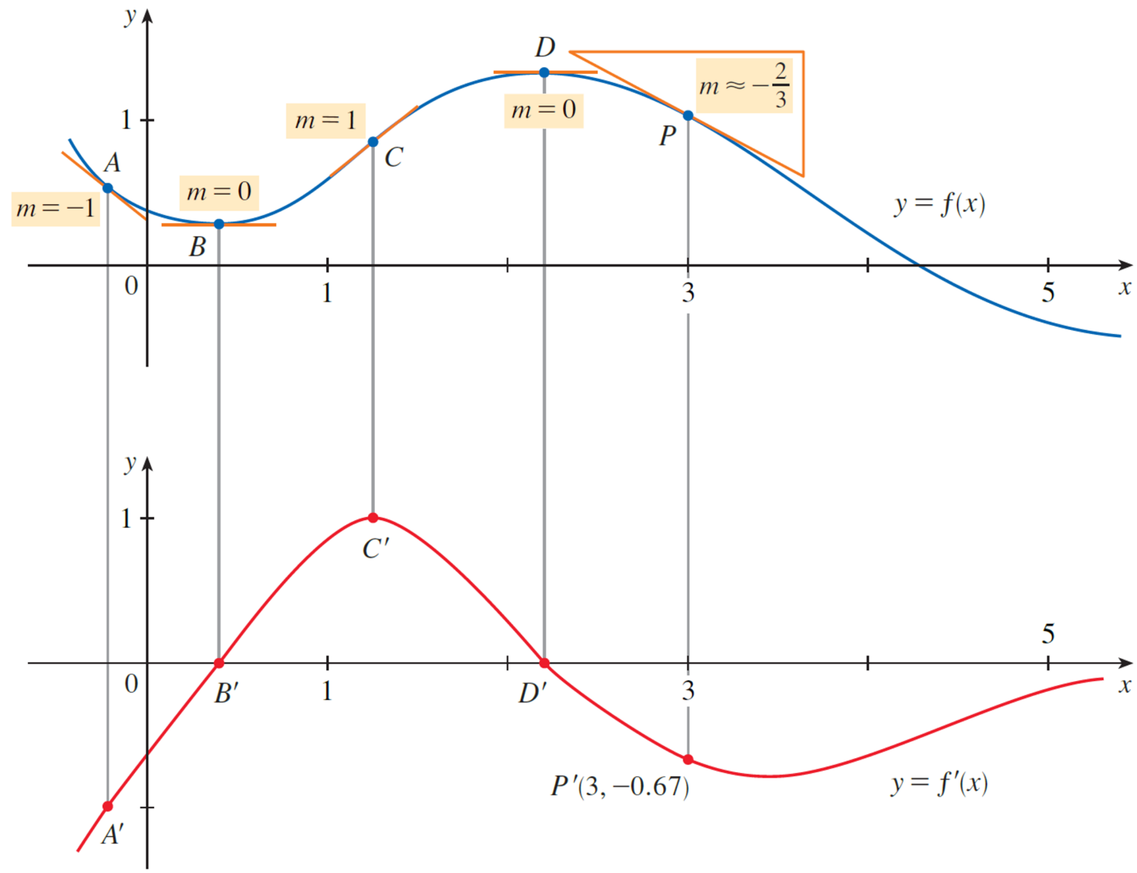
$y = \dfrac{1}{2}x + \dfrac{13}{2}$, normal line



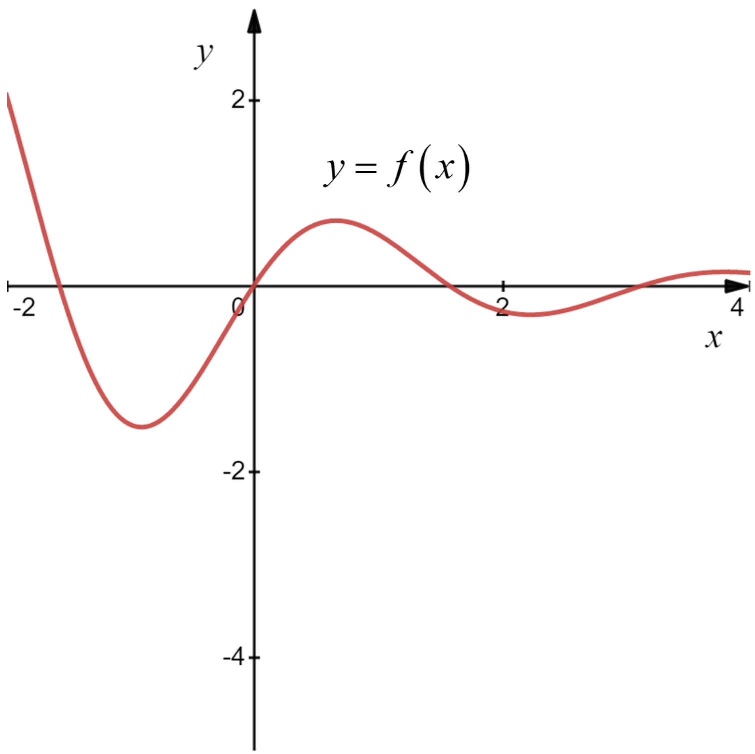
$y = f\,'(x)$ as a function.



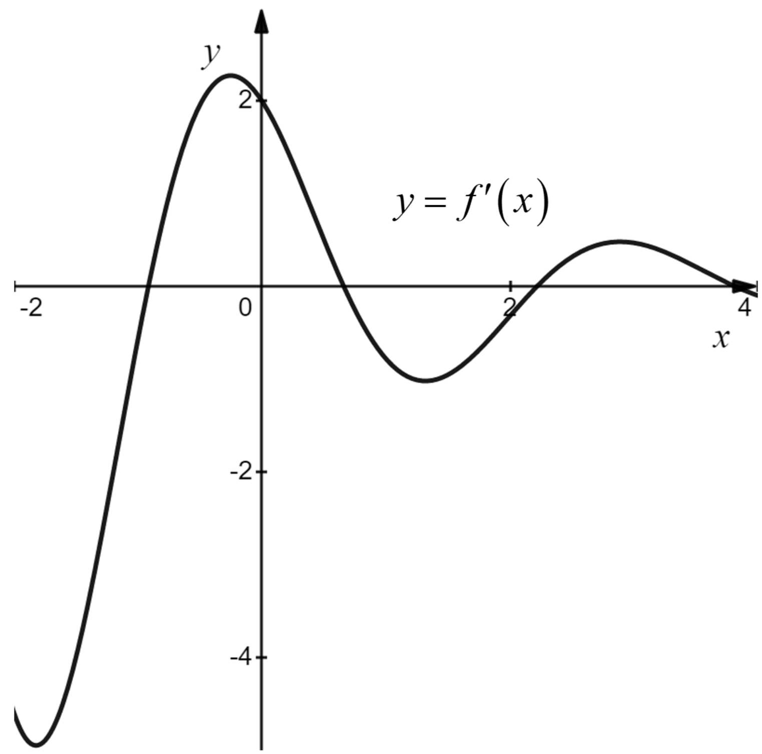
$y = f'(x)$ as a function.



Take $y = f(x)$ as a function.



$y = f'(x)$ as a function.



Example (a) If $f(x) = x + 1/x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

Solution (a) If $f(x) = x + 1/x$, find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ Provide the limit exists.

$$= \lim_{h \to 0} \frac{(x+h) + \dfrac{1}{x+h} - \left(x + \dfrac{1}{x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{(x+h)^2 + 1}{x+h} - \dfrac{x^2+1}{x}}{h} \cdot \frac{x(x+h)}{x(x+h)}$$

$$= \lim_{h \to 0} \frac{x[(x+h)^2 + 1] - (x+h)(x^2+1)}{h(x+h)x}$$

$$= \lim_{h \to 0} \frac{(x^3 + 2hx^2 + xh^2 + x) - (x^3 + x + hx^2 + h)}{h(x+h)x}$$

$$= \lim_{h \to 0} \frac{hx^2 + xh^2 - h}{h(x+h)x}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(x^2 + xh - 1)}{\cancel{h}(x+h)x}$$

$$= \lim_{h \to 0} \frac{x^2 + \cancel{xh}^{\,0} - 1}{(x + \cancel{h}_{\,0})x}$$

$$= \frac{x^2 - 1}{x^2}$$

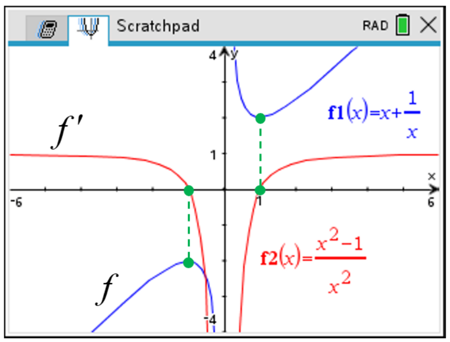
$$f'(x) = \frac{x^2 - 1}{x^2}$$

Example (a) If $f(x) = x + 1/x$, find $f'(x)$.

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

Solution (b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

$$f(x) = x + \frac{1}{x} \qquad f'(x) = \frac{x^2 - 1}{x^2}$$



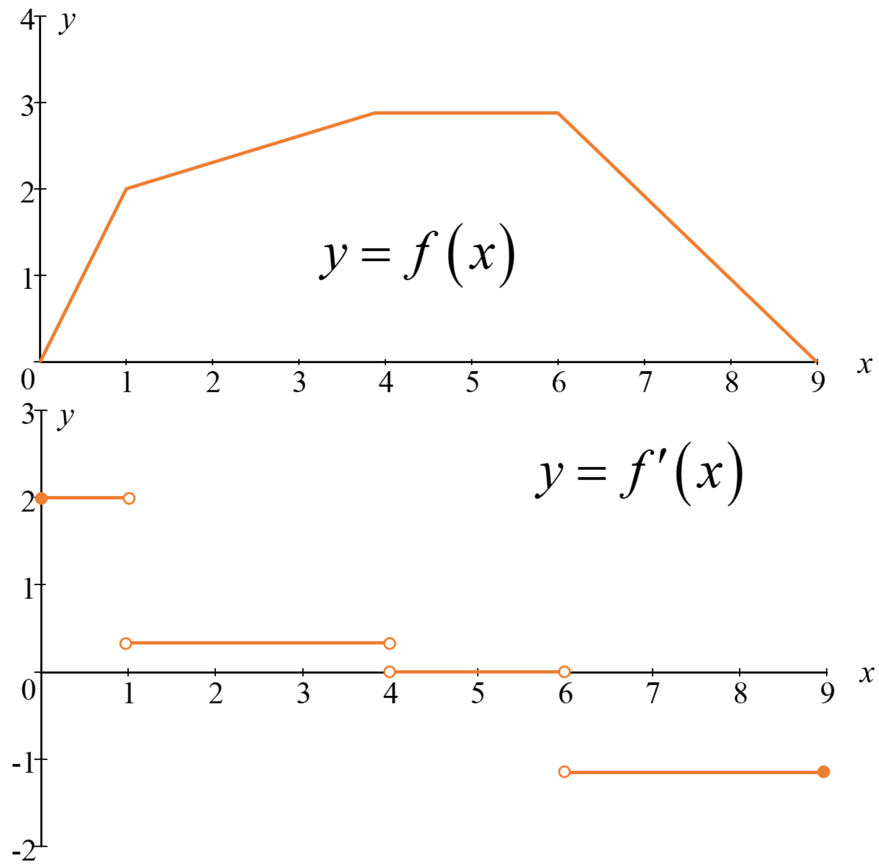
Notice that $f'(x) = 0$ when f has a horizontal tangent, $f'(x)$ is positive when the tangents have positive slope, and $f'(x)$ is negative when the tangents have negative slope. Both functions are discontinuous at $x = 0$.

Example

Draw the derivative of *f*.

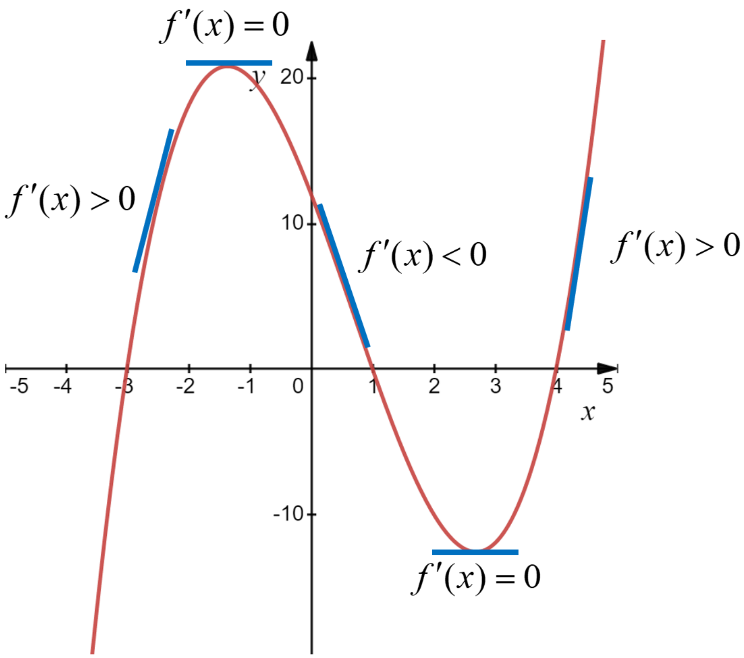
Solution

The derivative is the slope of the original function.



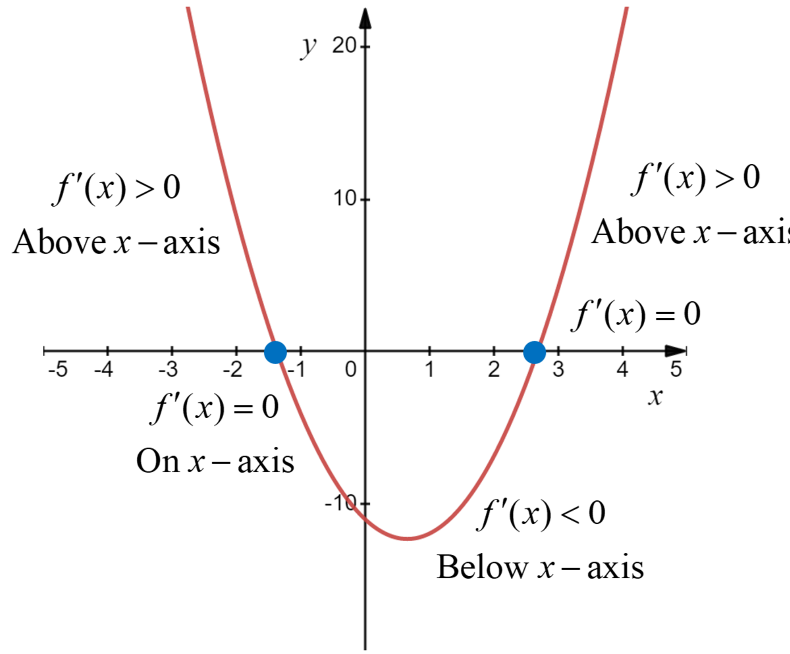
Example

The graph of $f(x)$



Solution

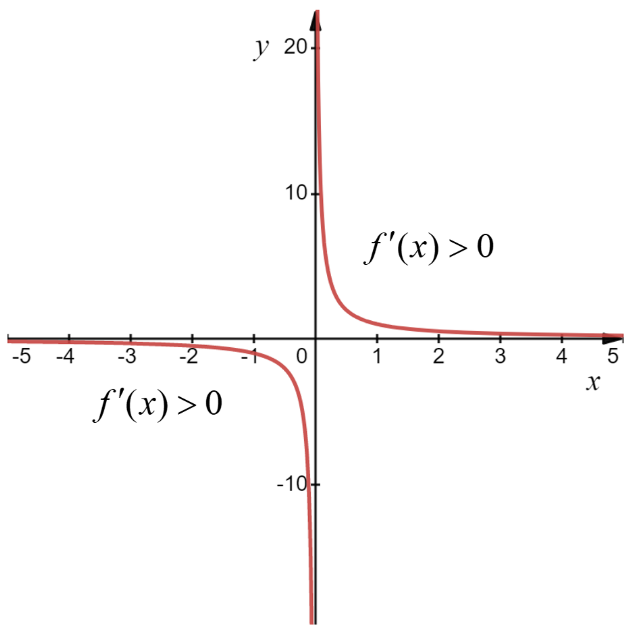
The graph of $f'(x)$



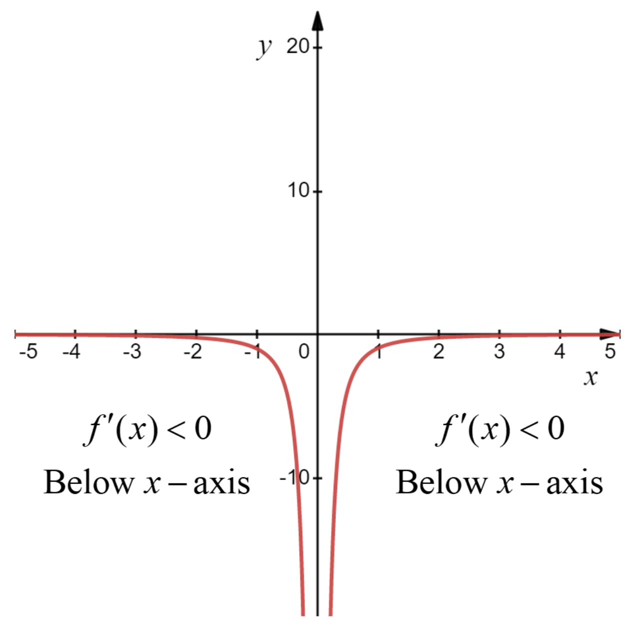
Example

Solution

The graph of $f(x)$



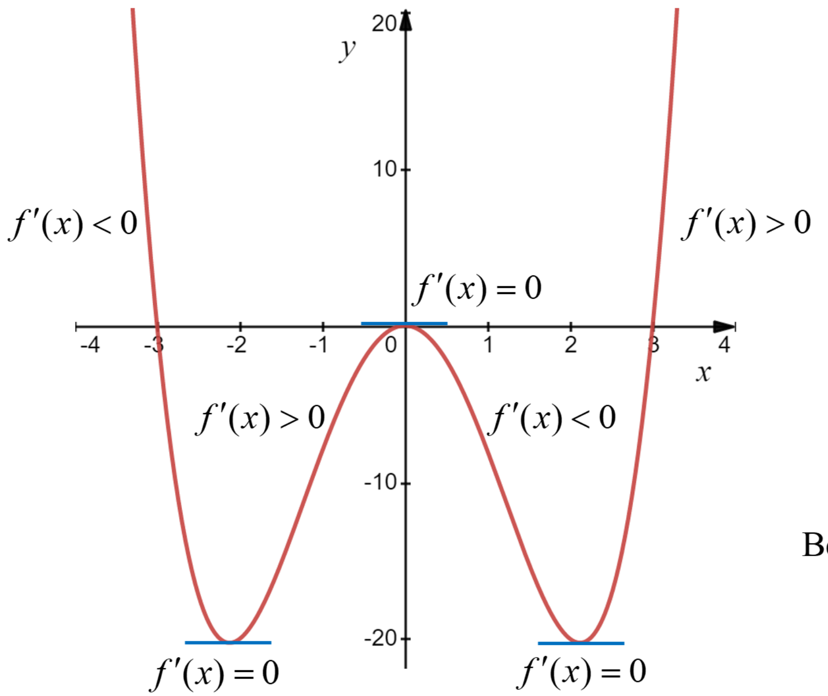
The graph of $f'(x)$



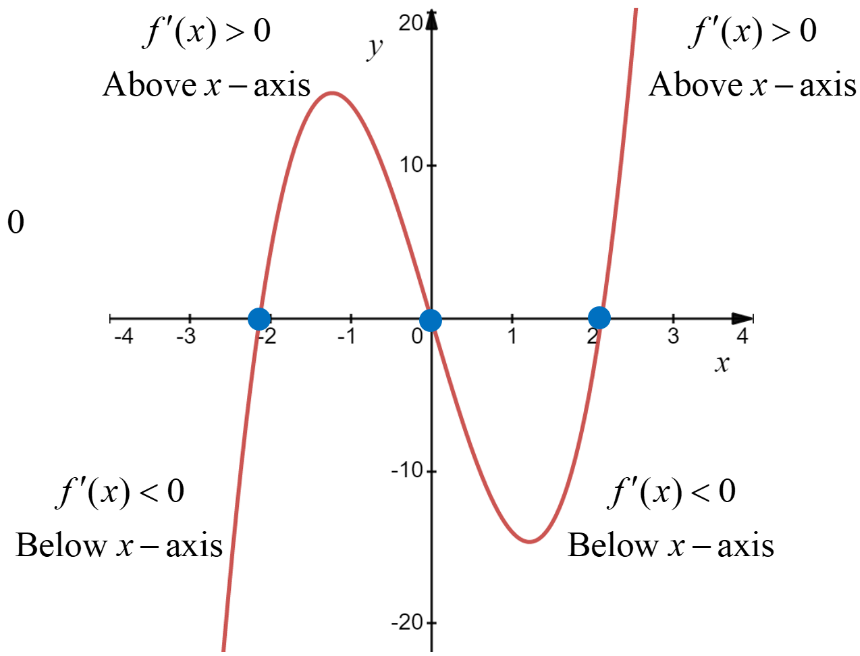
Example

Solution

The graph of $f(x)$



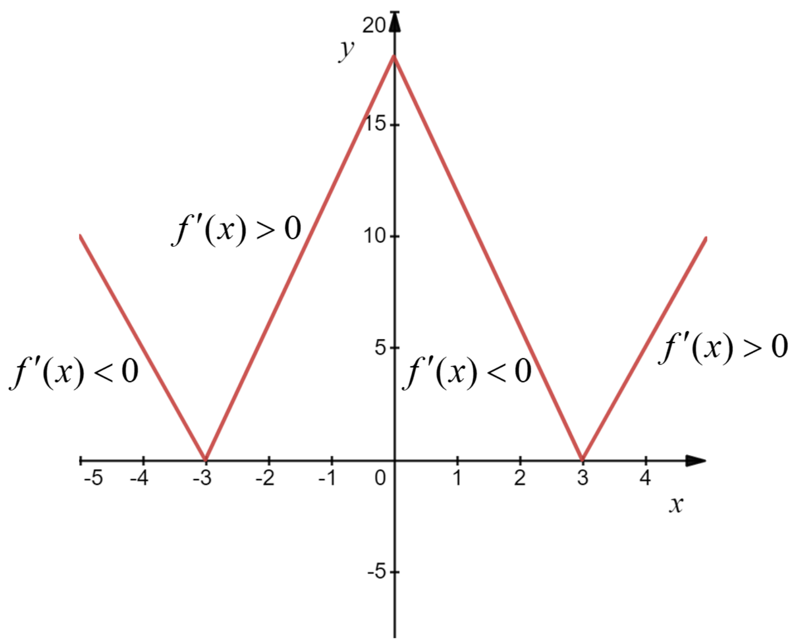
The graph of $f'(x)$



Example

Solution

The graph of $f(x)$



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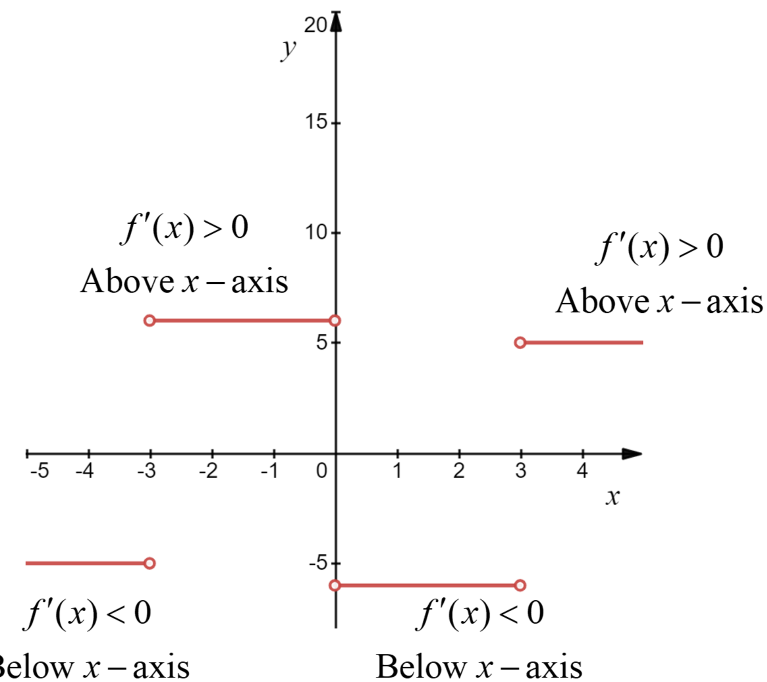
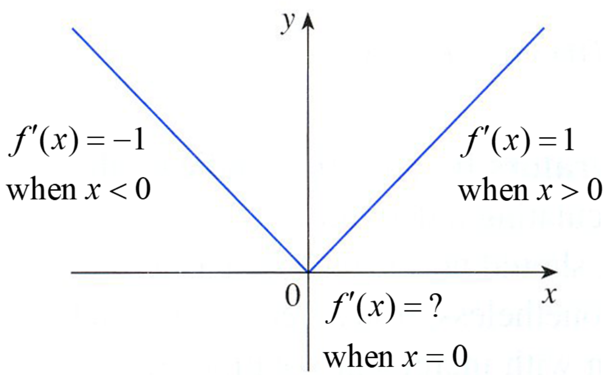


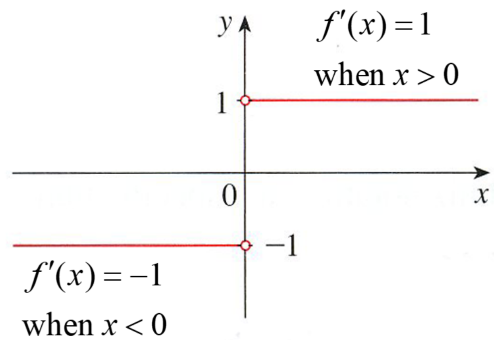
3 Definition A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Example

Where is the function $f(x) = |x|$ differentiable?



(a) $y = f(x) = |x|$



(b) $y = f'(x)$

$$\lim_{x\to 0^-} f'(x) \neq \lim_{x\to 0^+} f'(x)$$

$$\lim_{x\to 0} f'(x) \text{ DNE}$$

Thus, f is differentiable at all x except 0.

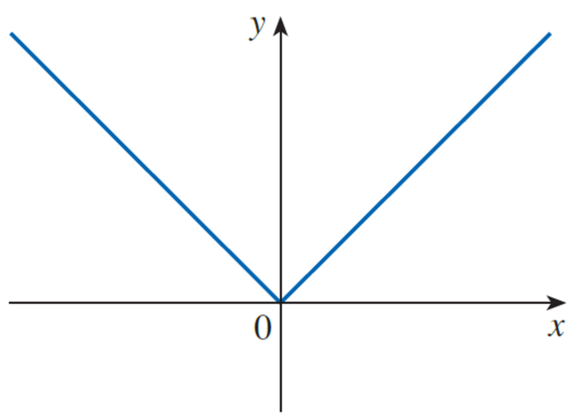
$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Both continuity and differentiability are desirable properties for a function to have. The following theorem shows how these properties are related.

4 Theorem If f is differentiable at a, then f is continuous at a.

NOTE The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable.



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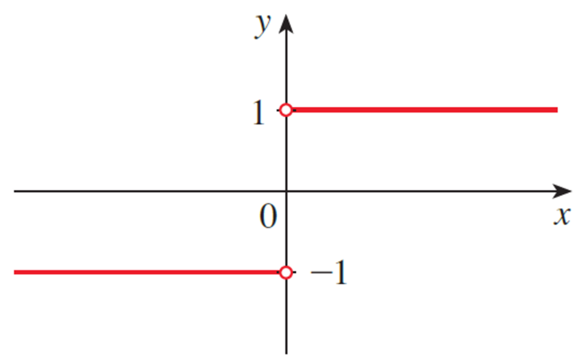
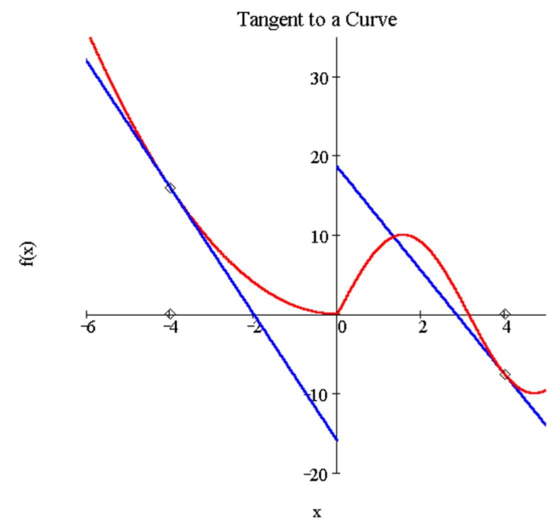


(b) $y = f'(x)$

Tangent_Line_from_the_Left = −8

Tangent_Line_from_the_Right = −6.536



$$\lim_{x\to 0^-} f'(x) \neq \lim_{x\to 0^+} f'(x)$$

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Definition **The Derivative**

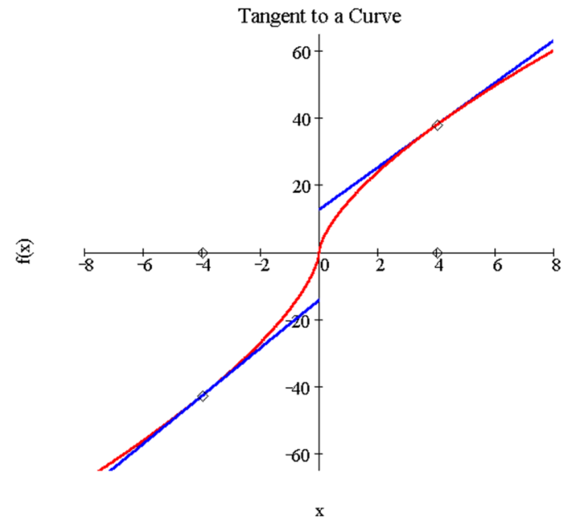
The **derivative** of f is the function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists and x is in the domain of f. If $f'(x)$ exists, we say f is **differentiable** at x. If f is differentiable at every point of an open interval I, we say that f is differentiable on I.

Tangent_Line_from_the_Left = 7.14

Tangent_Line_from_the_Right = 6.3



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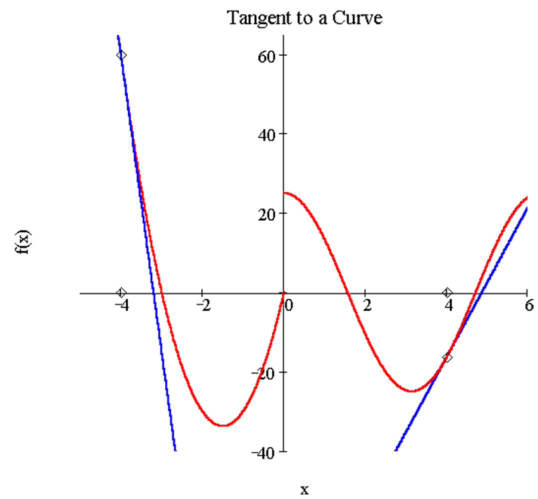
Tangent_Line_from_the_Left = −75

Tangent_Line_from_the_Right = 18.92



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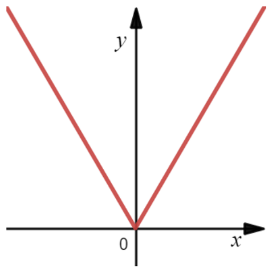
How Can a Function Fail To Be Differentiable?

To be differentiable, a function must be continuous and smooth.

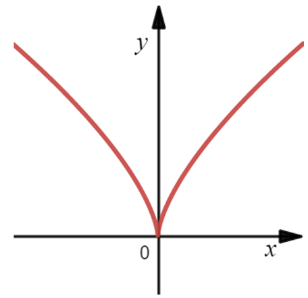
Derivatives will fail to exist at:

$f(x) = |x|$



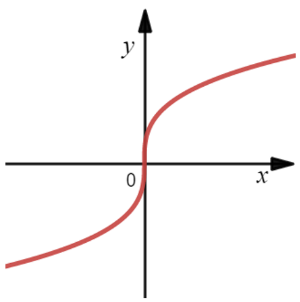
corner

$f(x) = x^{\frac{2}{3}}$



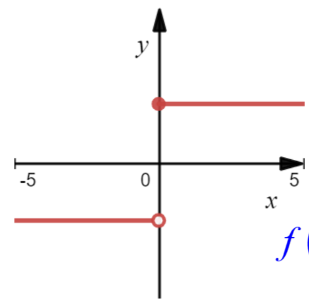
cusp

$f(x) = \sqrt[3]{x}$



vertical tangent

$$f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

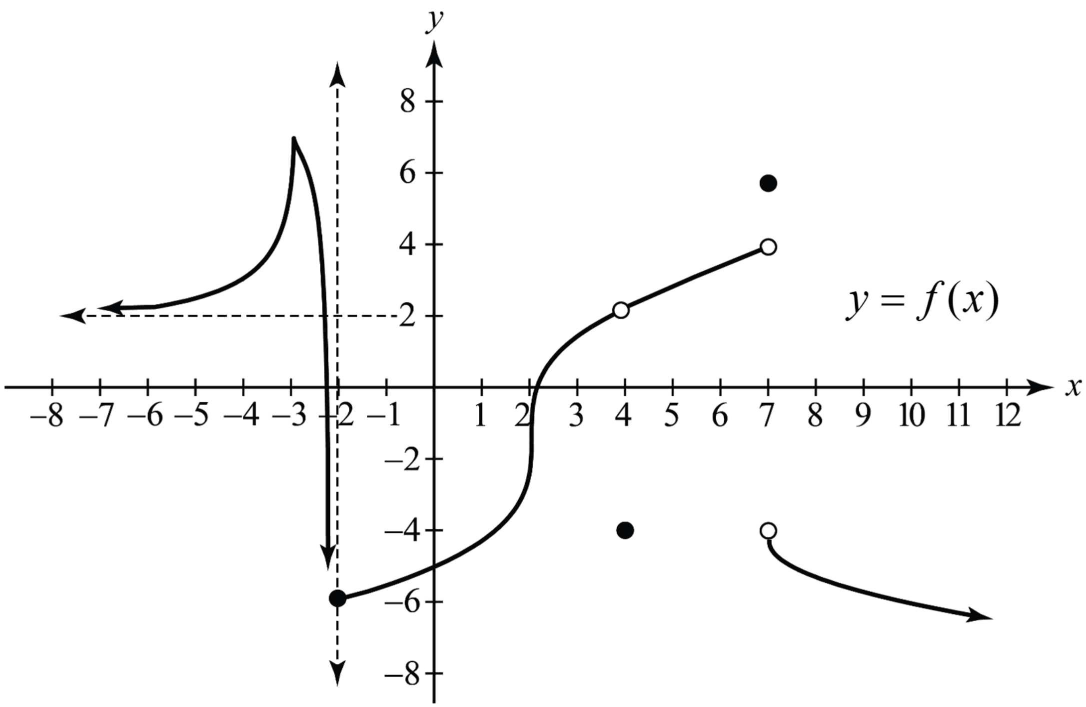


discontinuity

Example The graph of $y = f(x)$ is given below.

(a) At what numbers is $f(x)$ not continuous? Explain.

(b) At what numbers is $f(x)$ not differentiable? Explain.



Solution

(a) $f(x)$ not continuous at:

$x = -2$

infinite discontinuity

$x = 4$

removalable discontinuity

$x = 7$

jump discontinuity

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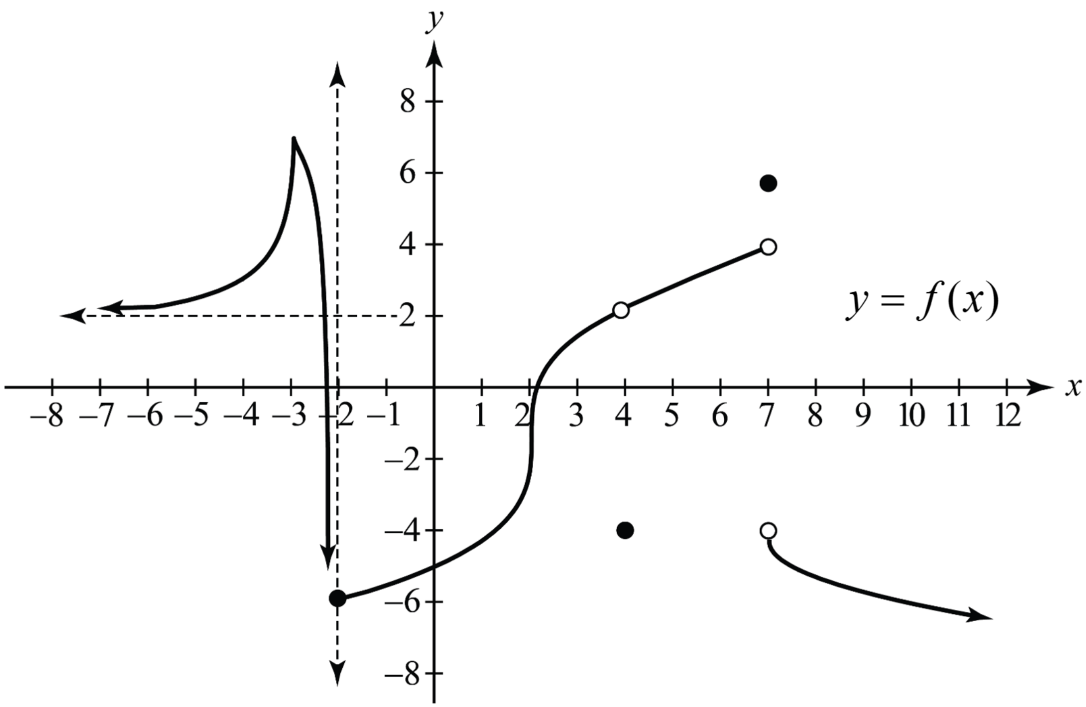
(a) At what numbers is $f(x)$ not continuous? Explain.

(b) At what numbers is $f(x)$ not differentiable? Explain.



Solution

(b) $f(x)$ not differentiable at:

$x = -3$

cusp, corner

$x = -2$

discontinuity

$x = 2$

vertical tangent line

$x = 4$

discontinuity

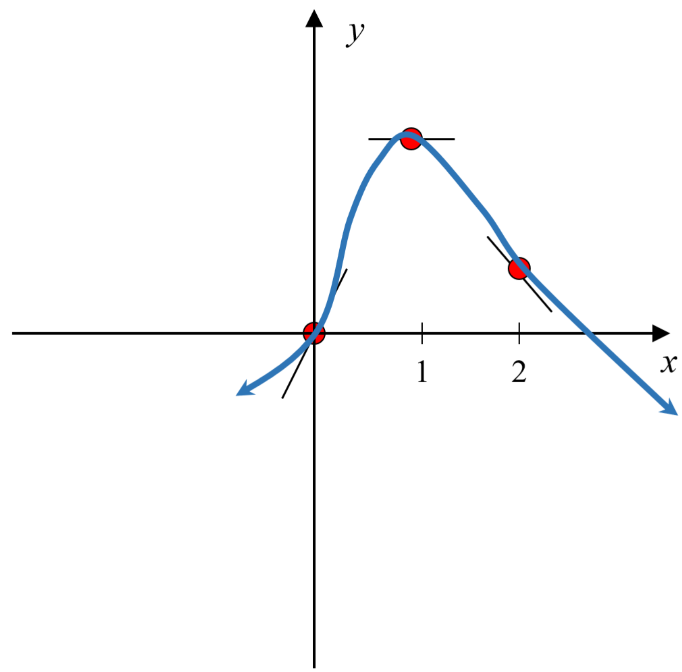
$x = 7$

discontinuity

Example Sketch the graph of a function f such that

$$f(0)=0,\ f'(0)=3,\ f'(1)=0,\ \text{and } f'(2)=-1.$$

Solution



HIGHER DERIVATIVES

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f because it is the derivative of the derivative of f. Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

The **third derivative** f''' is the derivative of the second derivative: $f''' = (f'')'$. So $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$. If $y = f(x)$, then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

The process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$. In general, the nth derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times. If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

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$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Example

The figure shows graphs of f, f', f''. Identify each curve.