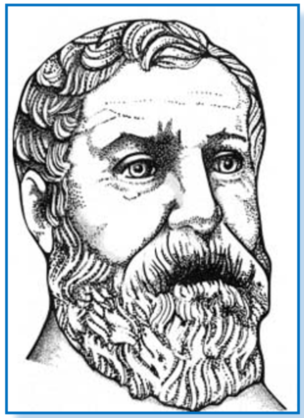
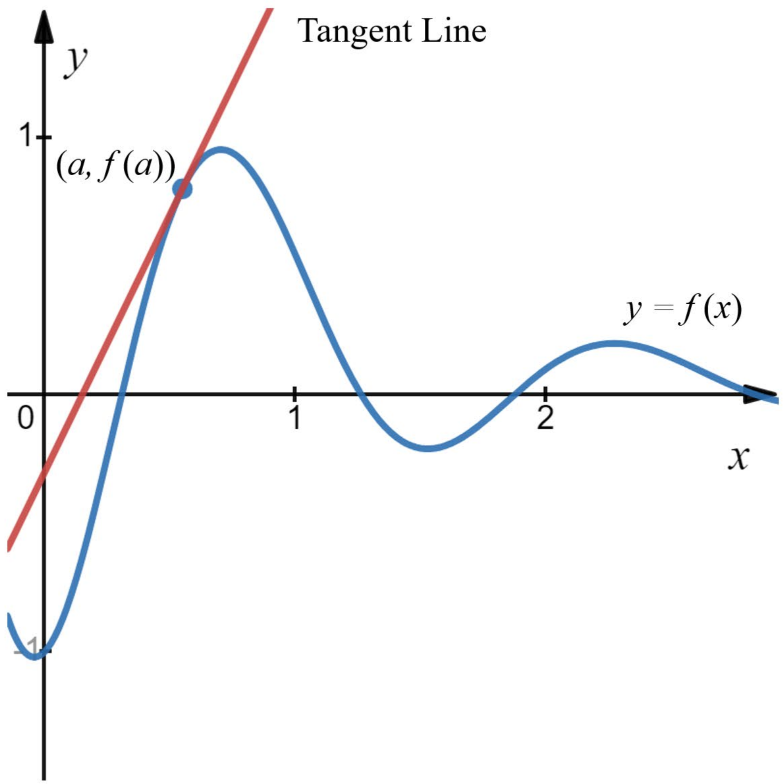
2.7 Derivatives and Rates of Change



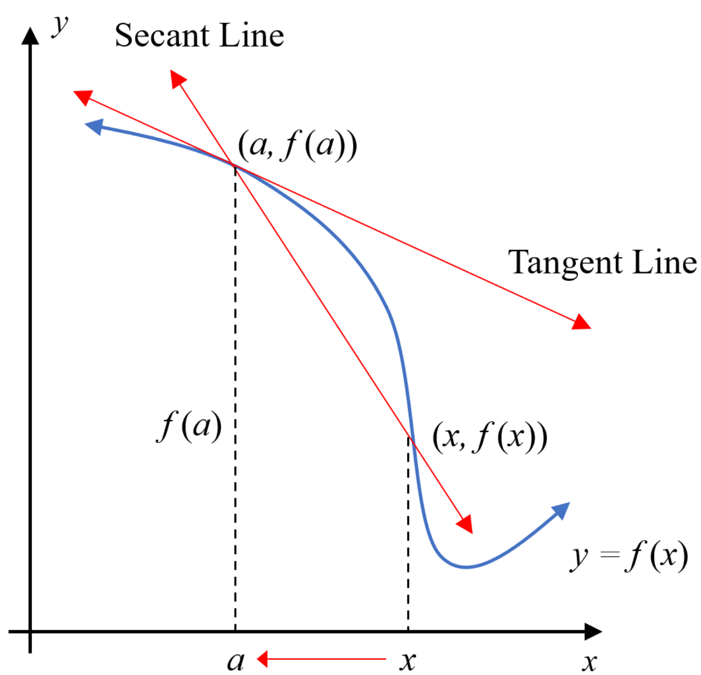
Heron of Alexandria
10 – 75 A.D.

Heron or **Hero of Alexandria** was an important geometer and worker in mechanics who invented many machines including a steam turbine. He's best-known mathematical work is the formula for the area of a triangle in terms of the lengths of its sides.

Tangents



Secant vs Tangent Lines Definition 1



$[a, x]$

Slope of Secant Line

$$m_{\text{secant}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Average Rate of Change

$$m_{\text{secant}} = \frac{f(x) - f(a)}{x - a}$$

Difference Quotient

$$m_{\text{secant}} = \boxed{\frac{f(x) - f(a)}{x - a}}$$

Instantaneous Rate of Change

$$m_{\text{tangent}} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

1 Definition **Rates of Change and the Tangent Line**

The **average rate of change** in f on the interval $[a, x]$ is the slope of the corresponding secant line:

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

The **instantaneous rate of change** in f at a is

$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \qquad (1)$$

which is also the **slope of the tangent line** at a, provided this limit exists. The **tangent line** at $x = a$ is the unique line through $(a, f(a))$ with slope $m_{\tan}$. Its equation is

$$y - f(a) = m_{\tan}(x - a). \qquad \text{Point Slope} \quad y - y_1 = m(x - x_1)$$

Example Using the definition above, find the slope of the tangent line and the equation of the tangent line for $f(x) = x^2 - 5x$ at $a = 3$.

Solution $m = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}$ Provided the limit exists.

$$m = \lim_{x \to 3} \frac{\overbrace{x^2 - 5x}^{f(x)} - \overbrace{\left(3^2 - 5(3)\right)}^{f(3)}}{x - 3}$$

$$m = \lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3}$$

$$m = \lim_{x \to 3} \frac{\cancel{(x-3)}(x-2)}{\cancel{x-3}}$$

$$m = \lim_{x \to 3} (x - 2)$$

$$m = 1$$

Find the equation of the tangent line at $a = 3$ for $f(x) = x^2 - 5x$.

1) Slope

$$m|_{a=3} = 1$$

2) Point of Tangency

$$(a, f(a)) = (3, f(3)) = (3, -6)$$

3) Formula (Point Slope)

$$y - y_1 = m(x - x_1)$$

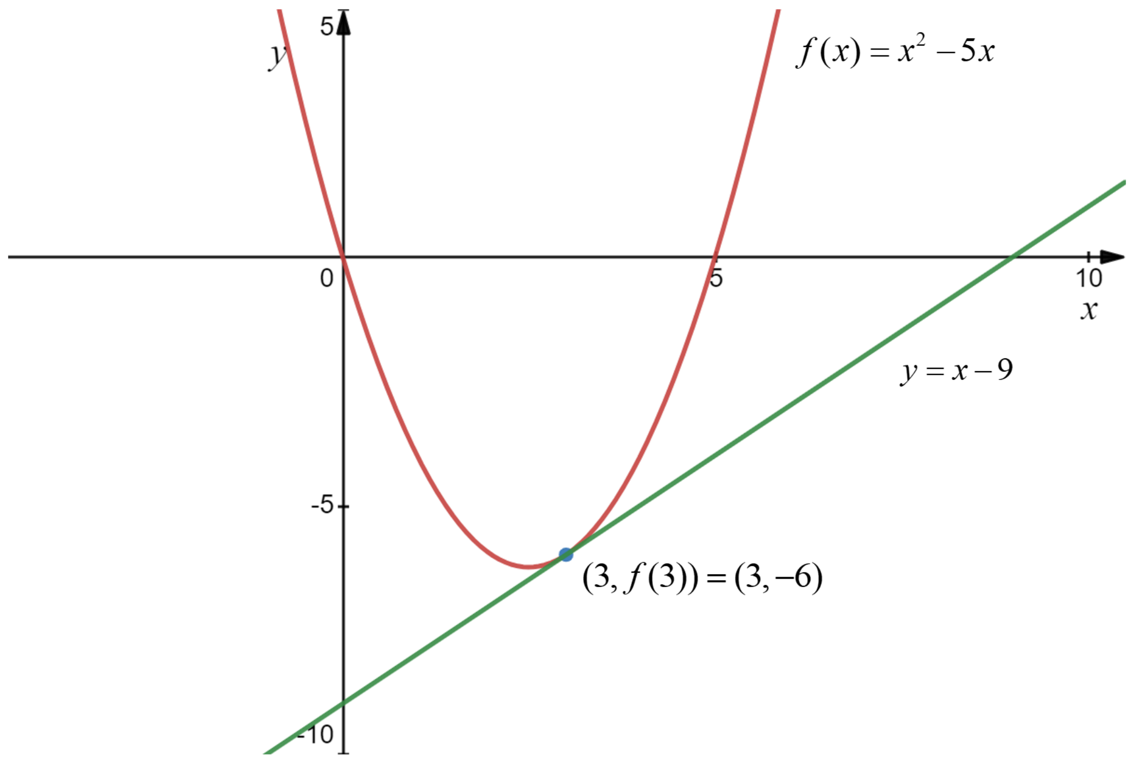
$$y - f(a) = m_{\tan}(x - a)$$

$$y - f(3) = 1 \cdot (x - 3)$$

$$y - (-6) = 1 \cdot (x - 3)$$

$$y + 6 = x - 3$$

$$y = x - 9$$



Example Using definition 1, find an equation of the tangent line to the curve at the given point.

$$y = \frac{x+2}{x-3}, \quad (2, -4)$$

Solution

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Provided the limit exists.

$$= \lim_{x \to 2} \frac{\frac{x+2}{x-3} - (-4)}{x-2}$$

$$= \lim_{x \to 2} \frac{\frac{x+2+4(x-3)}{x-3}}{x-2}$$

$$= \lim_{x \to 2} \frac{5x-10}{(x-2)(x-3)}$$

$$= \lim_{x \to 2} \frac{5\cancel{(x-2)}}{\cancel{(x-2)}(x-3)}$$

$$= \lim_{x \to 2} \frac{5}{x-3}$$

$$= \frac{5}{2-3}$$

$$= -5$$

Tangent line:

1) Slope

$$m|_{a=2} = -5$$

2) Point of Tangency

$$(a, f(a)) = (2, -4)$$

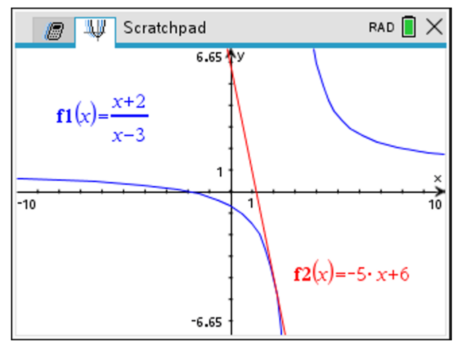
3) Formula (Point Slope)

$$y - y_1 = m(x - x_1)$$

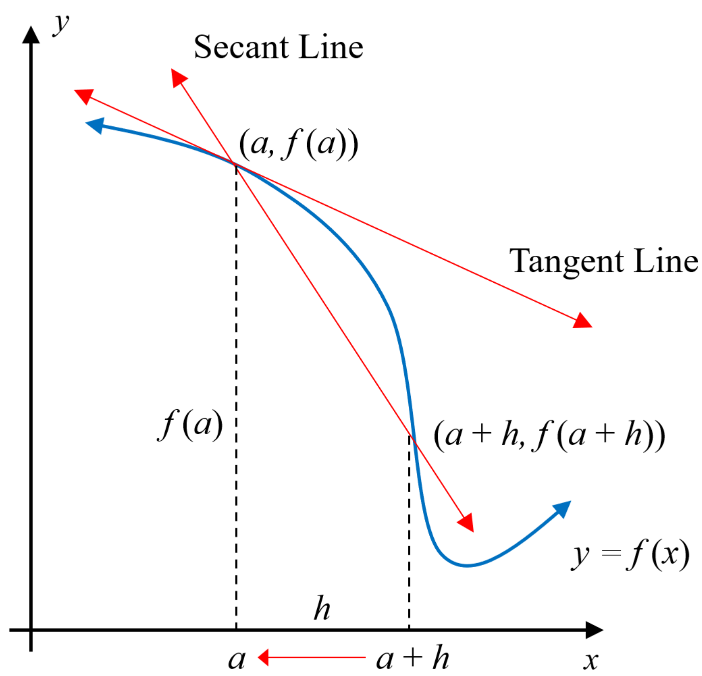
$$y - (-4) = -5(x - 2)$$

$$y + 4 = -5x + 10$$

$$y = -5x + 6$$



Secant vs Tangent Lines Definition 2



$$[a, a+h]$$

Slope of Secant Line

$$m_{\text{secant}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Average Rate of Change

$$m_{\text{secant}} = \frac{f(a+h) - f(a)}{a + h - a}$$

Difference Quotient

$$m_{\text{secant}} = \frac{f(a+h) - f(a)}{h}$$

Instantaneous Rate of Change

$$m_{\text{tangent}} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

2 Definition Rates of Change and the Slope of the Tangent Line

The **average rate of change** in f on the interval $[a, a+h]$ is the slope of the corresponding secant line:

$$m_{\text{sec}} = \frac{f(a+h) - f(a)}{h}.$$

The **instantaneous rate of change** in f at a is

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \qquad (2)$$

which is also the **slope of the tangent line** at $(a, f(a))$, provided this limit exists.

Example Using the definition above, find the slope of the tangent line for $f(x) = x^2 - 5x$ at $a = 3$.

Solution

$$m = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

Provided the limit exists.

$$m = \lim_{h \to 0} \frac{\overbrace{(3+h)^2 - 5(3+h)}^{f(3+h)} - \overbrace{\left(3^2 - 5(3)\right)}^{f(3)}}{h}$$

$$m = \lim_{h \to 0} \frac{9 + 6h + h^2 - 15 - 5h + 6}{h}$$

$$m = \lim_{h \to 0} \frac{h^2 + h}{h}$$

$$m = \lim_{h \to 0} \frac{\cancel{h}(h+1)}{\cancel{h}}$$

$$m = \lim_{h \to 0} (1 + h)$$

$$m = 1$$

Example

(a) Find the slope of the tangent to the curve $y = 3 + 4x^2 - 2x^3$ at the point where $x = a$.

(b) Find equations of the tangent lines at the points (1, 5) and (2, 3).

(c) Graph the curve and both tangents on a common screen.

Solution

(a) Using (2) with $y = f(x) = 3 + 4x^2 - 2x^3$, the slope of the tangent line is

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad \text{Provided the limit exists.}$$

$$= \lim_{h \to 0} \frac{3 + 4(a+h)^2 - 2(a+h)^3 - (3 + 4a^2 - 2a^3)}{h}$$

$$= \lim_{h \to 0} \frac{3 + 4(a^2 + 2ah + h^2) - 2(a^3 + 3a^2h + 3ah^2 + h^3) - 3 - 4a^2 + 2a^3}{h}$$

$$= \lim_{h \to 0} \frac{3 + 4a^2 + 8ah + 4h^2 - 2a^3 - 6a^2h - 6ah^2 - 2h^3 - 3 - 4a^2 + 2a^3}{h}$$

Example

(a) Find the slope of the tangent to the curve $y = 3 + 4x^2 - 2x^3$ at the point where $x = a$.

(b) Find equations of the tangent lines at the points (1, 5) and (2, 3).

(c) Graph the curve and both tangents on a common screen.

Solution

(a) Using (2) with $y = f(x) = 3 + 4x^2 - 2x^3$, the slope of the tangent line is

$$= \lim_{h \to 0} \frac{\cancel{3} + \cancel{4a^2} + 8ah + 4h^2 - \cancel{2a^3} - 6a^2h - 6ah^2 - 2h^3 - \cancel{3} - \cancel{4a^2} + \cancel{2a^3}}{h}$$

$$= \lim_{h \to 0} \frac{8ah + 4h^2 - 6a^2h - 6ah^2 - 2h^3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(8a + 4h - 6a^2 - 6ah - 2h^2)}{\cancel{h}}$$

$$= \lim_{h \to 0} (8a + \cancelto{0}{4h} - 6a^2 - \cancelto{0}{6ah} - \cancelto{0}{2h^2}) = 8a - 6a^2$$

Slope of tangent line at $x = a$ is
$m = 8a - 6a^2$

Example

(a) Find the slope of the tangent to the curve $y = 3 + 4x^2 - 2x^3$ at the point where $x = a$.

(b) Find equations of the tangent lines at the points (1, 5) and (2, 3).

(c) Graph the curve and both tangents on a common screen.

Solution

(b) At $(1, 5)$:

Tangent line:

1) Slope $m = 8a - 6a^2$

$$m|_{a=1} = 8(1) - 6(1)^2 = 2$$

2) Point of Tangency

$$(a, f(a)) = (1,5)$$

3) Formula (Point Slope)

$$y - y_1 = m(x - x_1)$$

$$y - 5 = 2(x - 1)$$

$$y = 2x + 3$$

At $(2, 3)$:

Tangent line:

1) Slope $m = 8a - 6a^2$

$$m|_{a=2} = 8(2) - 6(2)^2 = -8$$

2) Point of Tangency

$$(a, f(a)) = (2,3)$$

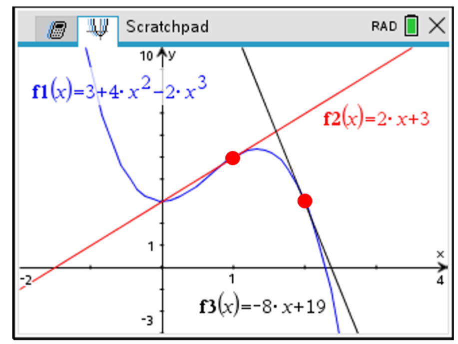
3) Formula (Point Slope)

$$y - y_1 = m(x - x_1)$$

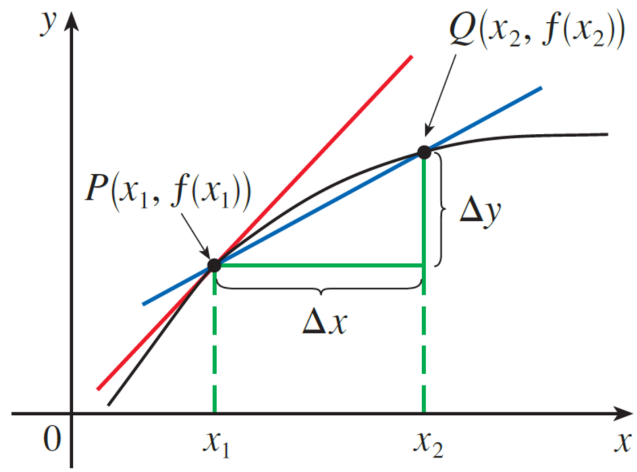
$$y - 3 = -8(x - 2)$$

$$y = -8x + 19$$

(c)



Secant vs Tangent Lines Definitions 3 and 4



average rate of change $= m_{PQ}$

instantaneous rate of change = slope of tangent at P

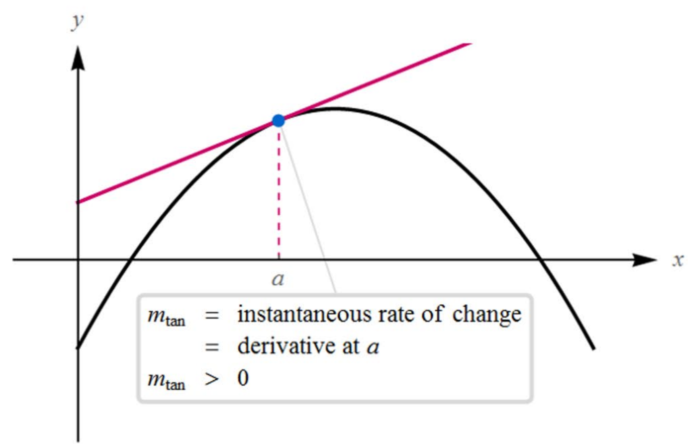
$$\text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Derivatives

4 Definition The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.



Example Consider the function $f(x)=\sqrt{x}$, find the derivative $f'(a)$ using Definition 4.

Solution $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ provided the limit exists.

$$=\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h}\cdot\frac{\sqrt{a+h}+\sqrt{a}}{\sqrt{a+h}+\sqrt{a}}$$

$$=\lim_{h\to 0}\frac{a+h-a}{h(\sqrt{a+h}+\sqrt{a})}$$

$$=\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{a+h}+\sqrt{a})}$$

$$=\frac{1}{\sqrt{a+0}+\sqrt{a}}$$

Hence,

$$f'(a)=\frac{1}{2\sqrt{a}}$$

Find the equation of the tangent line at $a = 4$.

1) Slope

$$f'(a) = \frac{1}{2\sqrt{a}}, \text{ so } m = f'(a) = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

2) Point of Tangency $(a, f(a))$ (POT)

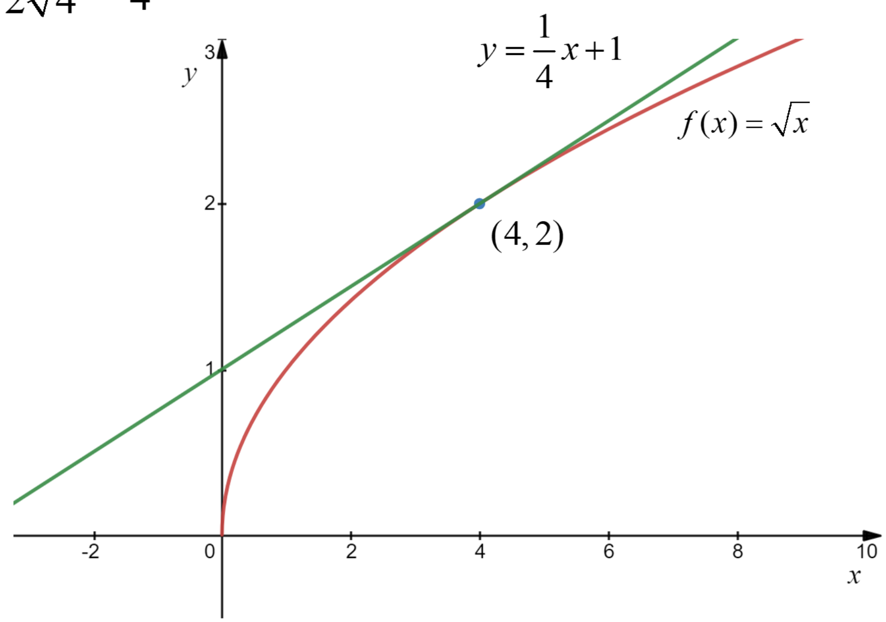
$$(4, f(4)) = (4, 2) \Rightarrow f(x) = \sqrt{x}$$

3) Formula

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x + 1$$



Example Consider the function $f(t) = \dfrac{1}{t^2+1}$.

a) Find $f'(a)$.

b) Find the slope of the tangent line and normal line to the curve when $a = 1$.

c) Find the equation of the tangent and normal lines to the curve when $a = 1$.

d) Graph the equation and the tangent and normal lines to the curve when $a = 1$.

Solution

a) Find $f'(a)$. Using Definition 4 with $f(t) = \dfrac{1}{t^2+1}$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \qquad \text{Provided the limit exists.}$$

$$= \lim_{h\to 0} \frac{\dfrac{1}{(a+h)^2+1} - \dfrac{1}{a^2+1}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{(a^2+1) - \left[(a+h)^2+1\right]}{\left[(a+h)^2+1\right](a^2+1)}}{h}$$

$$= \lim_{h\to 0} \frac{(a^2+1) - (a^2+2ah+h^2+1)}{h[(a+h)^2+1](a^2+1)}$$

$$= \lim_{h\to 0} \frac{-(2ah+h^2)}{h[(a+h)^2+1](a^2+1)}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}(2a+h)}{\cancel{h}[(a+h)^2+1](a^2+1)}$$

$$= \lim_{h\to 0} \frac{-(2a+\overset{0}{\cancel{h}})}{[(a+\underset{0}{\cancel{h}})^2+1](a^2+1)}$$

$$= \frac{-2a}{(a^2+1)(a^2+1)} = -\frac{2a}{(a^2+1)^2}$$

b) Find the slope of the tangent line and normal line to the curve when $a = 1$.

Slope of tangent line $m = f'(a) = -\dfrac{2a}{(a^2+1)^2}$

Tangent line:

1) Slope

$$m|_{a=1} = f'(1) = -\frac{2(1)}{((1)^2+1)^2} = -\frac{2}{4} = -\frac{1}{2}$$

Slope of normal line $m = -\dfrac{1}{f'(a)}$, negative reciprocal.

Normal line:

1) Slope

$$m|_{a=1} = -\frac{1}{f'(1)} = 2$$

c) Find the equation of the tangent and normal lines to the curve when $a = 1$.

2) Point of Tangency

$$(a, f(a)) = \left(1, \frac{1}{2}\right) \quad f(t) = \frac{1}{t^2+1} \quad f(1) = \frac{1}{1^2+1} = \frac{1}{2}$$

3) Formula (Point Slope)

$$y - y_1 = m(x - x_1)$$

$$y - \frac{1}{2} = -\frac{1}{2}(x-1)$$

$$y = -\frac{1}{2}x + 1, \text{ tangent line}$$

2) Point of Tangency

$$(a, f(a)) = \left(1, \frac{1}{2}\right)$$

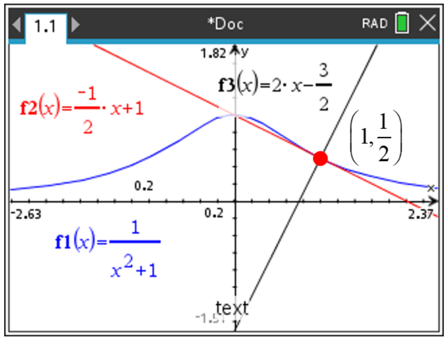
3) Formula (Point Slope)

$$y - y_1 = m(x - x_1)$$

$$y - \frac{1}{2} = 2(x-1)$$

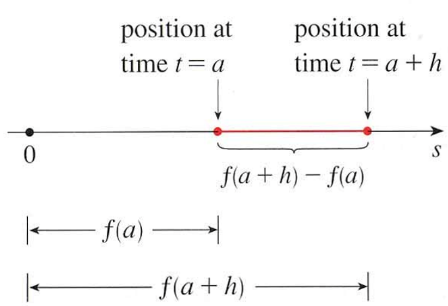
$$y = 2x - \frac{3}{2}, \text{ normal line}$$

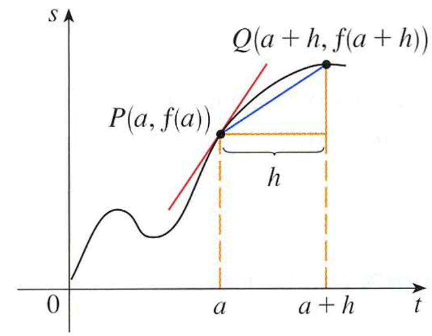
d) Graph the equation and the tangent and normal lines to the curve when $a = 1$.



VELOCITIES

Suppose a particle is moving along a line, and given the position function $s = f(t)$ we get,





$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$
= average velocity

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Hence velocity is the derivative of position function, that is $v(t) = s'(t)$.

Example

If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height (in meters) after t seconds is given by $H = 10t - 1.86t^2$.

(a) Find the velocity of the rock after one second.
(b) Find the velocity of the rock when $t = a$.
(c) When will the rock hit the surface?
(d) With what velocity will the rock hit the surface?

Solution (a) Find the velocity of the rock after one second.

$$v(a) = \lim_{h\to 0} \frac{H(a+h) - H(a)}{h}$$ Provided the limit exists.

$$v(1) = \lim_{h\to 0} \frac{H(1+h) - H(1)}{h}$$

$$= \lim_{h\to 0} \frac{\left[10(1+h) - 1.86(1+h)^2\right] - (10 - 1.86)}{h}$$

$$= \lim_{h\to 0} \frac{10 + 10h - 1.86(1 + 2h + h^2) - 10 + 1.86}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{10} + 10h - \cancel{1.86} - 3.72h - 1.86h^2 - \cancel{10} + \cancel{1.86}}{h}$$

$$= \lim_{h\to 0} \frac{6.28h - 1.86h^2}{h}$$

$$= \lim_{h\to 0} (6.28 - 1.\cancelto{0}{86h})$$

$$= 6.28$$

The velocity of the rock after one second is 6.28 m/s.

(b) Find the velocity of the rock when $t = a$.

$$v(a) = \lim_{h \to 0} \frac{H(a+h) - H(a)}{h}$$ Provided the limit exists.

$$= \lim_{h \to 0} \frac{\left[10(a+h) - 1.86(a+h)^2\right] - (10a - 1.86a^2)}{h}$$

$$= \lim_{h \to 0} \frac{10a + 10h - 1.86(a^2 + 2ah + h^2) - 10a + 1.86a^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{10a} + 10h - \cancel{1.86a^2} - 3.72ah - 1.86h^2 - \cancel{10a} + \cancel{1.86a^2}}{h}$$

$$= \lim_{h \to 0} \frac{10h - 3.72ah - 1.86h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(10 - 3.72a - 1.86h)}{\cancel{h}}$$

$$= \lim_{h \to 0} (10 - 3.72a - 1.86h \to 0)$$

$$v(a) = 10 - 3.72a$$

The velocity of the rock when $t = a$ is $(10 - 3.72a)$ m/s.

(c) When will the rock hit the surface?

The rock will hit the surface when $H = 0$

$H = 10t - 1.86t^2$

$10t - 1.86t^2 = 0$

$t(10 - 1.86t) = 0$

$t = 0$ or $1.86t = 10$

The rock hits the surface when $t = 10/1.86 \approx 5.4$ s.

(d) With what velocity will the rock hit the surface?

The velocity of the rock when it hits the surface is

$$v(a) = 10 - 3.72a$$

$$v\left(\tfrac{10}{1.86}\right) = 10 - 3.72\left(\tfrac{10}{1.86}\right)$$

$$= 10 - 20$$

$$= -10 \text{ m/s}$$