**Euclid of Alexandria**
**325 – 265 B.C.**

**Euclid** was a Greek mathematician best known for his treatise on geometry: *The Elements* .
This influenced the development of Western mathematics for more than 2000 years.

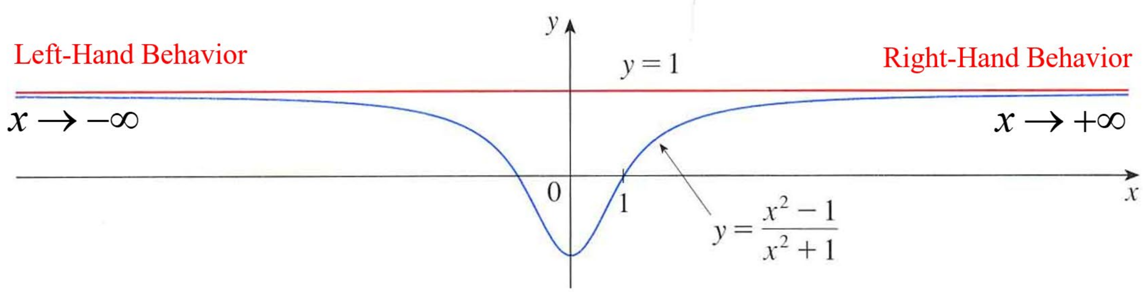
Let's begin by investigating the behavior of the function $f$ defined by

| $x$ | $f(x)$ |
|---|---|
| 0 | −1 |
| ±1 | 0 |
| ±2 | 0.600000 |
| ±3 | 0.800000 |
| ±4 | 0.882353 |
| ±5 | 0.923077 |
| ±10 | 0.980198 |
| ±50 | 0.999200 |
| ±100 | 0.999800 |
| ±1000 | 0.999998 |

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

So, we say, $\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$ and $\lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1$

These limits shows that $f(x)$ has a horizontal asymptote at $y = 1$.



**1 Intuitive Definition of a Limit at Infinity** Let $f$ be a function defined on some interval $(a, \infty)$. Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to $L$ by requiring $x$ to be sufficiently large.

Another notation for $\lim_{x \to \infty} f(x) = L$ is

$$f(x) \to L \quad \text{as} \quad x \to \infty$$

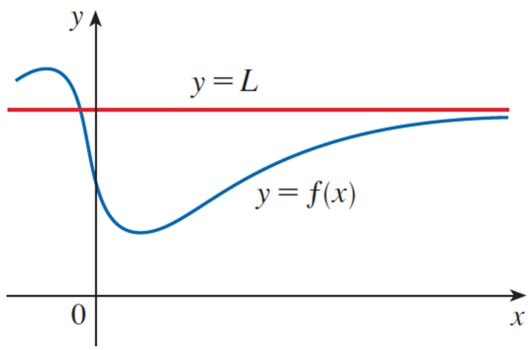
The symbol $\infty$ does not represent a number. Nonetheless, the expression $\lim_{x \to \infty} f(x) = L$ is often read as

"the limit of $f(x)$, as $x$ approaches infinity, is $L$"

or "the limit of $f(x)$, as $x$ becomes infinite, is $L$"

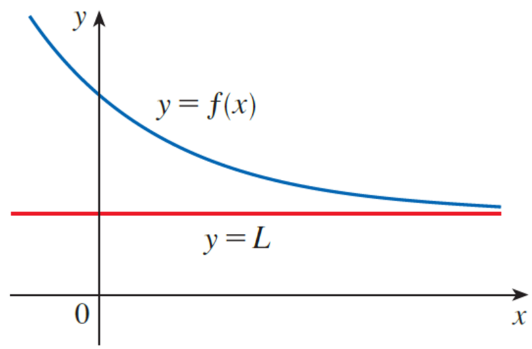
or "the limit of $f(x)$, as $x$ increases without bound, is $L$"

We take limits to infinity to analyze **end-behavior** of functions.



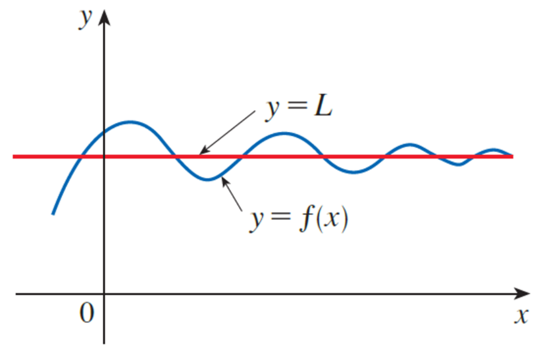

$$\lim_{x \to \infty} f(x) = L$$

Right-Hand Behavior




$$\lim_{x \to \infty} f(x) = L$$

Right-Hand Behavior



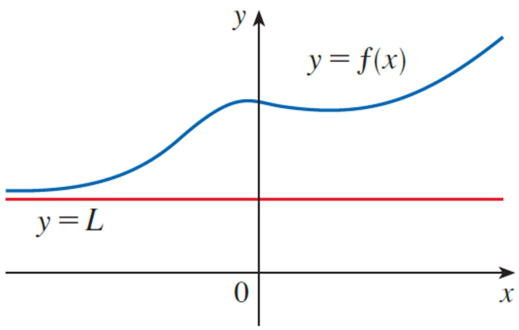

$$\lim_{x \to \infty} f(x) = L$$

Right-Hand Behavior

**2 Definition** Let $f$ be a function defined on some interval $(-\infty, a)$. Then

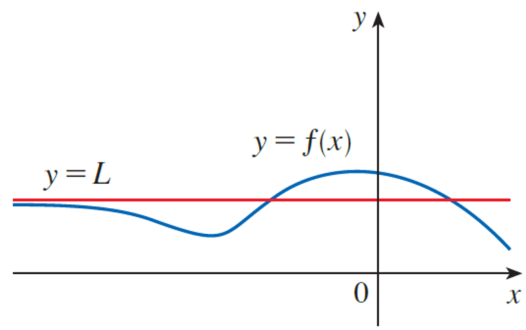
$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to $L$ by requiring $x$ to be sufficiently large negative.




$$\lim_{x \to -\infty} f(x) = L$$

Left-Hand Behavior



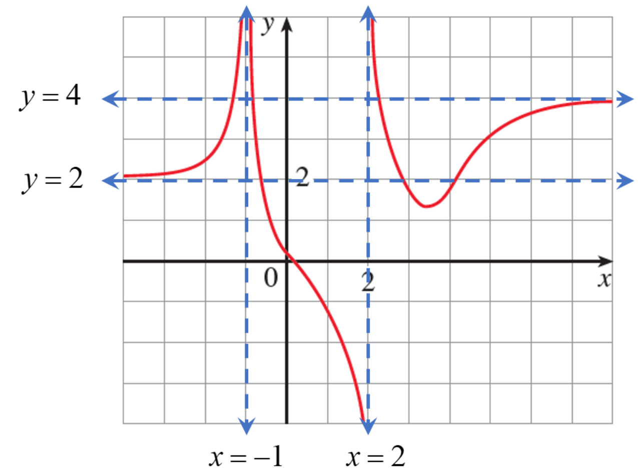

$$\lim_{x \to -\infty} f(x) = L$$

Left-Hand Behavior

**3** **Definition** The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

**Example** Find the infinite limits, limits at infinity, and asymptotes for the function $f$ whose graph is shown in the figure.




**Solution**

$$\lim_{x \to -1} f(x) = \infty$$

$$\lim_{x \to 2^-} f(x) = -\infty \qquad \text{and} \qquad \lim_{x \to 2^+} f(x) = \infty$$

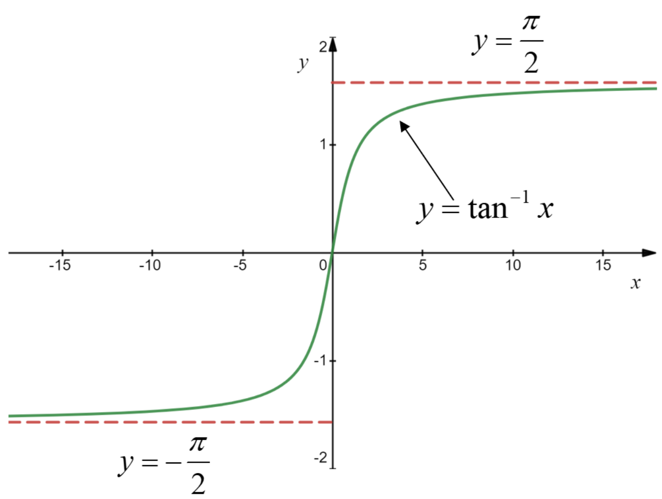
Thus both of the lines $x = -1$ and $x = 2$ are vertical asymptotes.

$$\lim_{x \to \infty} f(x) = 4 \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = 2$$

This means that both $y = 4$ and $y = 2$ are horizontal asymptotes.

**Example** Find $\lim\limits_{x\to\pm\infty} \tan^{-1} x$

**Solution**



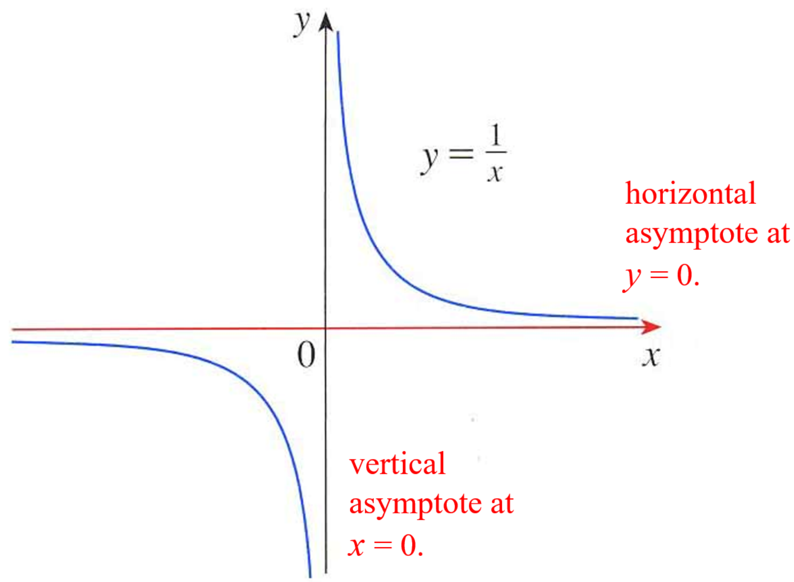

So, $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes for $y = \tan^{-1} x$.

$$\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2} \qquad \lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$$

Recall $f(x)=\dfrac{1}{x}$

$$\lim_{x\to 0^-}\frac{1}{x}=-\infty \qquad \lim_{x\to 0^+}\frac{1}{x}=+\infty$$

$$\lim_{x\to +\infty}\frac{1}{x}=0 \qquad \lim_{x\to -\infty}\frac{1}{x}=0$$




In 2.3 these limits showed $x = 0$ is a vertical asymptote.

In these limits, as the denominator gets larger, the value of the fraction gets closer to zero. So, by definition, $y = 0$ is a horizontal asymptote.

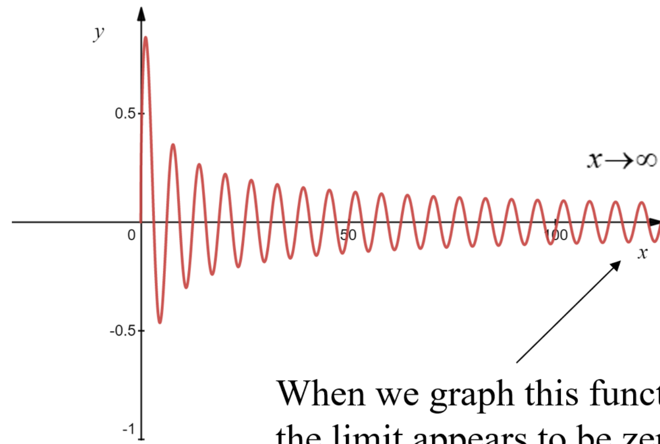
Example Find: $\lim_{x\to\infty} \frac{\sin x}{\sqrt{x}}$

Solution

We know that $-1 \le \sin x \le 1$.

Since $x > 0$ it follows that $\frac{1}{\sqrt{x}} > 0$,

So, $\frac{-1}{\sqrt{x}} \le \frac{\sin x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$ (1)




When we graph this function, the limit appears to be zero.

As $x \to \infty$, $\sqrt{x}$ becomes arbitrarily large, which means that

$$\lim_{x\to\infty} \frac{-1}{\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0. \qquad (2)$$

So from inequality (1) we get,

$$\lim_{x\to\infty} \frac{-1}{\sqrt{x}} \le \lim_{x\to\infty} \frac{\sin x}{\sqrt{x}} \le \lim_{x\to\infty} \frac{1}{\sqrt{x}}$$

Example Find: $\lim_{x\to\infty} \frac{\sin x}{\sqrt{x}}$

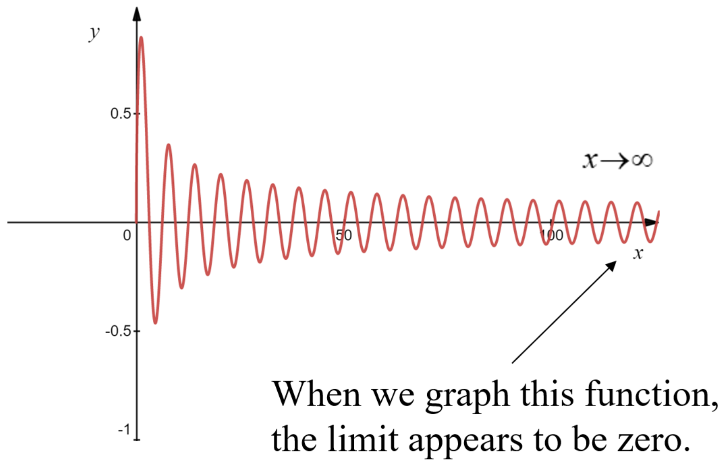
Solution

Applying the limits in (2) we get,

$$0 \le \lim_{x\to\infty} \frac{\sin x}{\sqrt{x}} \le 0.$$

Thus, by the Squeeze Theorem:

$\lim_{x\to\infty} \frac{\sin x}{\sqrt{x}} = 0$, and $y = 0$ is a horizontal asymptote.



## Evaluating Limits at Infinity

**5 Theorem** If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that $x^r$ is defined for all $x$, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

**Example** Evaluate the following limit and indicate which properties of limits are used at each stage.

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

**Solution**

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{\dfrac{3x^2 - x - 2}{x^2}}{\dfrac{5x^2 + 4x + 1}{x^2}}$$

$$\frac{+\infty}{+\infty}$$

$$= \lim_{x\to\infty} \frac{3 - \dfrac{1}{x} - \dfrac{2}{x^2}}{5 + \dfrac{4}{x} + \dfrac{1}{x^2}}$$

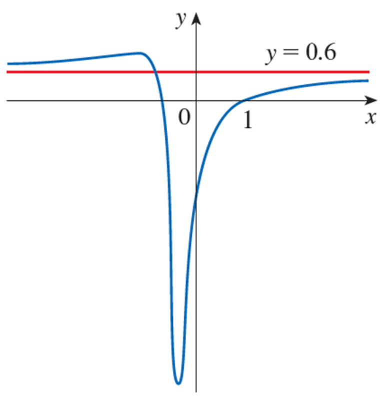
$$= \frac{\lim_{x\to\infty}\left(3 - \dfrac{1}{x} - \dfrac{2}{x^2}\right)}{\lim_{x\to\infty}\left(5 + \dfrac{4}{x} + \dfrac{1}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty} \dfrac{1}{x} - 2\lim_{x\to\infty} \dfrac{1}{x^2}}{\lim_{x\to\infty} 5 + 4\lim_{x\to\infty} \dfrac{1}{x} + \lim_{x\to\infty} \dfrac{1}{x^2}} \quad \text{(by 1, 2, and 3)}$$

$$= \frac{3 - 0 - 0}{5 + 0 + 0} \quad \text{(by 8 and Theorem 5)}$$

$$= \frac{3}{5}$$

The results of these calculations show that the graph of the given rational function approaches the horizontal asymptote $y = \frac{3}{5}$.



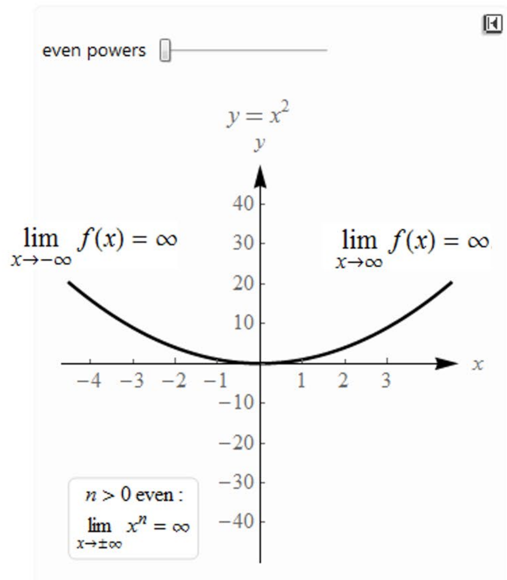
**4 Definition Infinite Limits at Infinity**

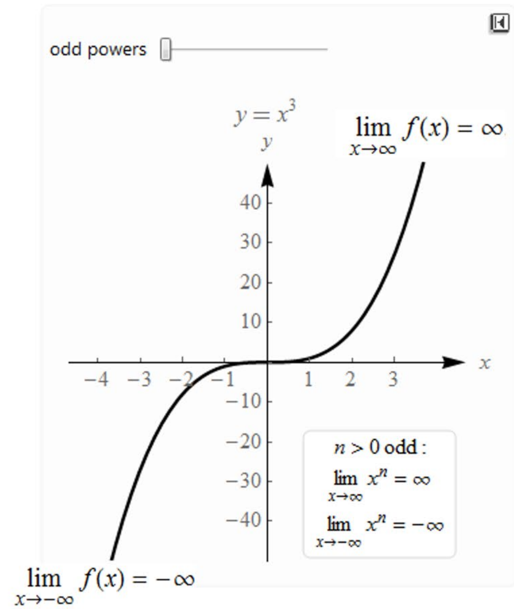
If $f(x)$ becomes arbitrarily large as $x$ becomes arbitrarily large, then we write

$$\lim_{x\to\infty} f(x) = \infty.$$

The limits $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$, and $\lim_{x\to-\infty} f(x) = -\infty$ are defined similarly.

For polynomials



**Theorem** **Limits at Infinity of Powers and Polynomials**

Let $n$ be a positive integer and let $p$ be the polynomial

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, where $a_n \neq 0$.

**1.** $\lim_{x \to \pm\infty} x^n = \infty$ when $n$ is even.

**2.** $\lim_{x \to \infty} x^n = \infty$ and $\lim_{x \to -\infty} x^n = -\infty$ when $n$ is odd.

**3.** $\lim_{x \to \pm\infty} \frac{1}{x^n} = \lim_{x \to \pm\infty} x^{-n} = 0$.

**4.** $\lim_{x \to \pm\infty} p(x) = \lim_{x \to \pm\infty} a_n x^n = \pm\infty$, depending on the degree of the polynomial and the sign of the leading coefficient $a_n$. Dominating Term Effect (DTE)

**Example** Evaluate the limits as $x \to \pm\infty$ of the following function

$$q(x) = -2x^3 + 3x^2 - 12$$

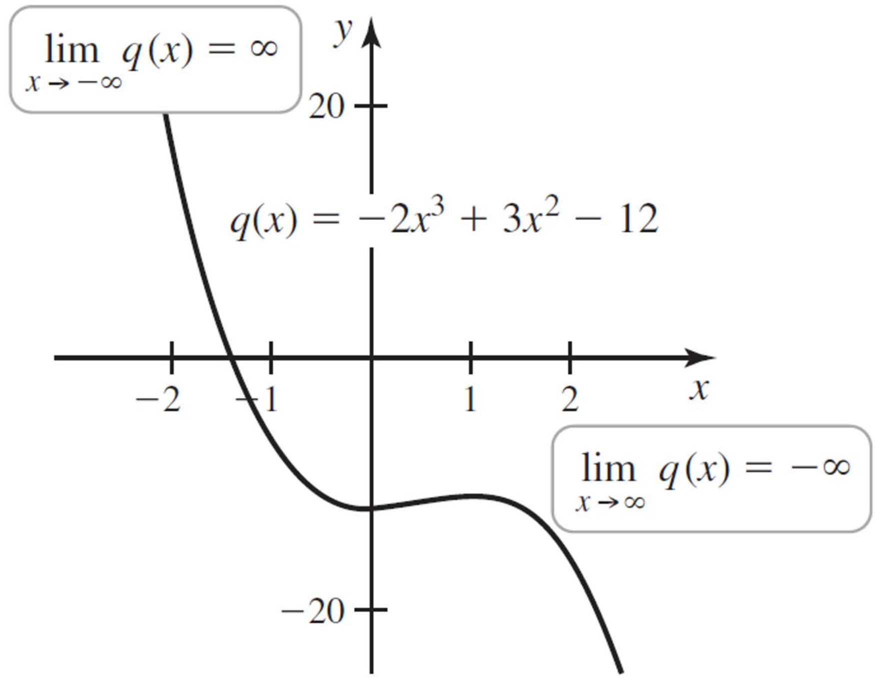
**Solution**

$$\lim_{x\to\infty}\left(\underbrace{-2x^3}+3x^2-12\right)\overset{\text{DTE}}{=}\lim_{x\to\infty}\left(-2\underbrace{x^3}_{\to\infty}\right)=-\infty$$

Dominating term

$$\lim_{x\to-\infty}\left(\overbrace{-2x^3}+3x^2-12\right)\overset{\text{DTE}}{=}\lim_{x\to-\infty}\left(-2\underbrace{x^3}_{\to-\infty}\right)=\infty$$

$q(x) \to \infty$ as $x \to -\infty$




$q(x) \to -\infty$ as $x \to \infty$

# Algebraic Limits

## Method I

$$\lim_{x\to\infty}\frac{3x^2-x-2}{5x^2+4x+1}=\lim_{x\to\infty}\frac{\frac{3x^2-x-2}{x^2}}{\frac{5x^2+4x+1}{x^2}}=\lim_{x\to\infty}\frac{3-\frac{1}{x}-\frac{2}{x^2}}{5+\frac{4}{x}+\frac{1}{x^2}}$$

$$\frac{+\infty}{+\infty}$$

$$=\frac{\lim_{x\to\infty}\left(3-\frac{1}{x}-\frac{2}{x^2}\right)}{\lim_{x\to\infty}\left(5+\frac{4}{x}+\frac{1}{x^2}\right)}$$

$$=\frac{\lim_{x\to\infty}3-\lim_{x\to\infty}\frac{1}{x}-2\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty}5+4\lim_{x\to\infty}\frac{1}{x}+\lim_{x\to\infty}\frac{1}{x^2}}$$

$$=\frac{3-0-0}{5+0+0}$$

$$=\frac{3}{5}$$

## Method II

$$\lim_{x\to\infty}\frac{3x^2-x-2}{5x^2+4x+1}$$

DTE

$$=\lim_{x\to\infty}\frac{3x^2}{5x^2}$$

$$=\lim_{x\to\infty}\frac{3}{5}$$

$$=\frac{3}{5}$$

Which method do you like better?

DTE can only be used with polynomials!

**Example** Find $\lim_{x\to+\infty} \dfrac{2x^5+x^4-x^2+1}{3x^2-5x+7}$

**Solution**

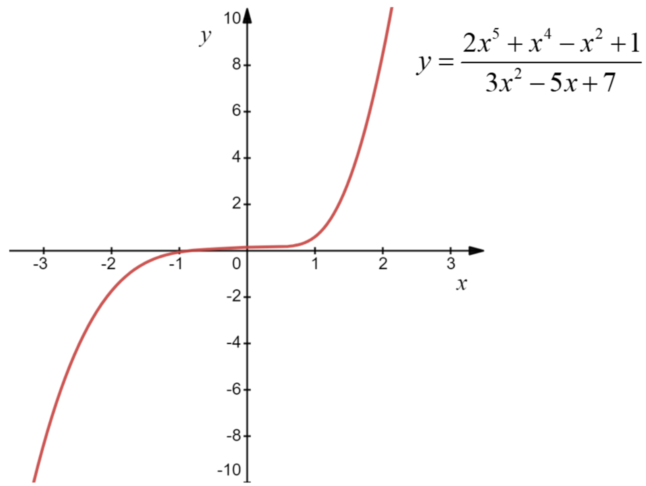
Dominating term effect for rational functions (DTE) we get,

$$\lim_{x\to+\infty} \frac{2x^5+x^4-x^2+1}{3x^2-5x+7} \overset{\text{DTE}}{=} \lim_{x\to+\infty} \frac{2x^5}{3x^2}$$

Dominant terms in numerator and denominator

$$= \lim_{x\to+\infty} \frac{2}{3}x^3$$

$$= +\infty$$



**Theorem** **End Behavior and Asymptotes of Rational Functions**

Suppose $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function, where

$$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_2 x^2 + a_1 x + a_0 \text{ and}$$
$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_2 x^2 + b_1 x + b_0$$

and $a_m \neq 0$ and $b_n \neq 0$.

**a.** If $m < n$, then $\lim_{x \to \pm\infty} f(x) = 0$, and $y = 0$ is a horizontal asymptote of $f$.

**b.** If $m = n$, then $\lim_{x \to \pm\infty} f(x) = a_m / b_n$, and $y = a_m / b_n$ is a horizontal asymptote of $f$.

**c.** If $m > n$, then $\lim_{x \to \pm\infty} f(x) = \infty$ or $-\infty$, and $f$ has no horizontal asymptote.

**d.** If $m = n + 1$, then $\lim_{x \to \pm\infty} f(x) = \infty$ or $-\infty$, $f$ has no horizontal asymptote, but $f$ has a slant asymptote.

**e.** Assuming that $f(x)$ is in reduced form ($p$ and $q$ share no common factors), vertical asymptotes occur at the zeros of $q$.

Recall:

$$\sqrt{x^2} = |x|$$

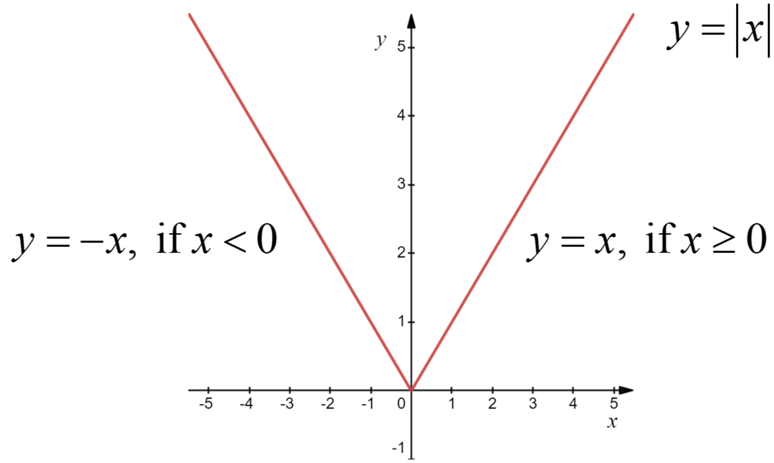
Example

$$|13| = 13$$

$$|-23| = -(-23) = 23$$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Piecewise Function



**Example** Determine the end behavior of $f(x) = \dfrac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$.

**Solution**

## Right-Hand Behavior

$$\lim_{x\to+\infty} \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}} \overset{\text{DTE}}{=} \lim_{x\to+\infty} \frac{10x^3}{\sqrt{25x^6}}$$

$$= \lim_{x\to+\infty} \frac{10x^3}{5|x^3|}$$

$$= \lim_{x\to+\infty} \frac{10x^3}{5x^3} \qquad |x^3| = x^3, \text{ since } x^3 > 0$$

$$= \lim_{x\to+\infty} 2$$

$$= 2$$

## Left-Hand Behavior

$$\lim_{x\to-\infty} \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}} \overset{\text{DTE}}{=} \lim_{x\to-\infty} \frac{10x^3}{\sqrt{25x^6}}$$

$$= \lim_{x\to-\infty} \frac{10x^3}{5|x^3|}$$

$$= \lim_{x\to-\infty} \frac{10x^3}{-5x^3} \qquad |x^3| = -x^3, \text{ since } x^3 < 0$$

$$= \lim_{x\to-\infty} 2$$

$$= 2$$

**Example** Determine the end behavior of $f(x) = \dfrac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$.

**Solution**

### Right-Hand Behavior

$$\lim_{x\to+\infty} \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}} \overset{\text{DTE}}{=} \lim_{x\to+\infty} \frac{10x^3}{\sqrt{25x^6}}$$

$$= \lim_{x\to+\infty} \frac{10x^3}{5|x^3|}$$

$$= \lim_{x\to+\infty} \frac{10x^3}{5x^3} \qquad |x^3| = x^3, \text{ since } x^3 > 0$$

$$= \lim_{x\to+\infty} 2$$

$$= 2$$

### Left-Hand Behavior

$$\lim_{x\to-\infty} \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}} \overset{\text{DTE}}{=} \lim_{x\to-\infty} \frac{10x^3}{\sqrt{25x^6}}$$

$$= \lim_{x\to-\infty} \frac{10x^3}{5|x^3|}$$

$$= \lim_{x\to-\infty} \frac{10x^3}{-5x^3} \qquad |x^3| = -x^3, \text{ since } x^3 < 0$$

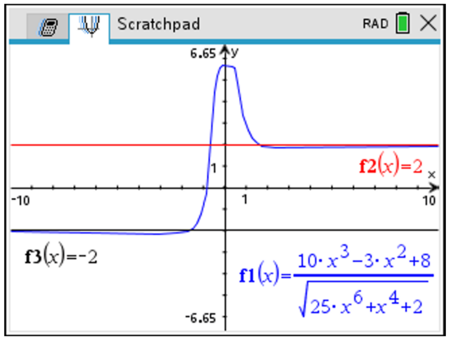
$$= \lim_{x\to-\infty} (-2)$$

$$= -2$$

**Example** Determine the end behavior of $f(x) = \dfrac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$.

**Solution**

So $y = 2$ and $y = -2$ are horizontal asymptotes.



Example Find the limit or show that it does not exist.

$$\lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)$$

Solution

$$\lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)=\lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)\left[\frac{\sqrt{4x^2+3x}-2x}{\sqrt{4x^2+3x}-2x}\right]$$

$$=\lim_{x\to-\infty}\frac{(4x^2+3x)-(2x)^2}{\sqrt{4x^2+3x}-2x}$$

$$=\lim_{x\to-\infty}\frac{3x}{\sqrt{4x^2+3x}-2x}$$

$$\overset{\text{DTE}}{=}\lim_{x\to-\infty}\frac{3x}{\sqrt{4x^2}-2x}$$

$$\overset{\text{DTE}}{=}\lim_{x\to-\infty}\frac{3x}{2|x|-2x}$$

Example Find the limit or show that it does not exist.

$$\lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)$$

Solution

$$\lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)\overset{\text{DTE}}{=}\lim_{x\to-\infty}\frac{3x}{2|x|-2x}$$

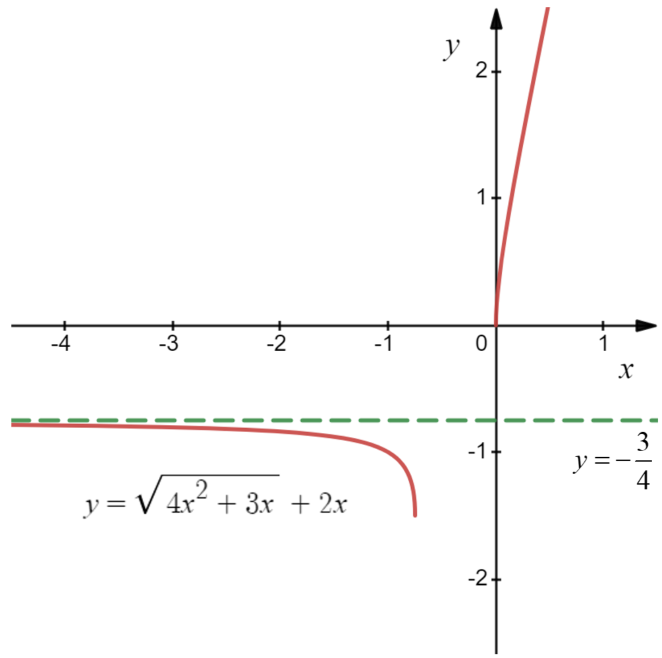
$$=\lim_{x\to-\infty}\frac{3x}{-2x-2x}\qquad |x|=-x,\ \text{since } x<0$$

$$=\lim_{x\to-\infty}\frac{3\cancel{x}}{-4\cancel{x}}$$

$$=\lim_{x\to-\infty}\left(-\frac{3}{4}\right)$$

$$=-\frac{3}{4}$$

So $y=-\dfrac{3}{4}$ is horizontal asymptotes.



## End behavior of transcendental functions

Example Determine the end behavior of the following transcendental functions.

**a.** $f(x) = e^x$ and $f(x) = e^{-x}$

**b.** $g(x) = \ln x$

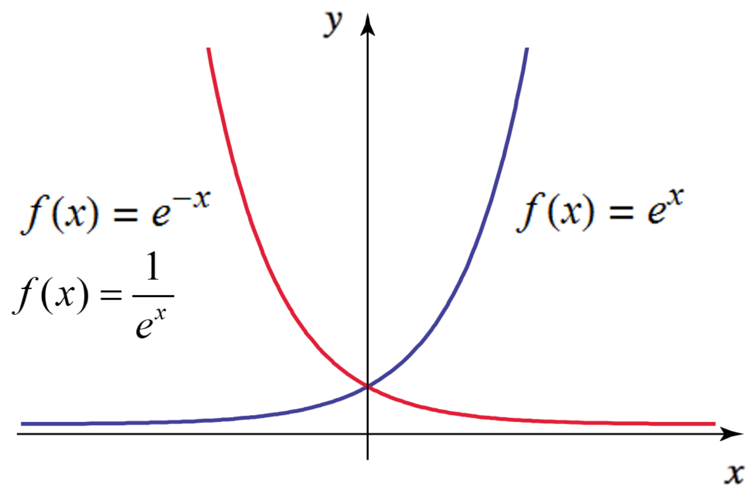
**c.** $f(x) = \cos x$

Solution

**a.** $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$

$\lim_{x\to\infty} e^{-x} = 0$ and $\lim_{x\to-\infty} e^{-x} = \infty$

Hence, $y = 0$ is a horizontal asymptote.



**b.** The domain of $\ln x$ is $\{x : x > 0\}$, so we evaluate $\lim_{x\to 0^+} \ln x$ and $\lim_{x\to\infty} \ln x$ to determine end behavior. For the first limit, recall that $\ln x$ is the inverse of $e^x$

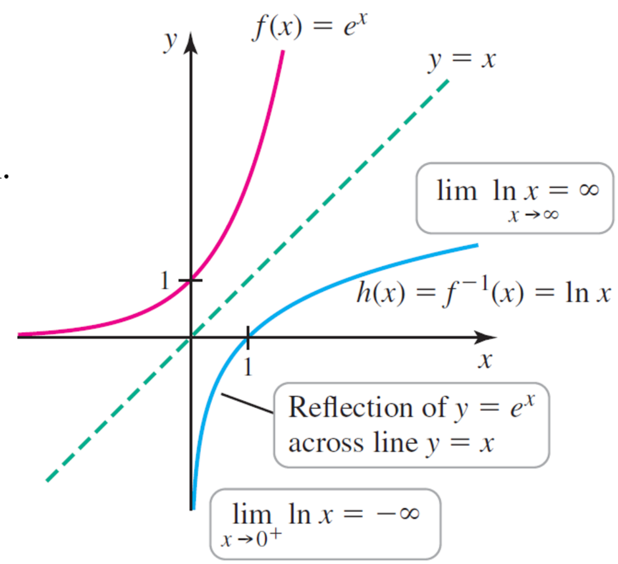
$$\lim_{x\to\infty} \ln x = \infty$$

There is no left-hand behavior because of the domain restriction.

Hence, there is no horizontal asymptote.

$$\lim_{x\to 0^+} \ln x = -\infty$$

So, $x = 0$ is a vertical asymptote.



**c.** The cosine function oscillates between $-1$ and 1 as $x$ approaches infinity (Figure 2.45). Therefore, $\lim_{x\to\infty} \cos x$ does not exist.

For the same reason, $\lim_{x\to-\infty} \cos x$ does not exist.

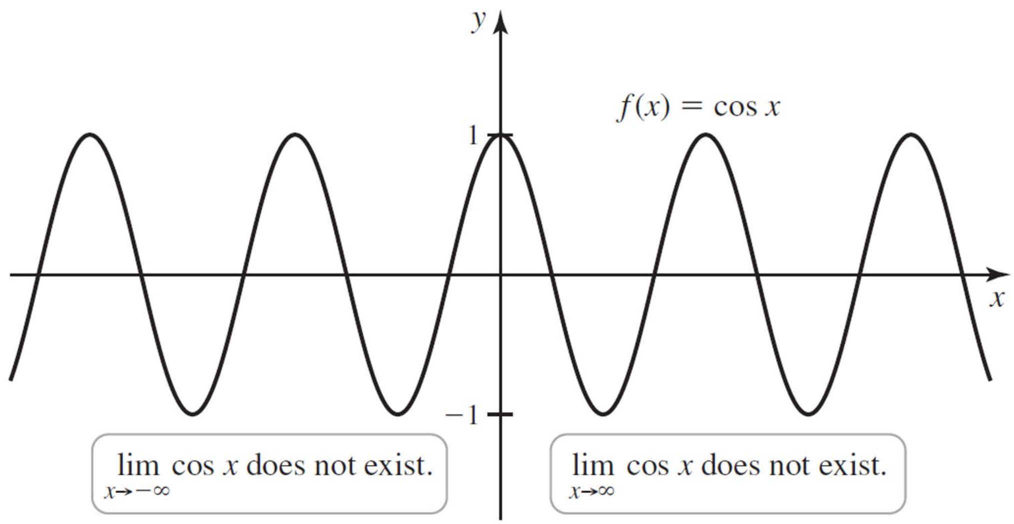



**Figure 2.45**

**Theorem** **End Behavior of $e^x$, $e^{-x}$, and $\ln x$**

The end behavior for $e^x$ and $e^{-x}$ on $(-\infty, \infty)$ and $\ln x$ on $(0, \infty)$ is given by the following limits:

$$\lim_{x \to \infty} e^x = \infty \quad \text{and} \quad \lim_{x \to -\infty} e^x = 0,$$

$$\lim_{x \to \infty} e^{-x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} e^{-x} = \infty,$$

$$\lim_{x \to 0^+} \ln x = -\infty \quad \text{and} \quad \lim_{x \to \infty} \ln x = \infty.$$