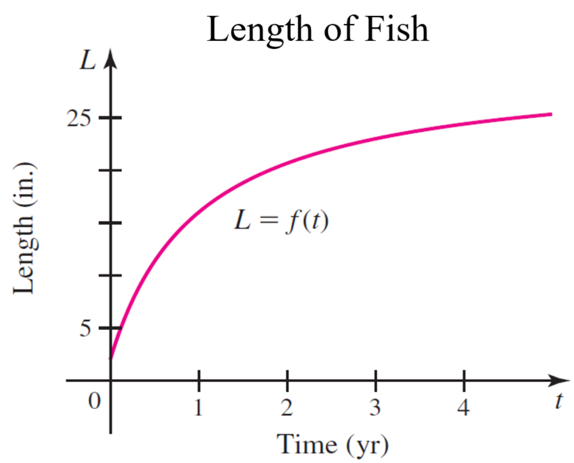
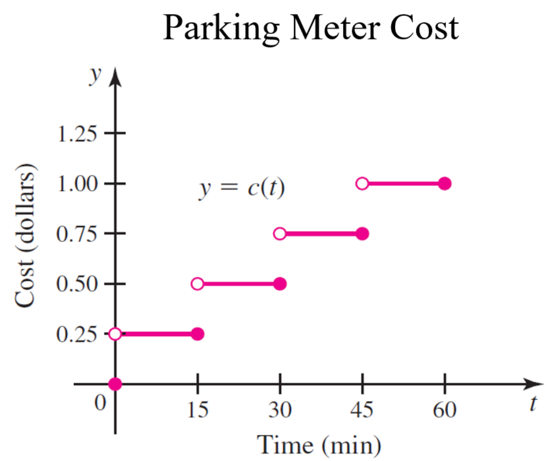
2.5 Continuity

Apollonius of Perga
262 – 190 B.C.

Apollonius was a Greek mathematician known as 'The Great Geometer'. His works had a very great influence on the development of mathematics and his famous book *Conics* introduced the terms parabola, ellipse and hyperbola.

Most of the techniques of calculus require that functions be continuous. A function is continuous if you can draw it in one motion without picking up your pencil.





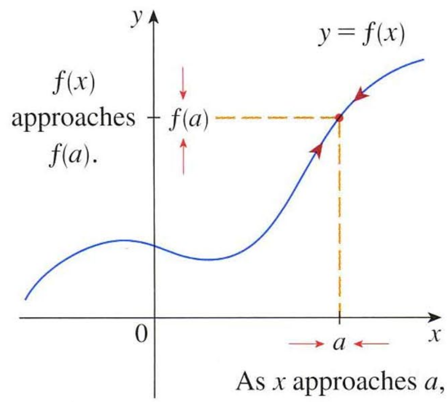
1 Definition A function f is **continuous at a number a** if

$$\lim_{x \to a} f(x) = f(a)$$

Continuity Checklist

In order for f to be continuous at a, the following three conditions must hold.

1. $f(a)$ is defined (a is in the domain of f).
2. $\lim_{x\to a} f(x)$ exists.
3. $\lim_{x\to a} f(x) = f(a)$ (the value of f equals the limit of f at a).



1. $f(a)$ is defined
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

Example Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

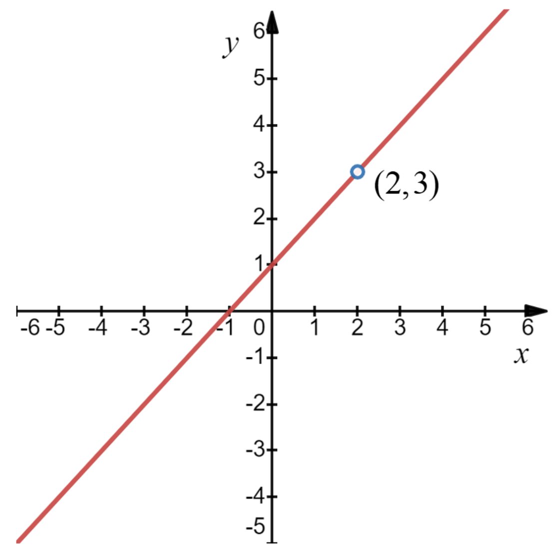
Solution

(a) From the Continuity Check list we get

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

1. $f(2)$ is undefined, division by zero. So, $f(x)$ is discontinuous at $x = 2$.

$f(x) = \dfrac{x^2 - x - 2}{x - 2}$



Undefined Discontinuity at $x = 2$.

Example Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

Solution

(b) From the Continuity Check list we get

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

1. $f(2) = 1$ is defined ✓

2.
$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2}$$
$$= \lim_{x \to 2} \frac{\cancel{(x - 2)}(x + 1)}{\cancel{x - 2}}$$
$$= \lim_{x \to 2} (x + 1)$$
$$= 3$$

So, the $\lim_{x \to 2} f(x) = 3$ exists. ✓

3. But
$$\lim_{x \to 2} f(x) \neq f(2)$$ ✗

So, f is not continuous at 2.

Example Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

$$f(r) = \sqrt[3]{4r^2 - 2r + 7}, \quad a = -2$$

Solution

From the Continuity Check list we get

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

1. $f(-2) = \sqrt[3]{4(-2)^2 - 2(-2) + 7}$

$= \sqrt[3]{27}$

$= 3$ is defined ✓

2. $\lim_{r \to -2} f(r) = \lim_{r \to -2} \sqrt[3]{4r^2 - 2r + 7}$

$= \sqrt[3]{\lim_{r \to -2} (4r^2 - 2r + 7)}$

$= \sqrt[3]{4(-2)^2 - 2(-2) + 7}$

$= \sqrt[3]{27}$

$= 3$ exists ✓

3. $\lim_{r \to -2} f(r) = f(-2)$ ✓

Hence, $f(r)$ is continuous at –2.

Example Explain why the function below is discontinuous at the given number a. Sketch the graph of the function.

$$f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases} \qquad a = 0$$

Solution

From the Continuity Check list we get

1. $f(a)$ is defined (that is, a is in the domain of f)

2. $\lim\limits_{x \to a} f(x)$ exists

3. $\lim\limits_{x \to a} f(x) = f(a)$

1. $f(0) = 0$ ✓

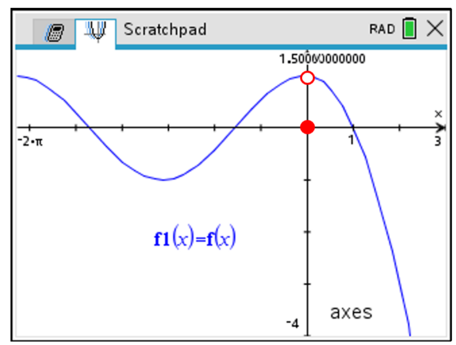
2. Since $\lim\limits_{x \to 0^-} \cos x = 1$ and $\lim\limits_{x \to 0^-} (1 - x^2) = 1$

then $\lim\limits_{x \to 0} f(x) = 1$ exists ✓

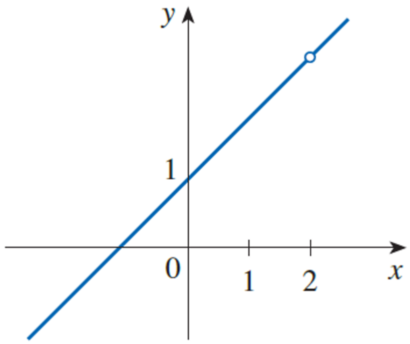
3. But

$$\lim_{x \to 0} f(x) \neq f(0) \quad ✗$$

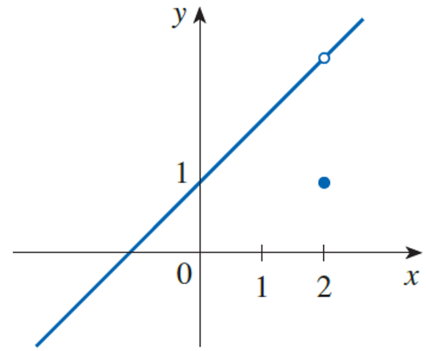
So, f is not continuous at $a = 0$.



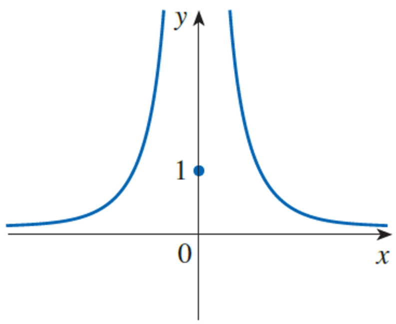
Types of Discontinuities



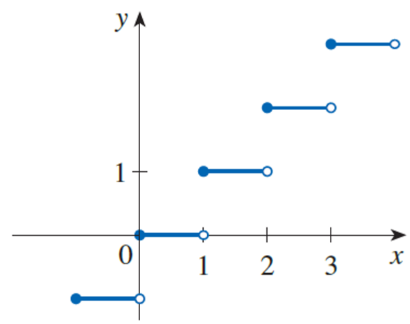
An undefined discontinuity



A removable discontinuity



An infinite discontinuity



A jump discontinuities

Example Consider the function

$$f(x)=\frac{x^3-1}{x-1}$$

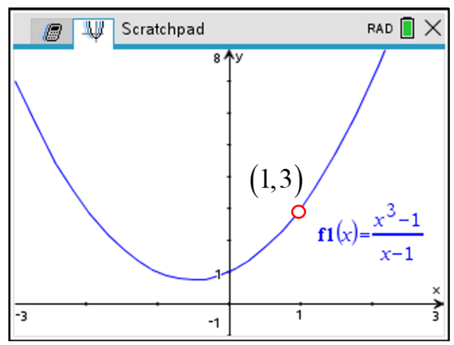
(a) Show that f has an undefined discontinuity at $x = 1$ (a hole in the graph).

(b) Redefine $f(x)$ as a piecewise function so that f is continuous at $x = 1$ (and thus "remove" the undefined discontinuity).

Solution Recall: $a^3-b^3=(a-b)(a^2+ab+b^2)$

$$\lim_{x\to 1}\frac{x^3-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1}=\lim_{x\to 1}(x^2+x+1)=1+1+1=3$$

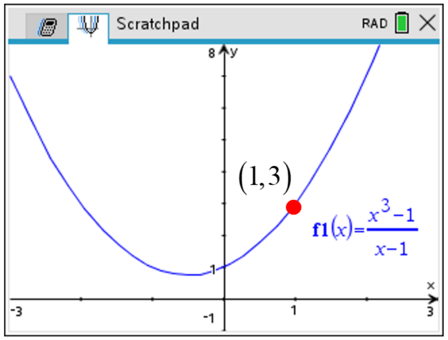
A hole at $(1,3)$, undefined discontinuity.



(b) Redefine $f(x)$ as a piecewise function so that f is continuous at $x = 1$ (and thus "remove" the undefined discontinuity).

Solution

$$f(x)=\begin{cases}\dfrac{x^3-1}{x-1}, & x\neq 1\\ 3, & x=1\end{cases}$$

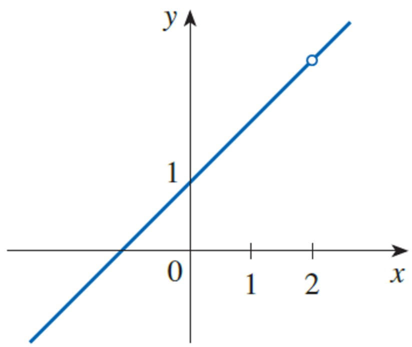


Take $x = 1$, it follows

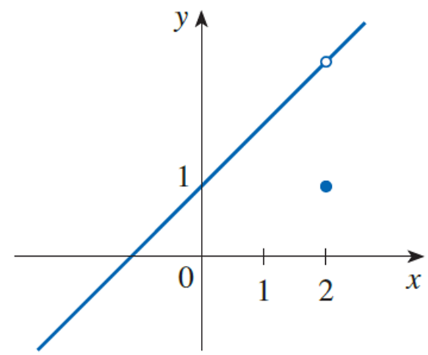
1. $f(1)=3$ exists
2. $\lim_{x\to 1} f(x)=3$ exists, shown in part (a)
3. $\lim_{x\to 1} f(x)=f(3)$

Hence, by definition f is continuous at $x = 1$.

Fixable Discontinuities

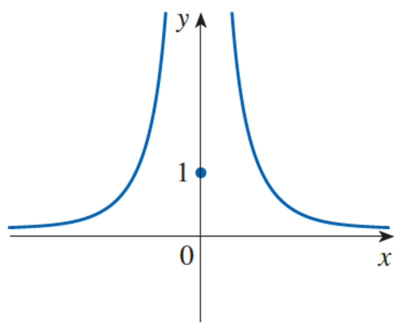


An undefined discontinuity

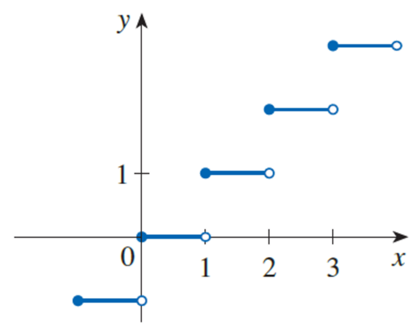


A removable discontinuity

Non-fixable Discontinuities



An infinite discontinuity



A jump discontinuities

2 Definition A function f is **continuous from the right at a number a** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at a** if

$$\lim_{x \to a^-} f(x) = f(a)$$

Continuity Checklist

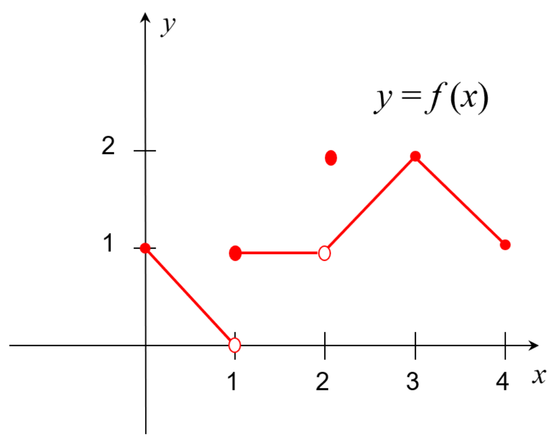
continuous from the right

1. $f(a)$ exists
2. $\lim_{x \to a^+} f(x)$ exists
3. $\lim_{x \to a^+} f(x) = f(a)$

continuous from the left

1. $f(a)$ exists
2. $\lim_{x \to a^-} f(x)$ exists
3. $\lim_{x \to a^-} f(x) = f(a)$

Example For the function below, discuss the integer values where $y = f(x)$ is continuous and explain.



Solution

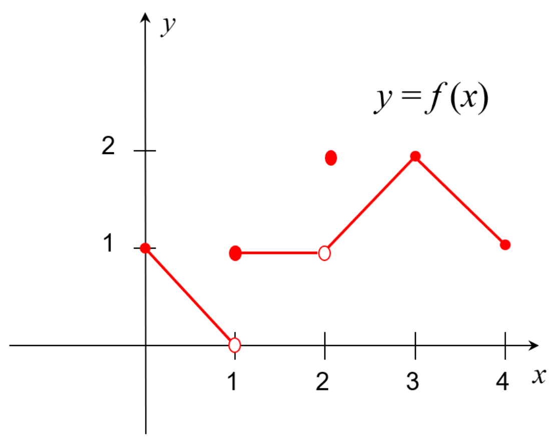
It is continuous at $x = 0$, $x = 3$ and $x = 4$.

Take $x = 0$, since

1. $f(0) = 1$ exists
2. $\lim_{x \to 0^+} f(x) = 1$ exists
3. $\lim_{x \to 0^+} f(x) = f(0)$

Hence, by definition f is continuous from the right at $x = 0$.
Sometimes called right-continuous.

Example For the function below, discuss the integer values where $y = f(x)$ is continuous and explain.



Solution

It is continuous at $x = 0$, $x = 3$ and $x = 4$.

Take $x = 3$, since

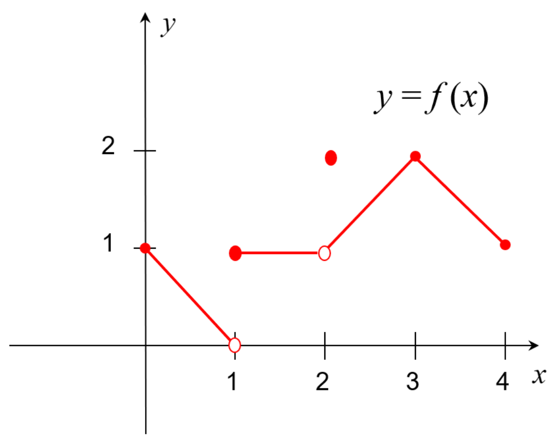
1. $f(3) = 2$ exists

2. $\lim_{x \to 3} f(x) = 2$ exists

3. $\lim_{x \to 3} f(x) = f(3)$

Hence, by definition f is continuous at $x = 3$.

Example For the function below, discuss the integer values where $y = f(x)$ is continuous and explain.



Solution

It is continuous at $x = 0$, $x = 3$ and $x = 4$.

Take $x = 4$, since

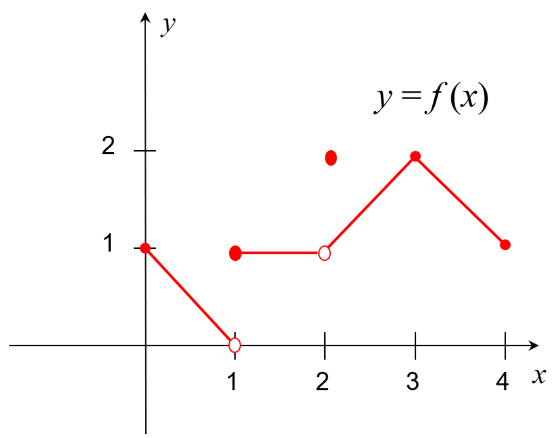
1. $f(4) = 1$ exists

2. $\lim_{x \to 4^-} f(x) = 1$ exists

3. $\lim_{x \to 4^-} f(x) = f(4)$

Hence, by definition f is continuous from the left at $x = 4$.
Sometimes called right-continuous.

Example For the function below, discuss the integer values where $y = f(x)$ is <u>not</u> continuous and explain.



Solution

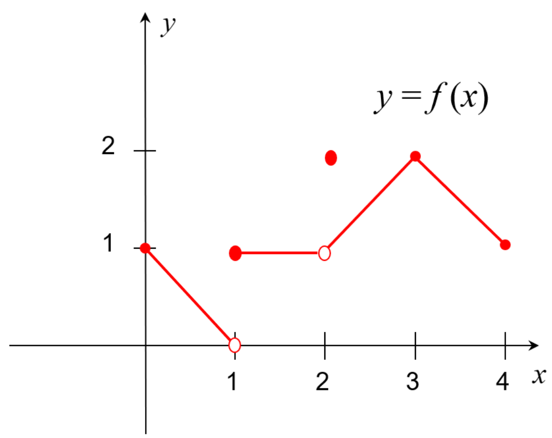
It is not continuous at $x = 1$ and $x = 2$.

Take $x = 1$, since

1. $f(1) = 1$ exists
2. $\lim_{x \to 1} f(x)$ DNE
3. $\lim_{x \to 1} f(x) \neq f(1)$

Hence, by definition f is discontinuous at $x = 1$.

Example For the function below, discuss the integer values where $y = f(x)$ is not continuous and explain.



Solution

It is not continuous at $x = 1$ and $x = 2$.

Take $x = 2$, since

1. $f(2) = 2$ exists

2. $\lim_{x \to 2} f(x) = 1$ exists

3. $\lim_{x \to 2} f(x) \neq f(2)$

Hence, by definition f is discontinuous at $x = 2$.

Determine the intervals where $y = f(x)$ is continuous.

$$x \in [0,1) \cup [1,2) \cup (2,4]$$

3 Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

Example Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

Solution If $-1 < a < 1$, then using the Limit Laws from Section 2.3, we have

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} \left(1 - \sqrt{1 - x^2}\right) \\
&= 1 - \lim_{x \to a} \sqrt{1 - x^2} && \text{(by Laws 2 and 8)} \\
&= 1 - \sqrt{\lim_{x \to a} (1 - x^2)} && \text{(by 7)} \\
&= 1 - \sqrt{1 - a^2} && \text{(by 2, 8, and 10)} \\
&= f(a)
\end{aligned}$$

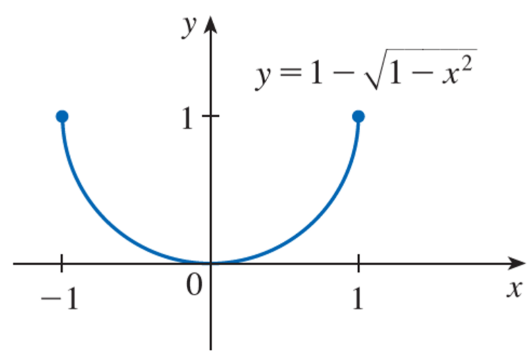
Thus, by Definition l, f is continuous at a if $-1 < a < 1$.

Example Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

Solution For the endpoints $x = -1$ and $x = 1$, similar calculations show that

$$\lim_{x \to -1^+} f(x) = 1 = f(-1) \qquad \text{and} \qquad \lim_{x \to 1^-} f(x) = 1 = f(1)$$

so f is continuous from the right at -1 and continuous from the left at 1. Therefore, according to Definition 3, f is continuous on $[-1, 1]$.



Properties of Continuous Functions

Some convenient theorems, which shows how to build up complicated continuous functions from simple ones.

4 Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$

5 Theorem

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

7 Theorem The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

8 Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Let's look at some examples.

Example Explain, using Theorems 4, 5, 7, and 9, why the function below is continuous at every number in its domain. State the domain.

$$L(v) = v \ln(1 - v^2)$$

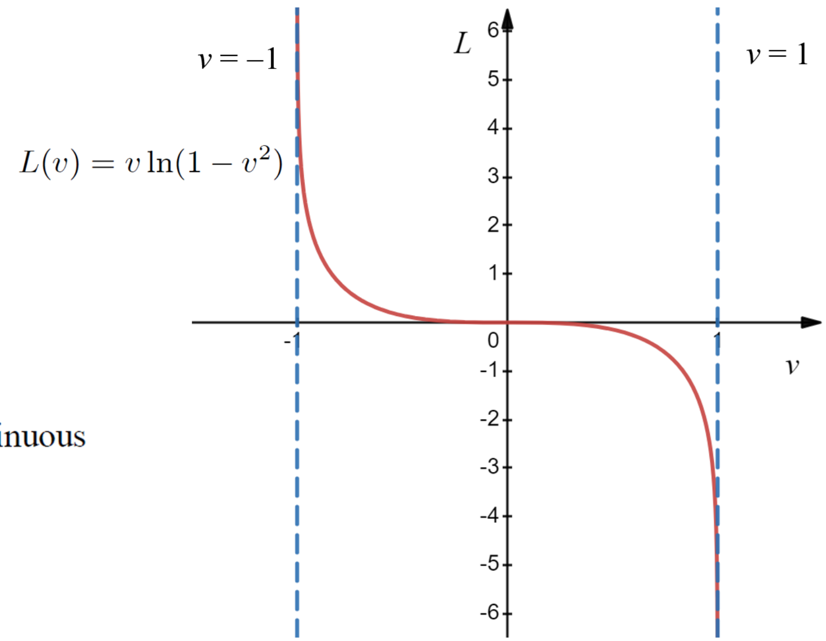
Solution

$L(v) = v \ln(1 - v^2)$ is defined when

$$1 - v^2 > 0$$
$$v^2 < 1$$
$$|v| < 1$$
$$-1 < v < 1$$

Thus, L(v) has domain $v \in (-1, 1)$.

Now v and the composite function $\ln(1 - v^2)$ are continuous on their domains by Theorems 7 and 9. Thus, by part 4 of Theorem 4, $L(v)$ is continuous on its domain.



Example Evaluate $\lim_{x \to 1} \arcsin\left(\frac{1 - \sqrt{x}}{1 - x}\right)$.

Solution Because arcsin is a continuous function, we can apply Theorem 8:

$$\begin{aligned}
\lim_{x \to 1} \arcsin\left(\frac{1 - \sqrt{x}}{1 - x}\right) &= \arcsin\left(\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x}\right) \\
&= \arcsin\left(\lim_{x \to 1} \frac{1 - \sqrt{x}}{(1 - \sqrt{x})(1 + \sqrt{x})}\right) \\
&= \arcsin\left(\lim_{x \to 1} \frac{1}{1 + \sqrt{x}}\right) \\
&= \arcsin \frac{1}{2} = \frac{\pi}{6}
\end{aligned}$$

The Intermediate Value Theorem

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. It is illustrated by Figure 8. Note that the value N can be taken on once [as in part (a)] or more than once [as in part (b)].

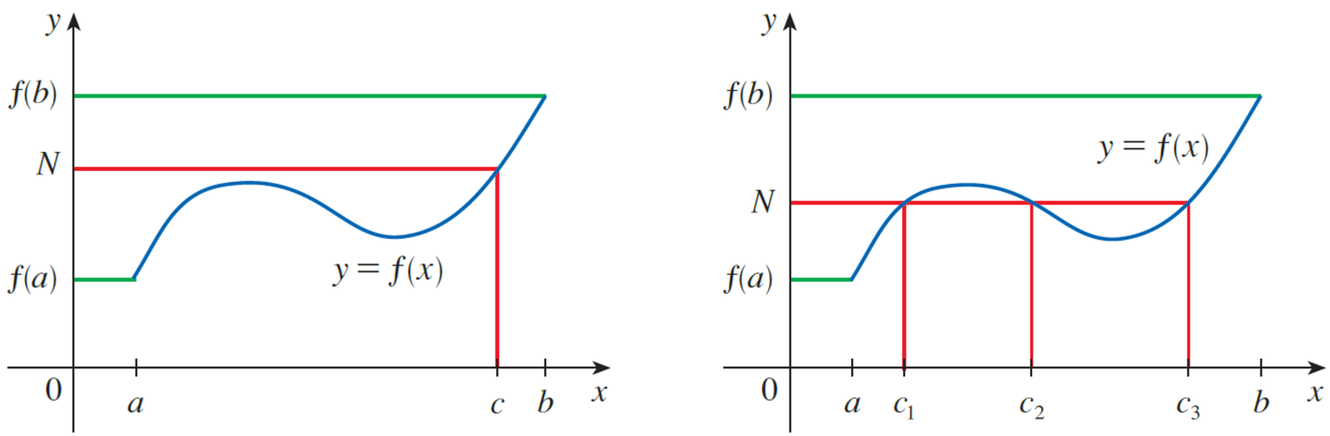


FIGURE 8 (a) (b)

Example Use the Intermediate Value Theorem to show that there is a solution of the given equation below in the specified interval.

$$e^x = 3 - 2x, \quad (0, 1)$$

Solution

The equation $e^x = 3 - 2x$ is equivalent to the equation $e^x + 2x - 3 = 0$.

$f(x) = e^x + 2x - 3$ is continuous on the interval $[0, 1]$,

$f(0) = -2$, and $f(1) = e - 1 \approx 1.72$.

Since $-2 < 0 < e - 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem.

Thus, there is a solution of the equation $e^x + 2x - 3 = 0$, or $e^x = 3 - 2x$, in the interval $(0, 1)$.