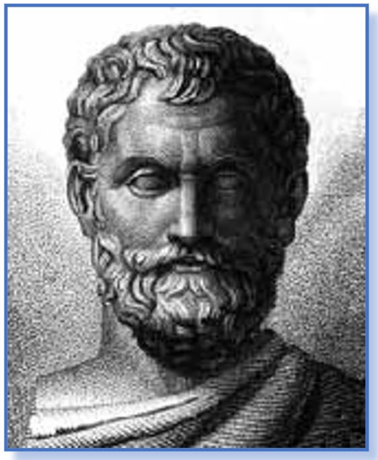
2.2 The Limit of a Function

The Ancients



Thales of Miletus
625 – 547 B.C.

i. A circle is bisected by any diameter.
ii. The base angles of an isosceles triangle are equal.
iii. The angles between two intersecting straight lines are equal.
iv. Two triangles are congruent if they have two angles and one side equal.
v. An angle in a semicircle is a right angle.

Thales was the first known Greek philosopher, scientist and mathematician. He is credited with five theorems of elementary geometry.

$$e^{i\pi}+1=0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{df}{dt} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

$$a^2 + b^2 = c^2$$

$$\log xy = \log x + \log y$$

$$F = G\frac{m_1 m_2}{d^2}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\hat{f}(\zeta) = \int_{-\infty}^{-\infty} f(x) e^{-2\pi i x \zeta} dx$$

$$\nabla \cdot E = 0 \quad \nabla \times E = -\frac{1}{c}\frac{\partial H}{\partial t}$$

$$\nabla \cdot H = 0 \quad \nabla \times H = \frac{1}{c}\frac{\partial E}{\partial t}$$

$$E = mc^2$$

$$i\hbar \frac{\partial}{\partial t} - \Psi = \hat{H}\Psi$$

$$y = f(x)$$

$$\lim_{x \to a} f(x) = L$$

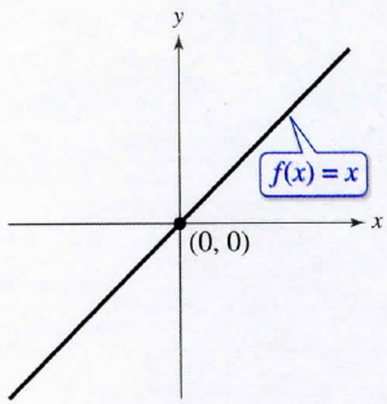
$$\frac{d}{dx} f(x) = f'(x)$$

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Library of Parent Functions

Linear Function

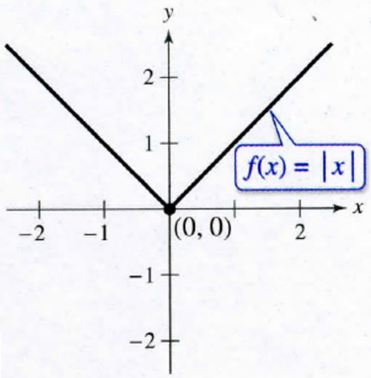
$f(x) = x$



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
Intercept: $(0, 0)$
Increasing

Absolute Value Function

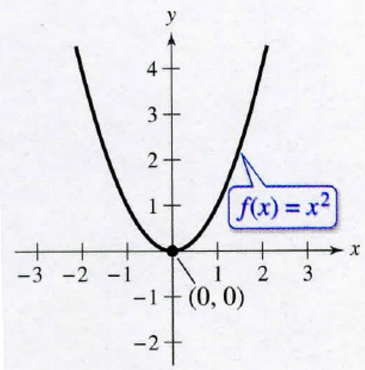
$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Intercept: $(0, 0)$
Decreasing on $(-\infty, 0)$
Increasing on $(0, \infty)$
Even function
y-axis symmetry

Quadratic Function

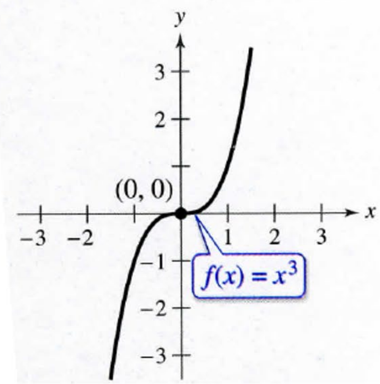
$f(x) = ax^2$



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Intercept: $(0, 0)$
Decreasing on $(-\infty, 0)$
Increasing on $(0, \infty)$
Even function
Axis of symmetry: $x = 0$
Relative minimum or vertex: $(0, 0)$

Cubic Function

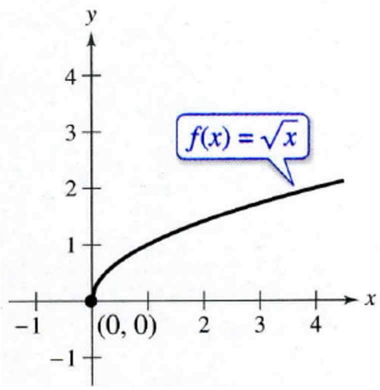
$f(x) = x^3$



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
Intercept: $(0, 0)$
Increasing on $(-\infty, \infty)$
Odd function
Origin symmetry

Square Root Function

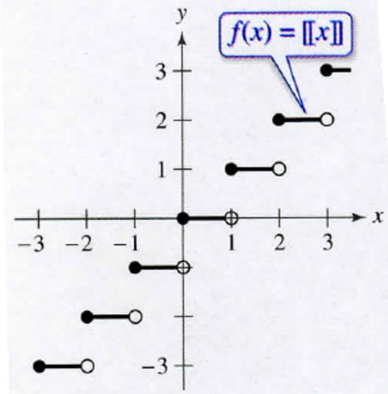
$f(x) = \sqrt{x}$



Domain: $[0, \infty)$
Range: $[0, \infty)$
Intercept: $(0, 0)$
Increasing on $(0, \infty)$

Greatest Integer Function

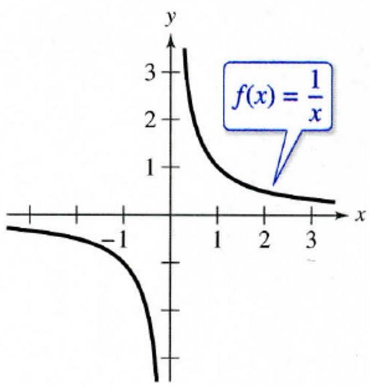
$f(x) = [\![x]\!]$



Domain: $(-\infty, \infty)$
Range: the set of integers
x-intercepts: in the interval $[0, 1)$
y-intercept: $(0, 0)$
Constant between each pair of consecutive integers
Jumps vertically one unit at each integer value

Rational Function

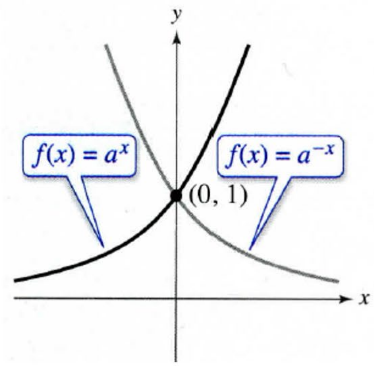
$f(x) = \frac{1}{x}$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$
No intercepts
Decreasing on $(-\infty, 0)$ and $(0, \infty)$
Odd function
Origin symmetry
Vertical asymptote: y-axis
Horizontal asymptote: x-axis

Exponential Function

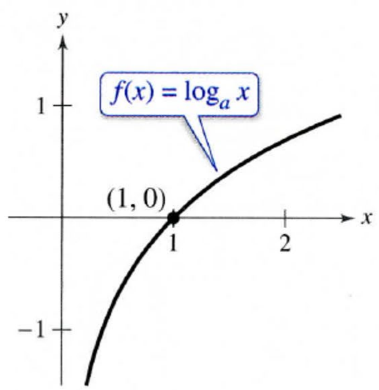
$f(x) = a^x,\ a > 0,\ a \neq 1$



Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Intercept: $(0, 1)$
Increasing on $(-\infty, \infty)$
for $f(x) = a^x$
Decreasing on $(-\infty, \infty)$
for $f(x) = a^{-x}$
x-axis is a horizontal asymptote
Continuous

Logarithmic Function

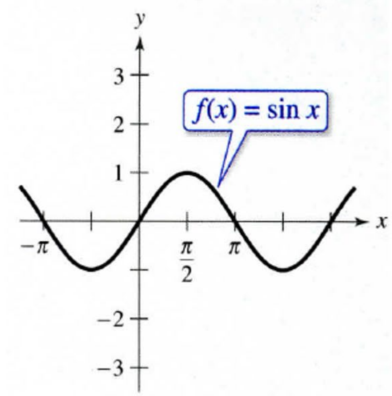
$f(x) = \log_a x, \ a > 0, \ a \neq 1$



Domain: $(0, \infty)$
Range: $(-\infty, \infty)$
Intercept: $(1, 0)$
Increasing on $(0, \infty)$
y-axis is a vertical asymptote
Continuous
Reflection of graph of $f(x) = a^x$ in the line $y = x$

Sine Function

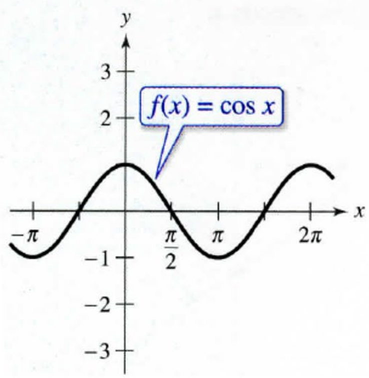
$f(x) = \sin x$



Domain: $(-\infty, \infty)$
Range: $[-1, 1]$
Period: 2π
x-intercepts: $(n\pi, 0)$
y-intercept: $(0, 0)$
Odd function
Origin symmetry

Cosine Function

$f(x) = \cos x$



Domain: $(-\infty, \infty)$

Range: $[-1, 1]$

Period: 2π

x-intercepts: $\left(\frac{\pi}{2} + n\pi, 0\right)$

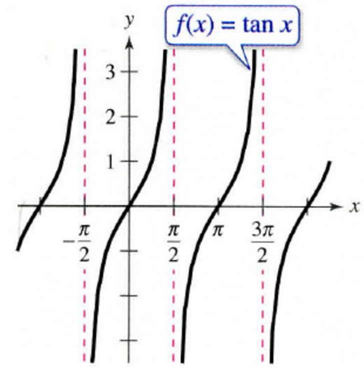
y-intercept: $(0, 1)$

Even function

y-axis symmetry

Tangent Function

$f(x) = \tan x$



Domain: $x \neq \frac{\pi}{2} + n\pi$

Range: $(-\infty, \infty)$

Period: π

x-intercepts: $(n\pi, 0)$

y-intercept: $(0, 0)$

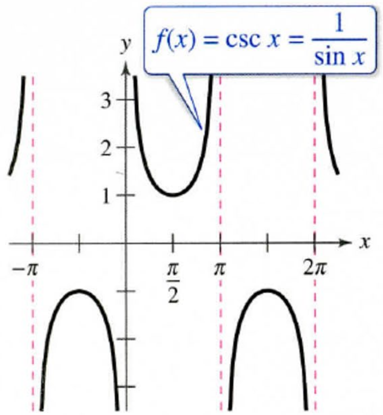
Vertical asymptotes: $x = \frac{\pi}{2} + n\pi$

Odd function

Origin symmetry

Cosecant Function

$f(x) = \csc x$



Domain: $x \neq n\pi$

Range: $(-\infty, -1] \cup [1, \infty)$

Period: 2π

No intercepts

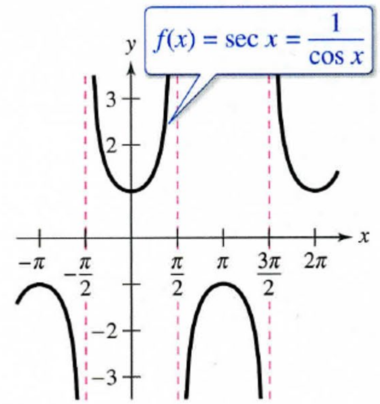
Vertical asymptotes: $x = n\pi$

Odd function

Origin symmetry

Secant Function

$f(x) = \sec x$



Domain: $x \neq \frac{\pi}{2} + n\pi$

Range: $(-\infty, -1] \cup [1, \infty)$

Period: 2π

y-intercept: $(0, 1)$

Vertical asymptotes:

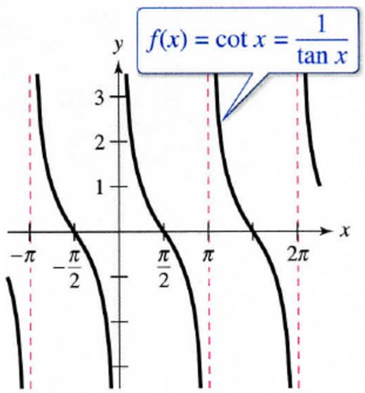
$$x = \frac{\pi}{2} + n\pi$$

Even function

y-axis symmetry

Cotangent Function

$f(x) = \cot x$



Domain: $x \neq n\pi$

Range: $(-\infty, \infty)$

Period: π

x-intercepts: $\left(\frac{\pi}{2} + n\pi, 0\right)$

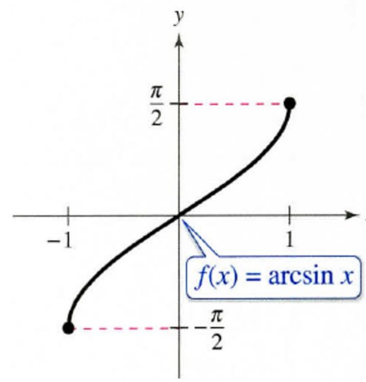
Vertical asymptotes: $x = n\pi$

Odd function

Origin symmetry

Inverse Sine Function

$f(x) = \arcsin x$



Domain: $[-1, 1]$

Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

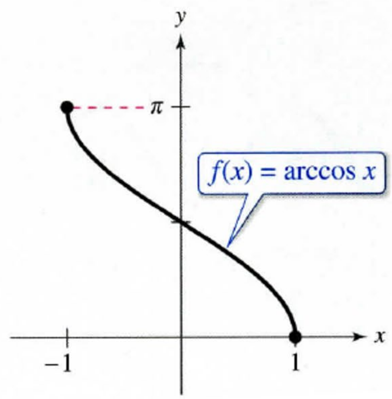
Intercept: $(0, 0)$

Odd function

Origin symmetry

Inverse Cosine Function

$f(x) = \arccos x$



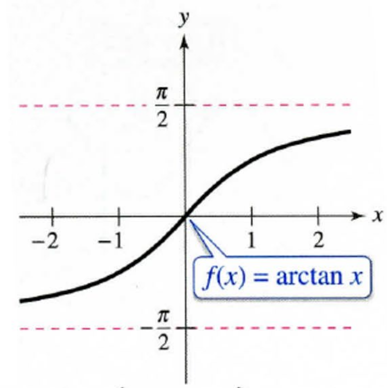
Domain: $[-1, 1]$

Range: $[0, \pi]$

y-intercept: $\left(0, \frac{\pi}{2}\right)$

Inverse Tangent Function

$f(x) = \arctan x$



Domain: $(-\infty, \infty)$

Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Intercept: $(0, 0)$

Horizontal asymptotes: $y = \pm\frac{\pi}{2}$

Odd function

Origin symmetry

Example Consider the function $f(x) = x^2 - 5x - 6$.

As x approaches 2, what is the behavior of $f(x)$?

Solution

There will be three strategies for analyzing this question.

1.) Graphically

2.) Numerically

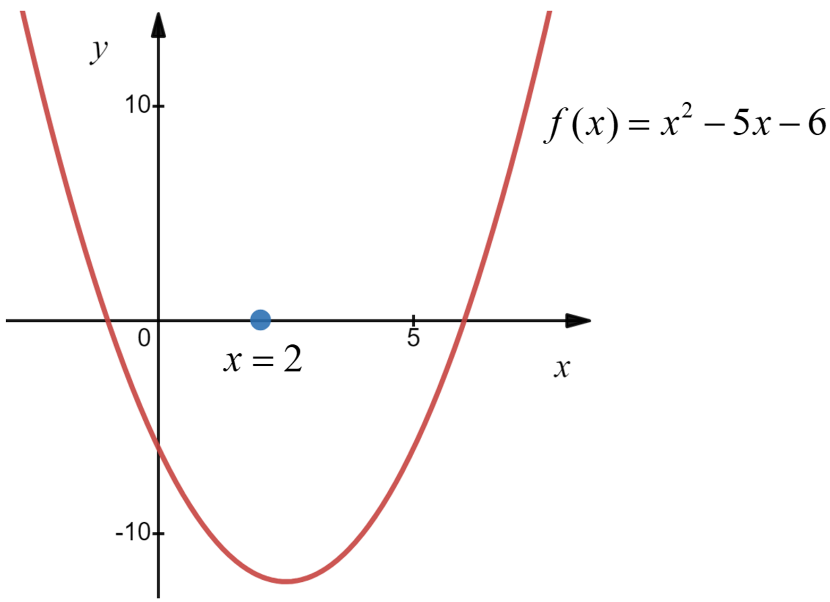
3.) Algebraically/Symbolically

In section 2.2, our analysis will be graphical and numerical, the warm and fuzzy way!

$f(x) = x^2 - 5x - 6$, so what happens to $f(x)$ as x approaches 2?

We write this question mathematically

$$\lim_{x \to 2} x^2 - 5x - 6.$$



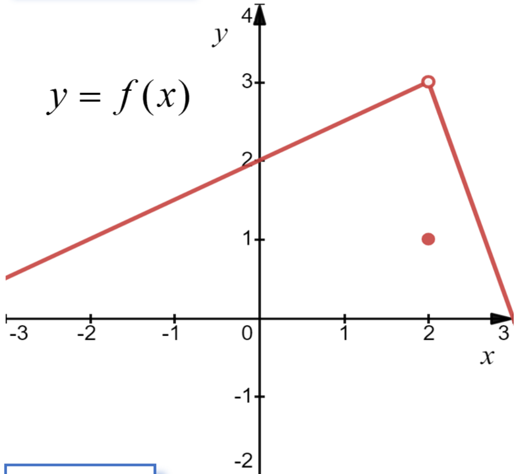
So we write

$$\lim_{x \to 2} x^2 - 5x - 6 = 12.$$

$x = 2$ is called the limit point.

The limit of a function refers to the value that the function approaches, not the actual value (if any).

Example Consider the function $y = f(x)$ below. Find $\lim_{x \to 2} f(x)$ and $f(2)$.



Solution

$$\lim_{x \to 2} f(x) = 3$$

$$f(2) = 1$$

1 Intuitive Definition of a Limit Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

An alternative notation for

$$\lim_{x \to a} f(x) = L$$

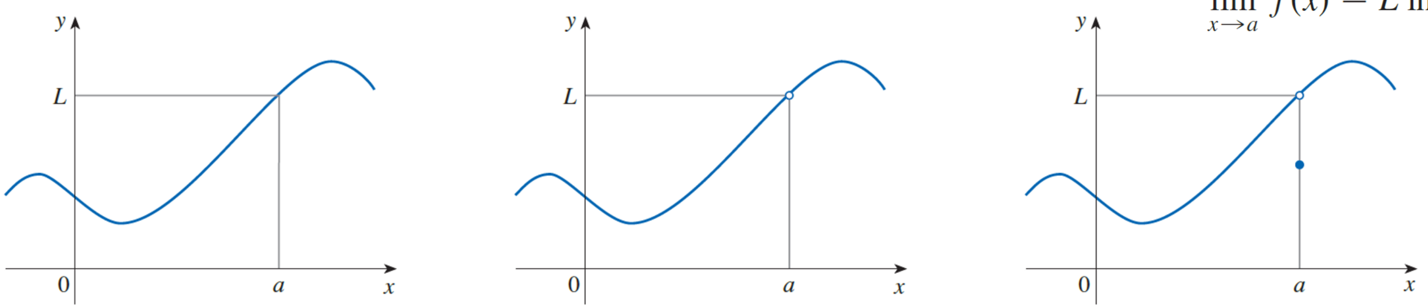
is

$$f(x) \to L \qquad \text{as} \qquad x \to a$$

which is usually read "$f(x)$ approaches L as x approaches a."

Notice the phrase "but x not equal to a" in the definition of limit. This means that in finding the limit of $f(x)$ as x approaches a, we never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. The only thing that matters is how f is defined *near* a.

The figure below shows the graphs of three functions. Note that in part (b), $f(a)$ is not defined and in part (c), $f(a) \neq L$. But in each case, regardless of what happens at a, it is true that $\lim_{x \to a} f(x) = L$.



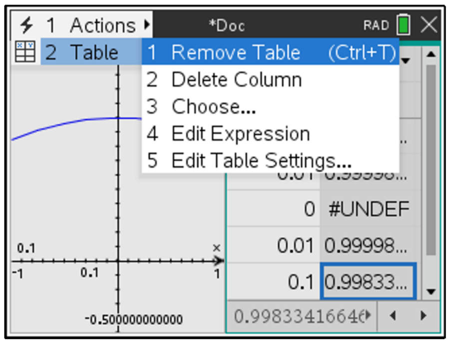
(a) (b) (c)

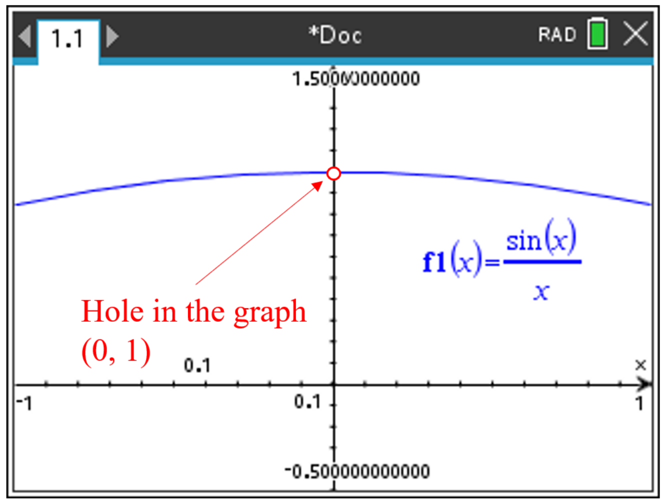
$\lim_{x \to a} f(x) = L$ in all three cases

Example

Guess the value of $\lim_{x\to 0}\frac{\sin x}{x}$.

Solution

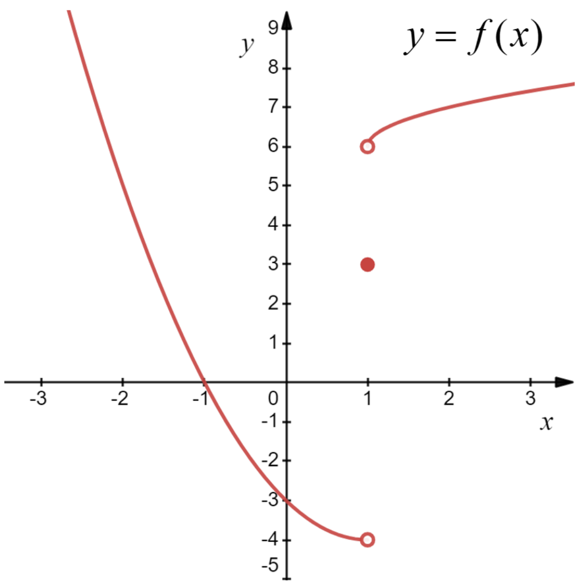




So it appears, $\lim_{x\to 0}\frac{\sin x}{x}=1$.

Example Consider the function $y = f(x)$ below. Find $\lim_{x \to 1} f(x)$.

Solution



So what is $\lim_{x \to 1} f(x)$?

<u>Limit from the left</u>

$$\lim_{x \to 1^-} f(x) = -4$$

From the left

<u>Limit from the right</u>

$$\lim_{x \to 1^+} f(x) = 6$$

From the right

These are called one-sided or directional limits.

Since,

$$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

We say,

$$\lim_{x \to 1} f(x) \text{ Does Not Exist}$$

Important Note,

$$\lim_{x \to 1} f(x) \text{ Does Not Exist.}$$

But,

$$f(1) = 3.$$

2 Intuitive Definition of One-Sided Limits We write

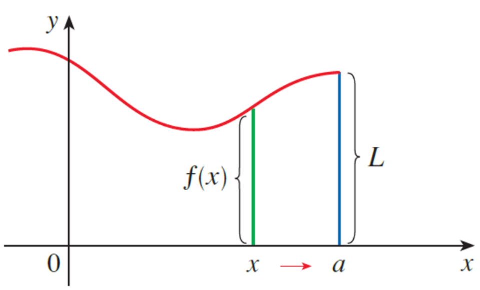
$$\lim_{x \to a^-} f(x) = L$$

and say that the **left-hand limit** of $f(x)$ as x approaches a [or the limit of $f(x)$ as x approaches a *from the left*] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a with x *less than* a.

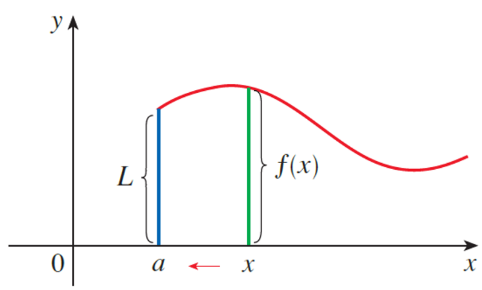
We write

$$\lim_{x \to a^+} f(x) = L$$

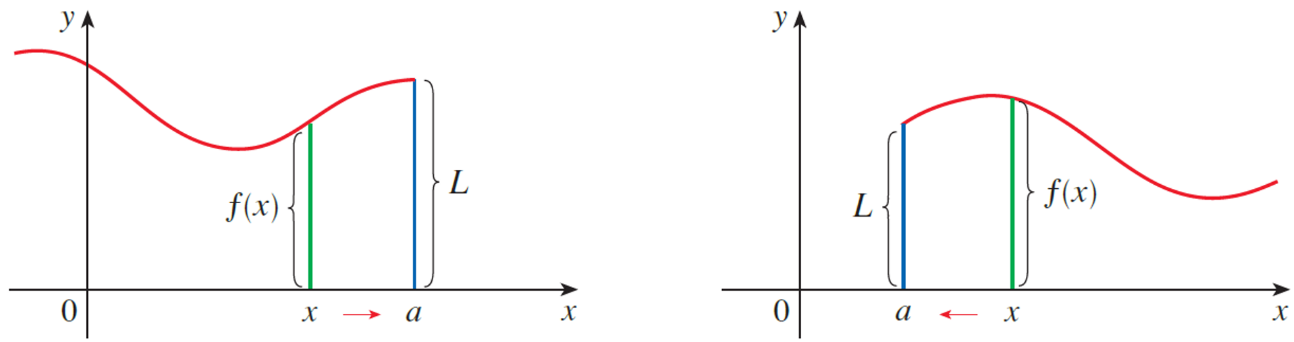
and say that the **right-hand limit** of $f(x)$ as x approaches a [or the limit of $f(x)$ as x approaches a *from the right*] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a with x *greater than* a.



(a) $\lim_{x \to a^-} f(x) = L$



(b) $\lim_{x \to a^+} f(x) = L$



(a) $\lim_{x\to a^-} f(x) = L$ (b) $\lim_{x\to a^+} f(x) = L$

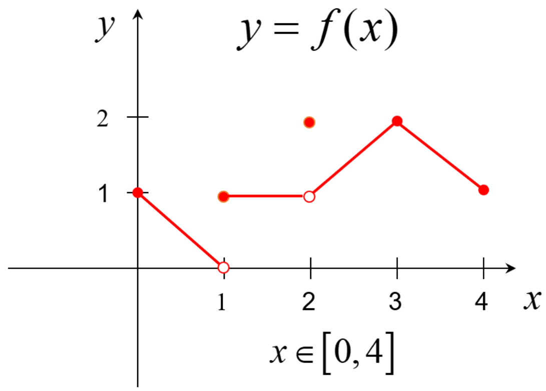
3 $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$

Non-directional limit

Directional limits
or
One-sided limits

Example Consider the function $y = f(x)$ on $[0,4]$ below. Find $\lim_{x \to a} f(x)$ for $a = 1, 2, 3.$

Solution



Interior points $a = 1$, $a = 2$, and $a = 3$.

At $a = 1$: $\lim_{x \to 1^-} f(x) = 0$ ← left hand limit

$\lim_{x \to 1^+} f(x) = 1$ ← right hand limit

$f(1) = 1$ ← value of the function

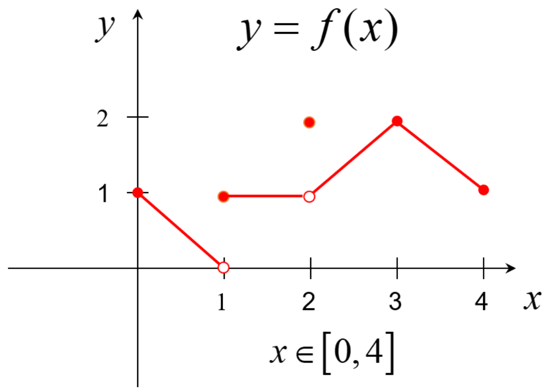
$\lim_{x \to 1} f(x)$ does not exist

because the left and right hand limits are not equal!

$$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

Example Consider the function $y = f(x)$ on $[0,4]$ below. Find $\lim_{x\to a} f(x)$ for $a = 1, 2, 3$.

Solution



Interior points $a = 1$, $a = 2$, and $a = 3$.

At $a = 2$: $\lim_{x\to 2^-} f(x) = 1$ ← left hand limit

$\lim_{x\to 2^+} f(x) = 1$ ← right hand limit

$f(2) = 2$ ← value of the function

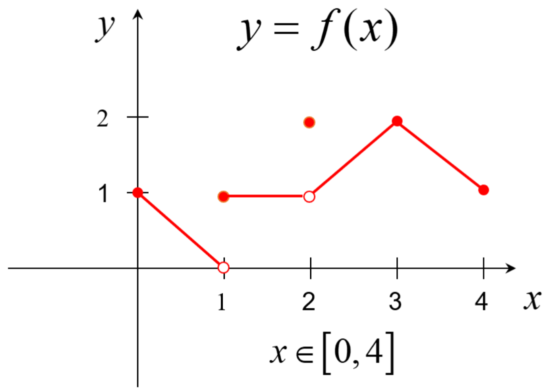
$$\lim_{x\to 2} f(x) = 1$$

because the left and right hand limits are equal.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$$

Example Consider the function $y = f(x)$ on $[0,4]$ below. Find $\lim_{x\to a} f(x)$ for $a = 1, 2, 3$.

Solution



Interior points $a = 1$, $a = 2$, and $a = 3$.

At $a = 3$: $\lim_{x\to 3^-} f(x) = 2$ ← left hand limit

$\lim_{x\to 3^+} f(x) = 2$ ← right hand limit

$f(2) = 2$ ← value of the function

$$\lim_{x\to 2} f(x) = 2$$

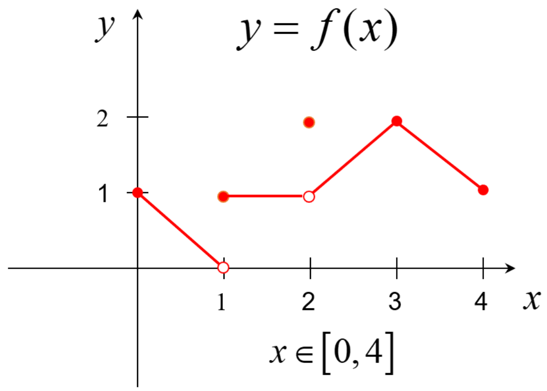
because the left and right hand limits are equal.

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$$

All three are equal! I wonder if that is something special?

Example Consider the function $y = f(x)$ on $[0,4]$ below. Find $\lim_{x \to a} f(x)$ for $a = 0, 4$.

Solution



Limits at the endpoints $a = 0$ and $a = 4$.

At $a = 0$: $\lim_{x \to 0^+} f(x) = 1$ ← right hand limit

At $a = 4$: $\lim_{x \to 4^-} f(x) = 1$ ← left hand limit

Limits can only approach through domain values.

Example Consider the function $f(x) = \dfrac{x^3 - 1}{x - 1}$.

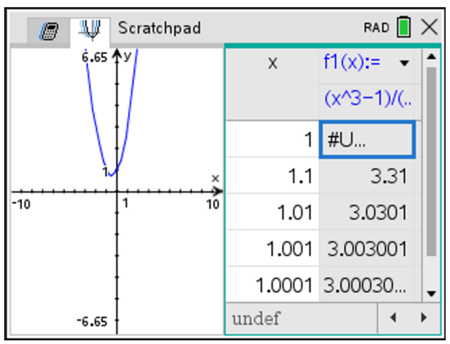
a. Calculate $f(x)$ for each value of x in the following table.

b. Make a conjecture about the value of $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1}$.

x	0.9	0.99	0.999	0.9999
$f(x) = \dfrac{x^3 - 1}{x - 1}$	2.71	2.970	2.997	2.9997
x	1.1	1.01	1.001	1.0001
$f(x) = \dfrac{x^3 - 1}{x - 1}$	3.31	3.0301	3.003001	3.00030

Solution

a.



b. $\lim\limits_{x \to 1^-} f(x) = 3$

$\lim\limits_{x \to 1^+} f(x) = 3$

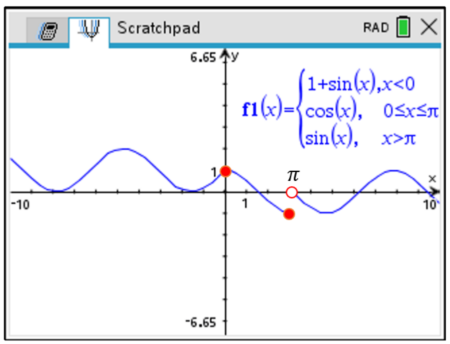
$\lim\limits_{x \to 1} f(x) = 3$

Example

Sketch the graph of the function and use it to determine the values from the domain for which the limit exists.

$$f(x)=\begin{cases}1+\sin x & \text{if } x<0\\ \cos x & \text{if } 0\le x\le \pi\\ \sin x & \text{if } x>\pi\end{cases}$$

Solution



We need to check $a=0$ and $a=\pi$.

At $a=0$: $\lim\limits_{x\to 0^-}(1+\sin x)=1$

$$\lim_{x\to 0^+}\cos x=1$$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)$$

So, $\lim\limits_{x\to 0} f(x)=1$ and the limit exists at $a=0$.

At $a=\pi$: $\lim\limits_{x\to \pi^-}\cos x=-1$

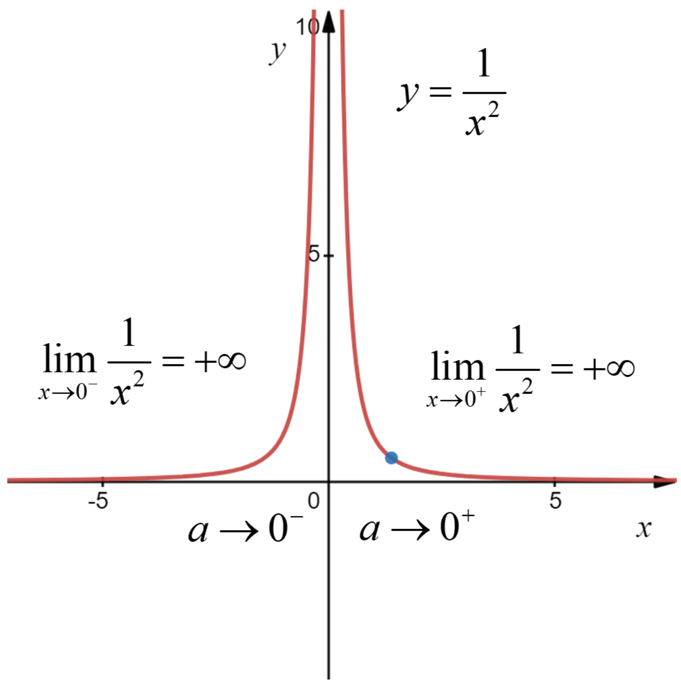
$$\lim_{x\to \pi^+}\sin x=0$$

$$\lim_{x\to \pi^-} f(x)\ne\lim_{x\to \pi^+} f(x)$$

So, $\lim\limits_{x\to \pi} f(x)$ DNE, hence the limit exists for values in the domain except at $a=\pi$.

Infinite Limits; Vertical Asymptotes

What is the $\lim_{x \to 0} \frac{1}{x^2}$?

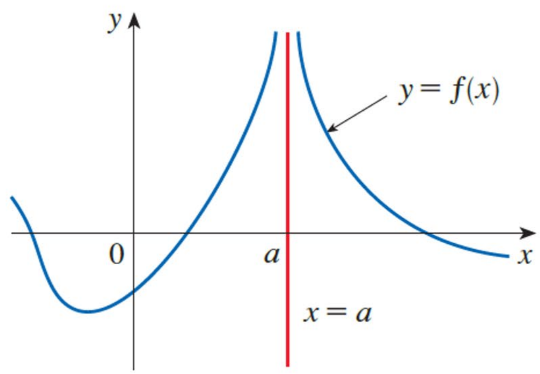


So we say, $\lim_{x \to 0} \frac{1}{x^2} = +\infty$.

4 Intuitive Definition of an Infinite Limit Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.



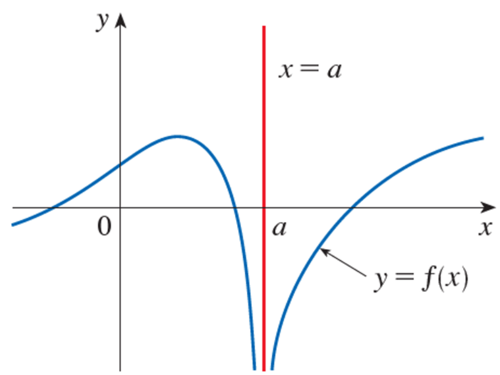
$$\lim_{x \to a} f(x) = \infty$$

A similar sort of limit, for functions that become large negative as x gets close to a, is defined in Definition 5.

5 Definition Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

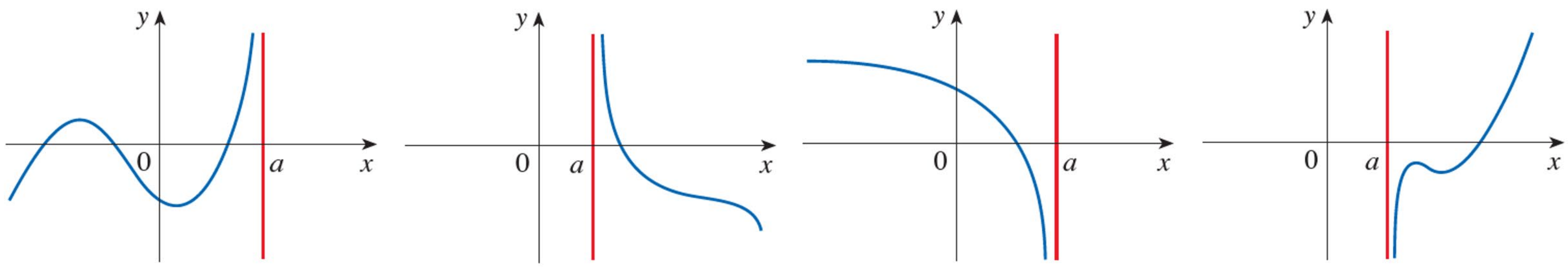


$$\lim_{x \to a} f(x) = -\infty$$

Similar definitions can be given for the one-sided infinite limits

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$



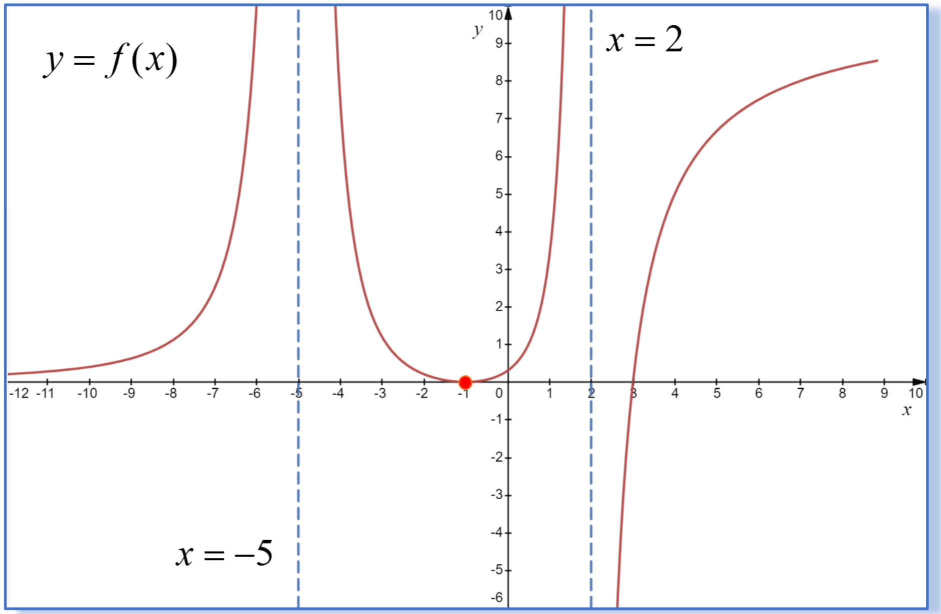
(a) $\lim_{x \to a^-} f(x) = \infty$ (b) $\lim_{x \to a^+} f(x) = \infty$ (c) $\lim_{x \to a^-} f(x) = -\infty$ (d) $\lim_{x \to a^+} f(x) = -\infty$

6 Definition The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Example Given the function $y = f(x)$ below, find the given limits and determine any vertical asymptotes.



a) $\lim_{x \to 2^-} f(x) = +\infty$

b) $\lim_{x \to 2^+} f(x) = -\infty$

c) $\lim_{x \to 2} f(x) =$ DNE

d) $f(-1) = 0$

e) $\lim_{x \to -5^-} f(x) = +\infty$

f) $\lim_{x \to -5^+} f(x) = +\infty$

g) $\lim_{x \to -5} f(x) = +\infty$

Example

Use a table of values to estimate the value of the limit. If you have a graphing device, use it to confirm your result graphically.

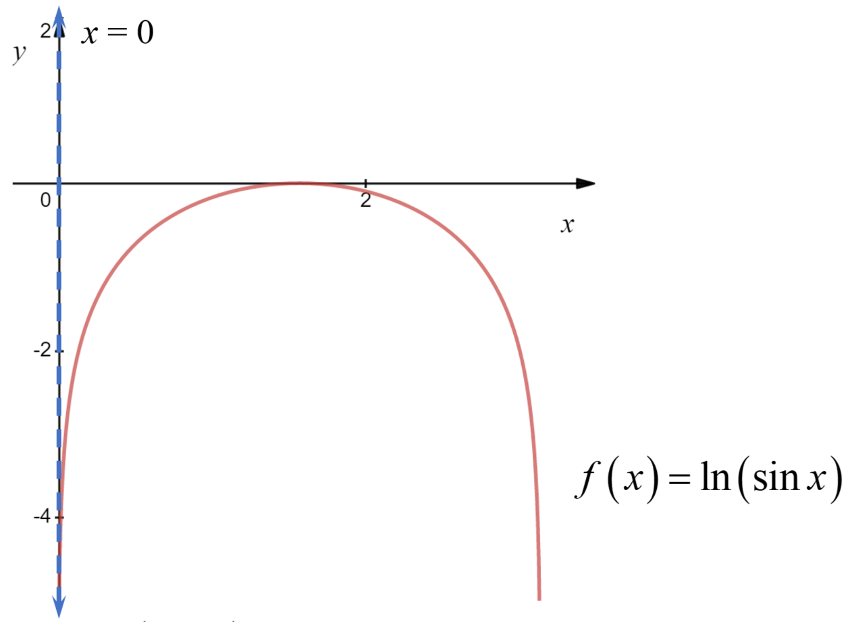
$$\lim_{x \to 0^+} \ln(\sin x)$$

Solution

Let $f(x) = \ln(\sin x)$

x	y
1	-0.1726
0.1	-2.30425
0.01	-4.60519
0.001	-6.90776
0.000001	-13.8155
1E-12	-27.631

$$\lim_{x \to 0^+} \ln(\sin x) = -\infty$$



$$\lim_{x \to 0^+} \ln(\sin x) = -\infty$$

Hence, there is a vertical asymptote $x = 0$.

Example

Use a table of values to estimate the value of the limit. If you have a graphing device, use it to confirm your result graphically.

$$\lim_{t \to 0} \frac{5^t - 1}{t}$$

Solution

$$\text{Let } f(t) = \frac{5^t - 1}{t}$$

$t \to 0^-$

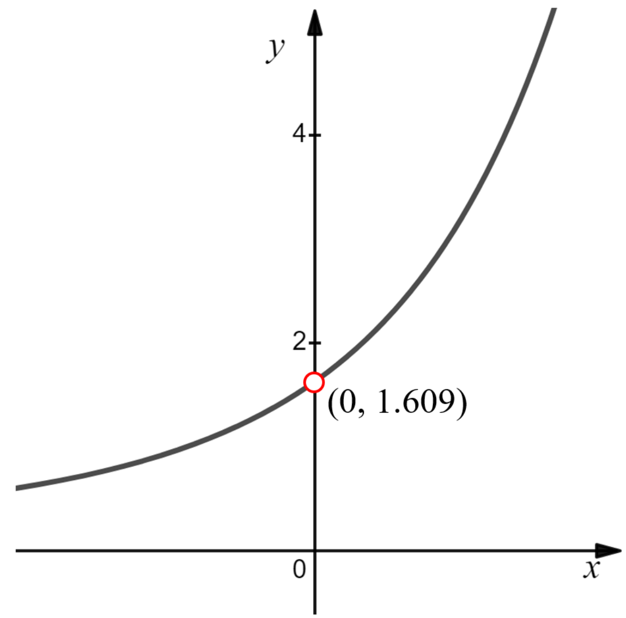
-1	0.8
-0.1	1.486601
-0.01	1.596556
-0.001	1.608143
-0.0001	1.609308
-0.00001	1.609425

$$\lim_{x \to 0^-} f(t) = 1.609$$

$t \to 0^+$

1	4
0.1	1.746189
0.01	1.622459
0.001	1.610734
0.0001	1.609567
0.00001	1.609451

$$\lim_{x \to 0^+} f(t) = 1.609$$



Hence, there is a hole (0, 1.609).