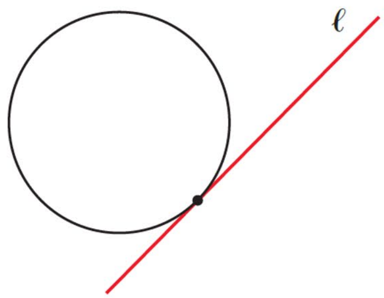
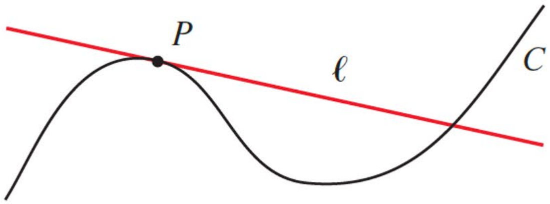
2.1 The Tangent and Velocity Problems





Archimedes of Syracuse
287 – 212 B.C.

Archimedes was the greatest mathematician of his age. His contributions in geometry revolutionized the subject and his methods anticipated the integral calculus 2,000 years before Newton and Leibniz.

Example

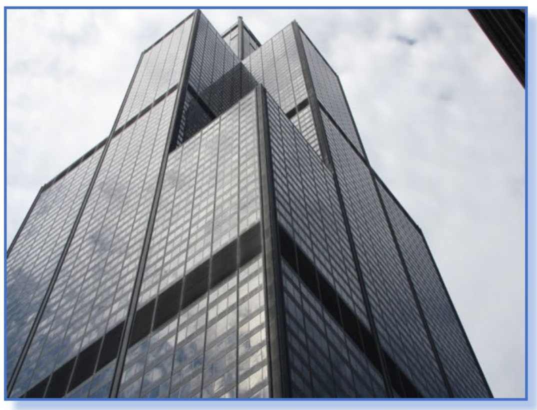
An object is dropped from the top of the Sears building in Chicago. The height of the building is 1450 feet. By Galileo's law we get function,

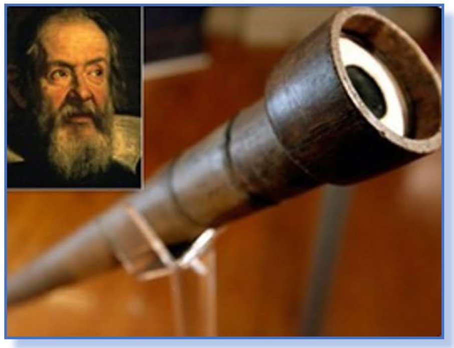
$$s(t) = -16t^2 + 1450 \qquad \text{Position Fuction}$$

Where t is time in seconds, and $s(t)$ is the position of the object above the ground, feet, after t seconds.

(a) Find the average velocity from 2 seconds to 6 seconds.

(b) Find the velocity at 6 seconds.





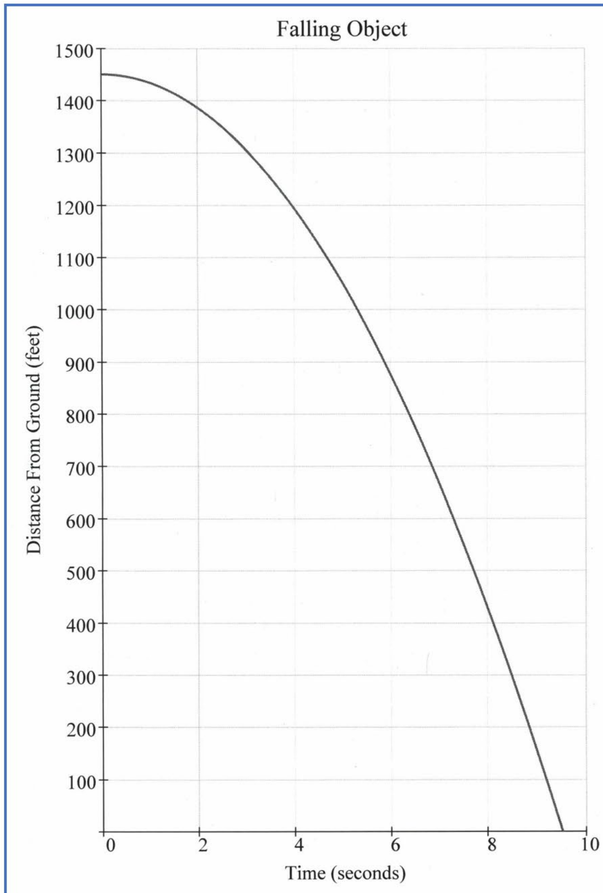
Galileo Galilei
1564 - 1642

Galileo was an Italian scientist who formulated the basic law of falling bodies, which he verified by careful measurements. He constructed a telescope with which he studied lunar craters and discovered four moons revolving around Jupiter and espoused the Copernican cause.

Example

$$s(t) = -16t^2 + 1450$$

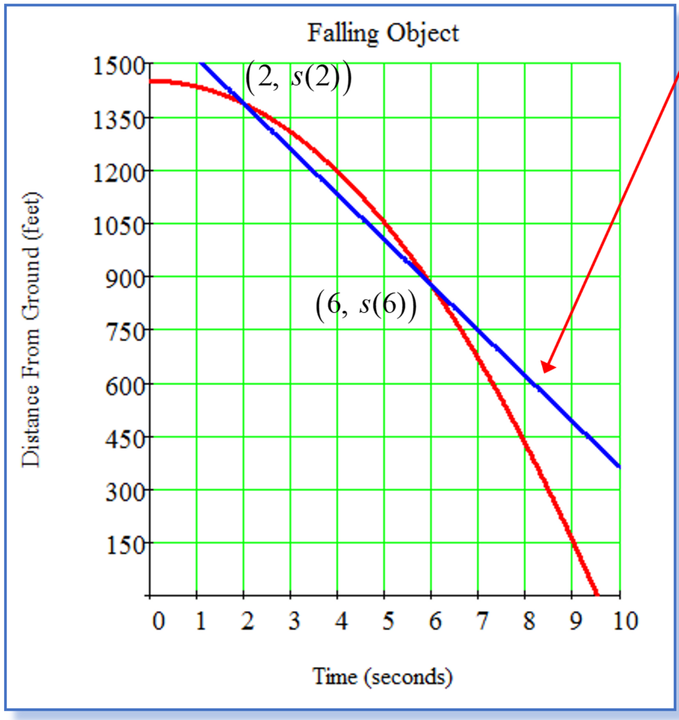
t =	$s(t)$ =
0	1450
1	1434
2	1386
3	1306
4	1194
5	1050
6	874
7	666
8	426
9	154



(a) Find the average velocity over the interval [2, 6] seconds.

Solution $s(t) = -16t^2 + 1450$

Secant Line



Its slope represents the average velocity we get

$$v_{\text{avg}} = \frac{s(2) - s(6)}{2 - 6}$$

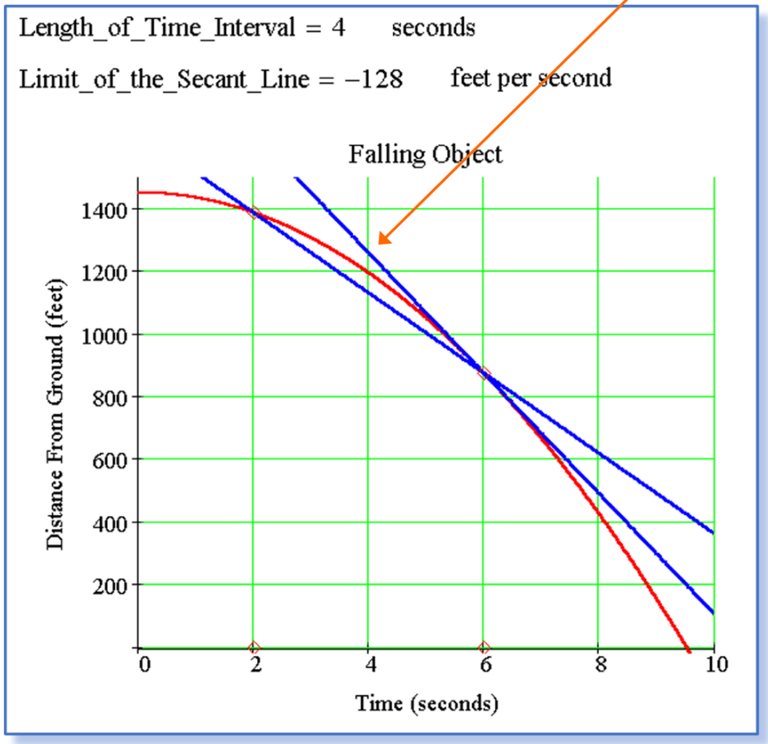
$$v_{\text{avg}} = \frac{1386 \text{ ft} - 874 \text{ ft}}{2 \text{ sec} - 6 \text{ sec}}$$

$$v_{\text{avg}} = -128 \text{ ft/sec}$$

So, the average velocity the object is falling over the time interval [2, 6] seconds is –128 ft/sec.

The negative sign indicates that the object is falling.

(b) Find the velocity at time 6 seconds. Tangent Line



The slope of the tangent line represents the exact velocity of the object at time 6 seconds.

What is the slope?

Let's examine the following time intervals for average velocity:

[5,6]

[5.5,6]

[5.9,6]

[5.99,6]

$$s(t) = -16t^2 + 1450$$

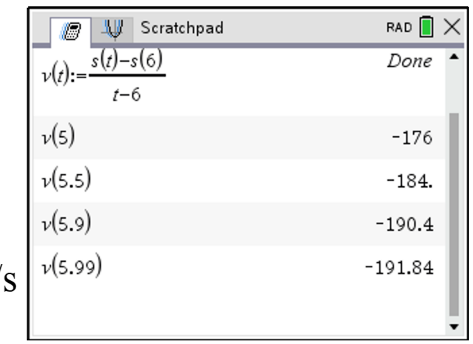
$$v_{avg}(t) = \frac{s(t) - s(6)}{t - 6}$$

$v_{avg}(5) = -176$ ft/s

$v_{avg}(5.5) = -184$ ft/s

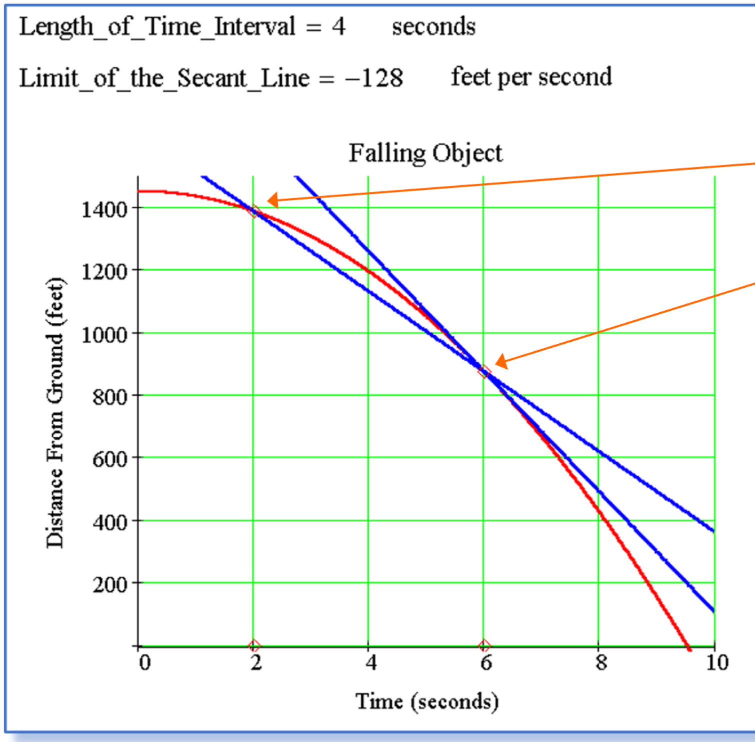
$v_{avg}(5.9) = -190.4$ ft/s

$v_{avg}(5.9) = -191.84$ ft/s



So it appear the velocity at 6 seconds is close to −192 ft/s.

(b) Find the velocity at time 6 seconds.



The slope of the tangent line represents the exact velocity of the object at time 6 seconds.

What is the exact slope?

$$(t,\ s(t))$$

$$(6,\ s(6))$$

$$v_{avg}(t) = \frac{s(t) - s(6)}{t - 6}$$

As t approaches 6, the limit of the secant line is the tangent line. We calculate the **Instantaneous Velocity** with the following limit

$$v_{inst} = \lim_{t \to 6} v_{avg}$$

(b) Find the velocity at time 6 seconds.

Using limits we get

$$v_{\text{avg}} = \frac{s(t) - s(6)}{t - 6}$$

$$v_{\text{inst}} = \lim_{t \to 6} v_{\text{avg}}$$

$$= \lim_{t \to 6} \frac{s(t) - s(6)}{t - 6}$$

$$= \lim_{t \to 6} \frac{-16t^2 + 1450 - 874}{t - 6}$$

$$= \lim_{t \to 6} \frac{-16t^2 + 576}{t - 6}$$

$$= \lim_{t \to 6} \frac{-16\left(t^2 - 36\right)}{t - 6}$$

$$= \lim_{t \to 6} \frac{-16\cancel{(t-6)}(t+6)}{\cancel{t-6}}$$

$$= \lim_{t \to 6} \; -16(t+6)$$

$$= -16(12)$$

$$= -192 \text{ ft/sec}$$

We say that the **Instantaneous Velocity** at $t = 6$ seconds is -192 ft/sec.

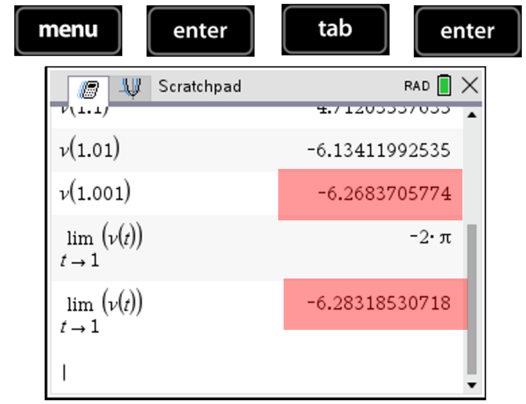
Example The displacement (in centimeters) of a particle moving back and forth along a straight line is given by the equation of motion $s = 2\sin \pi t + 3\cos \pi t$, where t is measured in seconds.

(a) Find the average velocity during each time period:

(i) $[1, 2]$ (ii) $[1, 1.1]$

(iii) $[1, 1.01]$ (iv) $[1, 1.001]$

(b) Estimate the instantaneous velocity of the particle when $t = 1$.



Solution

(a) (i) $s = s(t)$

$= 2\sin \pi t + 3\cos \pi t$

On the interval $[1, 2]$,

$$v_{avg}(t) = \frac{s(t) - s(1)}{t - 1}$$

$$v_{avg}(2) = \frac{s(2) - s(1)}{2 - 1}$$

$$v_{avg}(2) = 6 \text{ cm/s}$$

(ii) On the interval $[1, 1.1]$,

$$v_{avg}(1.1) = \frac{s(1.1) - s(1)}{1.1 - 1}$$

$$v_{avg}(1.1) = -4.712 \text{ cm/s}$$

(iii) On the interval $[1, 1.01]$,

$$v_{avg}(1.01) = \frac{s(1.01) - s(1)}{1.01 - 1}$$

$$v_{avg}(1.01) = -6.134 \text{ cm/s}$$

(iv) On the interval $[1, 1.001]$,

$$v_{avg}(1.001) = \frac{s(1.001) - s(1)}{1.001 - 1}$$

$$v_{avg}(1.001) = -6.268 \text{ cm/s}$$

(b) The instantaneous velocity of the particle when $t = 1$ appears to be about -6.268 cm/s.

$$v_{inst}(1) = \lim_{t\to 1} v_{avg}(t) = \lim_{t\to 1} \frac{s(t) - s(1)}{t - 1}$$

$$= -2\pi \text{ cm/s}$$